J. Math. Soc. Japan
Vol. 13, No. 1, 1961

# A property of a set of positive measure and its application

By Svetozar KUREPA

(Received Feb. 17, 1960)

Recently Z. Ciesielski has proved that a J-convex function of $m$-th order which is bounded on a set of strictly positive Lebesgue measure is continuous on some interval [**1**]. This is a generalisation of a well known result due to A. Ostrowski ($m=1$). On the other hand, T. Popoviciu has proved that the boundedness of a J-convex function of $m$-th order on some interval implies its continuity [**3**].

The main results of this paper are Theorems 1 and 2. In Theorem 1, we prove a property of a set of strictly positive Lebesgue measure in $n$-dimensional Euclidean space $E^n$, and, in Theorem 2, we use this result in order to prove that a function considered there which is bounded on a set $P \subseteqq E^n$ of strictly positive measure is bounded on some sphere. Since a J-convex function of $m$-th order satisfies the conditions of Theorem 2, we find, in Theorem 3, that the boundedness of a J-convex function on a set of positive measure implies its boundedness on some interval and (by the result of T. Popoviciu) its continuity on this interval. Theorem 4 is an application of Theorem 2. It is a generalisation of the well-known theorem according to which a measurable function $f$ such that:

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+ky) = 0$$

for all $x$ and $y$ is necessarily a polynomial of degree $\leqq m-1$.

NOTATIONS. An element $x \in E^n$ will be identified with a centered vector $x$ which has the terminal point $x$ and the initial point the origin. By $A-B$ ($A, B \subseteqq E^n$), we denote the set of all vectors $a-b$ with $a \in A$ and $b \in B$. For a real number $\alpha$ and a set $A \subseteqq E^n$, $\alpha A$ will denote the set of all $\alpha a$ with $a \in A$. The Lebesgue measure of a measurable set $A \subseteqq E^n$ is denoted by $mA$.

THEOREM 1. *Let $E^n$ be n-dimensional Euclidean space, $P \subseteqq E^n$ a set of strictly positive Lebesgue measure, and $\alpha_1, \alpha_2, \cdots, \alpha_m$ real numbers such that*

$$0 < |\alpha_k| \leqq 1 \qquad (k=1, 2, \cdots, m).$$

*If $x_0$ is a point of density of the set $P$, then there are two spheres $K(x_0, r')$ and $K(x_0, r)$ around $x_0$ with radius $r'$ resp. $r$ such that:*

a) $K(x_0, r') \subseteqq K(x_0, r)$

b) *For exery* $x \in K(x_0, r')$ *there is a sequence of vectors*

$$a_k(x) \in P \cap K(x_0, r) \qquad (k=1, 2, \cdots, m)$$

*and a vector* $h(x)$ *with the property that:*

$$a_1(x) = x + \alpha_1 h(x),$$
$$a_2(x) = x + \alpha_2 h(x),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_m(x) = x + \alpha_m h(x).$$

PROOF. I. Since $x_0$ is a point of density of the set $P$, we have ([**2**, p. 156])

$$\lim_{\rho \to 0} \frac{m[P \cap K(x_0, \rho)]}{mK(x_0, \rho)} = 1.$$

Hence, for

$$\varepsilon = \frac{1}{2\left(\frac{1}{|\alpha_1|} + \frac{1}{|\alpha_2|} + \cdots + \frac{1}{|\alpha_m|}\right)},$$

there is a sphere $K(x_0, r)$ $(r > 0)$ such that

$$mS \leqq \varepsilon mK,$$

where $K = K(x_0, r)$, $Q = K \cap P$ and $S = K \backslash Q$. We assert that the set

$$T = \frac{Q - x_0}{\alpha_1} \cap \frac{Q - x_0}{\alpha_2} \cap \cdots \cap \frac{Q - x_0}{\alpha_m}$$

has strictly positive measure. Otherwise, we should have $mT = 0$ which implies:

$$\begin{aligned} mK(0, r) &= m[K(0, r) \backslash T] \\ &\leqq m\left[K(0, r) \cap \frac{Q - x_0}{\alpha_1}\right] + m\left[K(0, r) \cap \frac{Q - x_0}{\alpha_2}\right] + \cdots \\ &\quad + m\left[K(0, r) \cap \frac{Q - x_0}{\alpha_m}\right]. \end{aligned}$$

But $0 < |\alpha| \leqq 1$ implies:

$$\frac{K(0, r)}{\alpha} = K\left(0, \frac{r}{|\alpha|}\right) \supseteqq K(0, r).$$

Using this we find:

$$\begin{aligned} mK(0, r) &\leqq \sum_{k=1}^{m} m\left[\frac{K(0, r)}{\alpha_k} \Big\backslash \frac{Q - x_0}{\alpha_k}\right] \\ &= \sum_{k=1}^{m} \frac{1}{|\alpha_k|} m[K(0, r) \backslash (Q - x_0)] \end{aligned}$$

$$=\sum_{k=1}^{m}\frac{1}{|\alpha_k|}m[(x_0+K(0,r))\setminus Q]$$

$$=\Big(\sum_{k=1}^{m}\frac{1}{|\alpha_k|}\Big)m[K(x_0,r)\setminus Q].$$

Hence

$$mK(x_0,r)=mK(0,r)\leqq\Big(\sum_{k=1}^{m}\frac{1}{|\alpha_k|}\Big)mS\leqq\frac{mK}{2}$$

which is impossible. Thus the set $T$ has strictly positive measure.

II. The function

$$\eta(x)=m\Big[\frac{Q-x}{\alpha_1}\cap\frac{Q-x}{\alpha_2}\cap\frac{Q-x}{\alpha_m}\Big]$$

is continuous. In order to see this denote by $\chi(x;S)$ the characteristic function of the set $S$. Using some simple properties of such functions we find:

$$\eta(x)=\int_{E^n}\chi\Big(y;\frac{Q-x}{\alpha_1}\cap\frac{Q-x}{\alpha_2}\cap\cdots\cap\frac{Q-x}{\alpha_m}\Big)dy$$

$$=\int_{E^n}\chi\Big(y;\frac{Q-x}{\alpha_1}\Big)\chi\Big(y;\frac{Q-x}{\alpha_2}\Big)\cdots\chi\Big(y;\frac{Q-x}{\alpha_m}\Big)dy$$

$$=\int_{E^n}\chi(x+\alpha_1y;Q)\chi(x+\alpha_2y;Q)\cdots\chi(x+\alpha_my;Q)dy.$$

This and $0\leqq\chi\leqq1$ implies:

$$|\eta(x')-\eta(x)|\leqq\sum_{k=1}^{m}\int_{E^n}|\chi(x'+\alpha_ky;Q)-\chi(x+\alpha_ky;Q)|dy.$$

But

$$\int_{E^n}|\chi(x'+\alpha y;Q)-\chi(x+\alpha y;Q)|dy$$

$$=\frac{1}{|\alpha|}\int_{E^n}|\chi(x'+y;Q)-\chi(x+y;Q)|dy\to0$$

$$\text{as } x'\to x.$$

Thus the function $\eta$ is continuous on $E^n$. Since

$$\eta(x_0)=mT>0,$$

there is a sphere $K(x_0,r')$ such that $r'\leqq r$ and

$$\eta(x)>0$$

for every $x\in K(x_0,r')$. This implies that the set

$$\frac{Q-x}{\alpha_1} \cap \frac{Q-x}{\alpha_2} \cap \cdots \cap \frac{Q-x}{\alpha_m}$$

is not empty for any $x \in K(x_0, r')$. If $h(x)$ denotes an element of this set, then

$$h(x) = \frac{a_1(x)-x}{\alpha_1} = \frac{a_2(x)-x}{\alpha_2} = \cdots = \frac{a_m(x)-x}{\alpha_m}$$

with $a_1(x), a_2(x), \cdots, a_m(x) \in Q$. Thus for every $x \in K(x_0, r')$ there are vectors

$$a_k(x) \in Q = P \cap K(x_0, r) \qquad (k = 1, 2, \cdots, m)$$

such that

$$a_k(x) = x + \alpha_k h(x). \qquad \text{Q. E. D.}$$

THEOREM 2. *Let $f(x)$ be a real-valued function which is defined in a sphere $K \subset E^n$ and let $\gamma_0, \gamma_1, \cdots, \gamma_m$ and $\beta_0 < \beta_1 < \cdots < \beta_m$ be two sequences of real numbers such that $\gamma_0 \cdot \gamma_1 < 0$.*

*Further suppose that*

$$\sum_{k=0}^{m} \gamma_k f(x+\beta_k h) \geqq 0 \tag{1}$$

*for every $x$ and $h$ for which $x+\alpha_k h \in K$ $(k = 0, 1, \cdots, m)$.*

*If the function $f$ is bounded on a set $P \subseteqq K$ of strictly positive Lebesgue measure then $f$ is bounded in some sphere $K' \subseteqq K$.*

PROOF. From (1) we have:

$$-\gamma_0 f(x+\beta_0 h) \leqq \sum_{k=1}^{m} \gamma_k f(x+\beta_k h) \tag{2}$$

and

$$-\gamma_1 f(x+\beta_1 h) \leqq \gamma_0 f(x+\beta_0 h) + \sum_{k=2}^{m} \gamma_k f(x+\beta_k h). \tag{3}$$

Setting $y = x+\beta_0 h$ in (2) and $y = x+\beta_1 h$ in (3), we find:

$$-\gamma_0 f(y) \leqq \sum_{k=1}^{m} \gamma_k f[y+(\beta_k-\beta_0)h] \tag{2'}$$

and

$$-\gamma_1 f(y) \leqq \gamma_0 f[y+(\beta_0-\beta_1)h] + \sum_{k=2}^{m} \gamma_k f[y+(\beta_k-\beta_1)h]. \tag{3'}$$

Now set:

$$\alpha_k = \frac{\beta_k-\beta_0}{\beta} \qquad \text{for} \quad k = 1, 2, \cdots, m,$$

$$\alpha_0 = \frac{\beta_0-\beta_1}{\beta} \qquad \text{and}$$

$$\alpha_{m+k} = \frac{\beta_k-\beta_1}{\beta} \qquad \text{for} \quad k = 2, 3, \cdots, m,$$

where $\beta = \max_k \{|\beta_k-\beta_0|, |\beta_k-\beta_1|\}$.

Since $mP>0$, there is a point $x_0 \in P \subseteqq K$ which is a density point of the set $P$. We take a sphere $K(x_0, r)$ around $x_0$ such that $K(x_0, r) \subseteqq K$. Now $0<|\alpha_k| \leqq 1$ and the sphere $K(x_0, r)$ satisfy all conditions of Theorem 1. There is therefore a sphere $K(x_0, r') \subseteqq K(x_0, r)$ $(r'>0)$ with the property that $y \in K(x_0, r')$ implies the existence of $a_k(y) \in P \cap K(x_0, r)$ and a vector $h(y)$ such that

$$a_k(y) = y + \alpha_k h(y)\,.$$

For a given $y \in K(x_0, r')$ we set

$$h = \frac{h(y)}{\beta}\,.$$

If $-\gamma_0 f(y) \geqq 0$, then (2′) and the assumption

$$M = \sup_{y \in P} |f(y)| < +\infty$$

imply:

$$\begin{aligned} -\gamma_0 f(y) = |\gamma_0 f(y)| &\leqq \sum_{k=1}^{m} |\gamma_k|\ |f[y+(\beta_k-\beta_0)h]| \\ &= \sum_{k=1}^{m} |\gamma_k|\ |f[y+\alpha_k h(y)]| = \sum_{k=1}^{m} |\gamma_k|\ |f[a_k(y)]| \\ &\leqq M \sum_{k=1}^{m} |\gamma_k|\,, \qquad \text{i. e.,} \end{aligned}$$

$$|f(y)| \leqq M \sum_{k=1}^{m} \left| \frac{\gamma_k}{\gamma_0} \right| \leqq M \sum_{k=0}^{m} \left| \frac{\gamma_k}{\gamma_0} \right|. \tag{4}$$

If $-\gamma_0 f(y) \leqq 0$, then $-\gamma_1 f(y) \geqq 0$ and (3′) lead to

$$|f(y)| \leqq M \sum_{k=0}^{m} \left| \frac{\gamma_k}{\gamma_1} \right|. \tag{5}$$

From (4) and (5) we deduce

$$\sup |f(y)| < +\infty \qquad (y \in K(x_0, r'))\,,$$

i. e., the function $f$ is bounded in the sphere $K(x_0, r')$. Q. E. D.

THEOREM 3. *Let $f(x)$ be a real valued function of a real variable $x \in (a, b) = \Delta$ $(a<b)$. The function $f$ is called* J-*convex of the $m$-*th *order* (*i. e. convex in the Jensen sense*) [**1**] *on $\Delta$ if*

$$\Delta_h^{m+1} f(x) \geqq 0$$

*for all $x$ and $h$ for which*

$$x, x+h, \cdots, x+(m+1)h \in \Delta\,,$$

*where*

$$\Delta_h^k f(x) = \Delta_h^{k-1} f(x+h) - \Delta_h^{k-1} f(x)\,,$$

$$\Delta_h^0 f(x) = f(x)\,.$$

*If $f$ is bounded on a set $P \subseteqq \Delta$ of strictly positive Lebesgue measure, then $f$ is continuous in some interval* ([**1**, Theorem 1, p. 3]).

PROOF. It is readily seen that $f$ satisfies all conditions of Theorem 2. Thus $f$ is bounded on some subinterval $\delta$ of $\Delta$. By a result of T. Popoviciu [**3**], a J-convex function of order $m$ bounded on $\delta$ is also continuous in $\delta$. Thus $f$ is a continuous function on some subinterval of $\Delta$. Q. E. D.

THEOREM 4. *Let $f$ be a real-valued function which is defined on the set of all real numbers and let $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ ($\alpha_k \neq 0$, $k = 1, 2, \cdots$) be a sequence of real numbers such that*

$$\sum_{k=0}^{m} \gamma_k f(x + \alpha_k y) = 0 \tag{6}$$

*holds for all $x$ and $y$, where $\gamma_0, \gamma_1, \cdots, \gamma_m$ are some real numbers such that $\gamma_0 \cdot \gamma_1 < 0$.*

*If the function $f$ is measurable, then $f$ is a polynomial of degree $\leqq m-1$.*

PROOF. Being measurable, $f$ is bounded on a set $P$ of strictly positive Lebesgue measure. This and Theorem 2 imply that $f$ is bounded on some interval which obviously leads to boundedness of $f$ on every finite interval. Thus $f$ is summable on every finite interval. If in (6) we replace $x + \alpha_0 y$ by $x$ we find:

$$\gamma_0 f(x) + \sum_{k=1}^{m} \gamma_k f(x + \beta_k y) = 0\,, \tag{7}$$

where $\beta_k = \alpha_k - \alpha_0$ ($k = 1, 2, \cdots, m$). Now we integrate (7) with respect to $y$ from 0 to 1. We get:

$$\gamma_0 f(x) = -\sum_{k=1}^{m} \frac{\gamma_k}{\beta_k} \int_x^{x+\beta_k} f(y)dy\,. \tag{8}$$

From (8) and $\gamma_0 \neq 0$, we conclude that $f(x)$ is a continuous function on the set of real numbers. But from (8) we see that $f$ is also derivable and that its derivative is a sum of derivable functions. This implies the existence of $f', f''$, etc., i. e., $f$ possesses derivatives of all orders.

If we take the $p$-th derivative of (6) with respect to $y$ and if we set $y = 0$, we get:

$$\Big(\sum_{k=1}^{m} \gamma_k \alpha_k^{p}\Big) f^{(p)}(x) = 0\,.$$

If $f^{(m)}(x)$ is different from zero in at least one point, then we have:

$$\sum_{k=1}^{m} \gamma_k \alpha_k^{p} = 0 \qquad p = 0, 1, 2, \cdots, m\,. \tag{9}$$

Since $\gamma_0 \cdot \gamma_1 < 0$, the system (6) has non-trivial solution $\gamma$. Thus:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_m \\ \alpha_0{}^2 & \alpha_1{}^2 & \cdots & \alpha_m{}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_0{}^m & \alpha_1{}^m & \cdots & \alpha_m{}^m \end{vmatrix} = 0\,,$$

which contradicts the assumption that all $\alpha$'s are different one from another. Thus

$$f^{(m)}(x) \equiv 0\,,$$

i. e., $f$ is a polinomial of degree $\leqq m-1$. Q. E. D.

Department of Mathematics
Zagreb, Yugoslavia

## References

[1] Z. Ciesielski, Some properties of convex function of higher orders, Ann. Pol. Math., 7 (1959), 1-7.
[2] N. Dunford and J. Schwartz, Linear operators I, 1958.
[3] T. Popoviciu., Sur quelques propriétés des fonctions d'une ou deux variables réelles, Mathematica, 8 (1934), 1-85.