NOTES

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A Property of Parallelograms Inscribed in Ellipses

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1. INTRODUCTION. The following surprising property of ellipses was observed by the physicist Jean-Marc Richard, in connection with a problem from ballistics [**2**, p. 843].

Theorem 1. *Let $\mathcal{E}$ be an ellipse and $f(d, d')$ the function of two diameters given by the perimeter of the parallelogram with vertices $d \cap \mathcal{E}$ and $d' \cap \mathcal{E}$ (Figure 1). Then*

$$f(d) := \sup_{d'} f(d, d')$$

is constant (independent of d).

In other words, the maximal perimeter of a parallelogram inscribed in a given ellipse can be realized by a parallelogram with one vertex at any prescribed point of the ellipse.

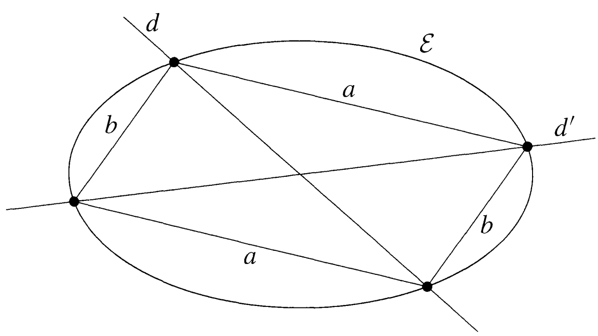

Figure 1. $f(d, d') = 2a + 2b$.

Richard proved this result by a direct computational verification. Jean-Pierre Bourguignon told us about the theorem and asked whether one could give a more enlightening proof. In this note we give two simple proofs, one geometric and the other algebraic, as well as a small generalization. We also describe briefly a connection with billiards that was pointed out to us by Sergei Tabachnikov. A different proof of Theorem 1 is given in [**1**, p. 350].

2. GEOMETRIC PROOF. We begin with a proof using classical geometry. The first step is to show that the theorem follows from a simple observation:

Fact. *The map S that associates to d the diameter $d' = S(d)$ where $f(d, d')$ reaches its (unique) maximum is an involution: $S^2(d) = d$ for all d.*

This implies Theorem 1. Indeed, with $d' = S(d)$ one has $\partial/\partial d' \, f(d, d') = 0$, but then since S is involutive one also has $\partial/\partial d \, f(d, d') = 0$. Thus the gradient of $f(d, d')$ vanishes and the derivative of $f(d) := \sup_{d'} f(d, d') = f(d, S(d))$ also vanishes.[1] ■

Both the "Fact" and the uniqueness of the maximum follow from the following geometric lemma, in which we write d as $[P, -P]$ and $\tau(d)$ for the tangent to the ellipse at P (the two choices are parallel):

Lemma. *Let d and d' be two diameters of $\mathcal{E}$. Then $d' = S(d)$ if and only if $\tau(d)$ is perpendicular to $\tau(d')$.*

Proof. This is quite easy and has many proofs. We give one that uses only the purest definition of a conic, namely, Pascal's projective characterization: six points $p_1, \dots, p_6$ of the plane are on the same conic if and only if the three points

$$[p_1, p_2] \cap [p_4, p_5], \qquad [p_2, p_3] \cap [p_5, p_6], \qquad [p_3, p_4] \cap [p_6, p_1]$$

are collinear. Taking $p_1 = p_2 = P$, $p_3 = -P$, $p_4 = p_5 = P'$, and $p_6 = -P'$, we conclude that

$$[-P, P'] \| [O, Y],$$

where $Y = [p_1, p_2] \cap [p_4, p_5]$ is the intersection of the tangents, $O = [p_2, p_3] \cap [p_5, p_6]$ is the center of the ellipse, and $[p_3, p_4] \cap [p_6, p_1]$ is the point at infinity in the direction $[-P, P']$ (see Figure 2). In particular, OY bisects PP'. But at a maximum of $f(d, d')$ the tangent to the ellipse at P' has equal angles with $[P', P]$ and $[P', -P]$, so the intersection point of OY and PP' is also equidistant from P' and Y. Thus Y belongs to the circle with diameter $[P, P']$, making $\widehat{PYP'}$ a right angle. ■

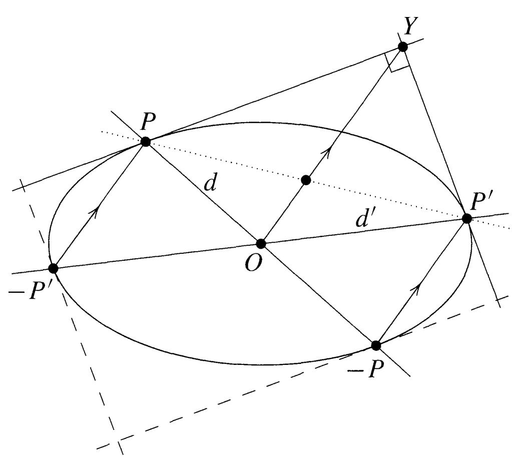

Figure 2. $f(d, d') =$ maximum if and only if $\angle PYP' = 90^o$.

[1] It is easy to justify the smoothness of the functions involved since one stays away from the diagonal $d = d'$ on which $f(d, d')$ is minimal.

Remarks. 1. The well-known geometric object to which the foregoing discussion relates is the *orthoptic circle* of Monge, who proved that the locus of points from which one sees an ellipse at right angles is a circle. The parallelism of $[-P, P']$ and $[O, Y]$ shows that $f(d)$ is four times the distance from 0 to Y. This allows us to bypass the first step using Monge's theorem, or (much better) to prove Monge's theorem using that differential geometric step.

2. Another geometric proof of the lemma, due to Marcel Berger [**1**], is to show, by projective reduction to the circle, that for a parallelogram inscribed in an ellipse the directions of the tangents to the ellipse at the vertices are in harmonic division with the directions of the sides.

3. ALGEBRAIC PROOF AND GENERALIZATION. In this section we prove a somewhat stronger version of the original theorem, in which the two diameters d and d' are allowed to belong to different ellipses with the same foci.

Define a "modified distance function" in $\mathbb{R}^2$ by

$$F(P, P') = \|P - P'\| + \|P + P'\| \qquad (P, P' \in \mathbb{R}^2).$$

Clearly this depends only on the images of P and P' in $\mathbb{R}^2/\{\pm 1\}$. In the situation of Theorem 1, if $\mathcal{E}$ has its center at $O = (0, 0)$ in $\mathbb{R}^2$ and $d = [-P, P]$ and $d' = [-P', P']$ are two diameters, then $f(d, d') = 2F(P, P')$.

Theorem 2. *Let $\mathcal{E}$ be an ellipse in $\mathbb{R}^2$ with center at the origin, and let $\mathcal{E}'$ be an arbitrary ellipse confocal with $\mathcal{E}$. Then the function $\mathbb{R}^2 \to \mathbb{R}_+$ defined by $P' \mapsto \max\{F(P, P') : P \in \mathcal{E}\}$ is constant on $\mathcal{E}'$.*

In the special case when $\mathcal{E} = \mathcal{E}'$ this reduces to Theorem 1, while in the limiting case when $\mathcal{E}$ is a degenerate ellipse consisting of the line joining two points P_0 and $-P_0$ in $\mathbb{R}^2$, it reduces to the standard definition of an ellipse with foci P_0 and $-P_0$ as a level curve of the function $F(\cdot, P_0)$.

Proof of Theorem 2. Without loss of generality we can assume that the major and minor axes of $\mathcal{E}$ and $\mathcal{E}'$ are the x- and y-axes. Then these ellipses have equations

$$\mathcal{E} : \frac{x^2}{\lambda} + \frac{y^2}{\mu} = 1, \qquad \mathcal{E}' : \frac{x'^2}{\lambda'} + \frac{y'^2}{\mu'} = 1 \tag{1}$$

for some $\lambda, \mu, \lambda', \mu' > 0$ with $\lambda - \mu = \lambda' - \mu'$. Set $C = \lambda + \mu' = \lambda' + \mu$. We claim that

$$F(P, P') \leq 2\sqrt{C} \tag{2}$$

for any $P = (x, y)$ in $\mathcal{E}$ and $P' = (x', y')$ in $\mathcal{E}'$, with equality if and only if P and P' satisfy

$$\frac{xx'}{\lambda\lambda'} + \frac{yy'}{\mu\mu'} = 0. \tag{3}$$

Since (3) has a solution (unique up to sign) $P = (x, y)$ in $\mathcal{E}$ for any $P' = (x', y')$ in $\mathcal{E}'$, this shows that $\max\{F(P, P') : P \in \mathcal{E}\} = 2\sqrt{C}$ for all P' in $\mathcal{E}'$. Note that (3) reduces to the lemma of Section 2 if $\mathcal{E} = \mathcal{E}'$.

To establish the claim, we first find by an easy calculation that $\|P \pm P'\| = \sqrt{C}\left(1 \pm xx'/\lambda\lambda'\right)$ if (3) is satisfied, so that $F(P, P') = 2\sqrt{C}$. To see that this is an extremal value (it is then easy to check that it is a maximum), we observe that a point near to P on $\mathcal{E}$ has the form $P + \varepsilon P^* + O(\varepsilon^2)$ with $P^* = (\lambda y, -\mu x)$ and ε small. The equation

$$\|P + \varepsilon P^* + O(\varepsilon^2) \pm P'\| = \|P \pm P'\| + \varepsilon \frac{(P \pm P', P^*)}{\|P \pm P'\|} + O(\varepsilon^2)$$

shows that, for P' fixed, the function $P \mapsto F(P, P')$ is stationary at P if and only if the vector

$$\frac{P - P'}{\|P - P'\|} + \frac{P + P'}{\|P + P'\|}$$

is orthogonal to P^*. Using the formulas for $\|P \pm P'\|$ given earlier, one checks that this property is true under assumption (3). ■

This algebraic proof is parallel to, but less transparent than, the geometric argument given in the previous section. However, now that we know the value $2\sqrt{C}$ for the maximum of $F(P, P')$, we can give a new argument that makes the theorem nearly obvious, as follows:

Let ellipses $\mathcal{E}$ and $\mathcal{E}'$ be given by the equations (1). As already mentioned, they are confocal if and only if $\lambda - \mu = \lambda' - \mu'$, their foci (assuming that $c := \lambda - \mu$ is nonnegative) then being at P_0 and $-P_0$, where $P_0 = (\sqrt{c}, 0)$, so the standard definition of an ellipse as the set of points with a given sum of distances from the two foci gives

$$P \in \mathcal{E} \Leftrightarrow F(P, P_0) = 2\sqrt{\lambda}, \qquad P' \in \mathcal{E}' \Leftrightarrow F(P', P_0) = 2\sqrt{\lambda'}.$$

The inequality (2) therefore can be restated as

$$F(P, P')^2 \leq F(P, P_0)^2 + F(P', P_0)^2 - F(P_0, P_0)^2,$$

with equality on a 1-dimensional subvariety of $\mathcal{E} \times \mathcal{E}'$. If we define $D : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ by

$$D(P, P') = \frac{1}{2} F(P, P')^2 - \|P\|^2 - \|P'\|^2 = \|P - P'\| \, \|P + P'\| \qquad (P, P' \in \mathbb{R}^2),$$

then this inequality translates into the triangle inequality $D(P, P') \leq D(P, P_0) + D(P_0, P')$. But this is now obvious, because if we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way and then identify $\mathbb{R}^2/\{\pm 1\} = \mathbb{C}/\{\pm 1\}$ with $\mathbb{C} = \mathbb{R}^2$ via $P \mapsto z = P^2$, then

$$D(P, P') = |P - P'| \, |P + P'| = |P^2 - P'^2| = |z - z'|$$

is just the usual distance with respect to the Euclidean metric in $\mathbb{R}^2$. It is now also clear that equality holds if and only if the points z and z' lie on opposite sides of the point $z_0 = P_0^2 = (c, 0)$ (see Figure 3); in particular, for each z it holds for a unique z'.

Remarks. 1. A simple calculation reveals that the images under the squaring map $P \mapsto P^2$ of the family of confocal ellipses with foci at $\sqrt{c} = (\sqrt{c}, 0)$ and $-\sqrt{c} = (-\sqrt{c}, 0)$ is the family of confocal ellipses with foci at $0 = (0, 0)$ and $c = (c, 0)$.

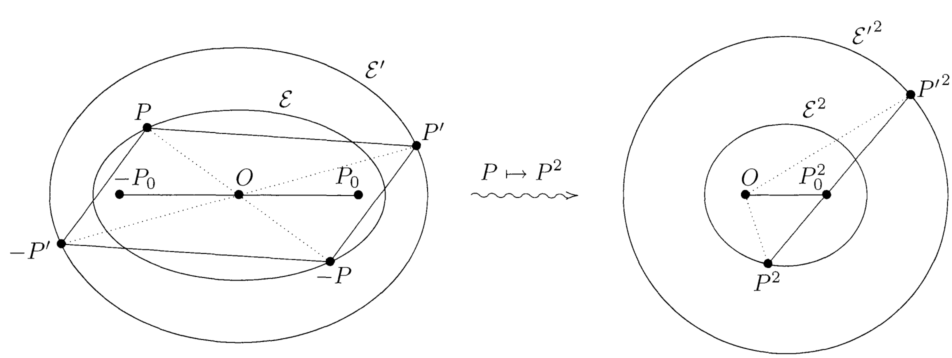

Figure 3. $F(P, P')$ maximal if and only if P_0^2 lies between P^2 and P'^2.

2. It would be interesting to see whether the more general assertion of Theorem 2 implies a generalization of Monge's theorem and of the notion of orthoptic circle to the case of two confocal ellipses. A first step would be to give a geometric interpretation of equation (3) generalizing the statement that the tangents $\tau(d)$ and $\tau(d')$ are orthogonal in the case $\mathcal{E} = \mathcal{E}'$.

4. A CONNECTION WITH BILLIARDS. In the proof of Theorem 1 given in Section 2, we mentioned that if d and d' are diameters of $\mathcal{E}$ such that d' maximizes $f(d, \cdot)$, then the tangent to $\mathcal{E}$ at P' forms equal angles with $[P', P]$ and $[P', -P]$. Moreover, we saw that in this case d also maximizes $f(\cdot, d')$, so the tangent to $\mathcal{E}$ at $-P$ also makes equal angles with $[-P, P']$ and $[-P, -P']$. Hence the polygonal path with vertices $\dots, P, P', -P, -P', P, \dots$ is a billiard trajectory (i.e., the path of a light ray inside a curved mirror) of period 4. The theorem proved in this note is therefore clearly closely related to the theory of billiards in an ellipse. In fact, it can be deduced from the main theorem of this theory, the so-called complete integrability of the system (see [**3**, Theorem 2.1.2] or [**4**, Theorem 4.4]), which asserts the following: if $\mathcal{E}$ is an ellipse centered at 0 with foci P_0 and $-P_0$, if $[P, P']$ is a chord of $\mathcal{E}$ not intersecting the line $[-P_0, P_0]$, and if $[P', P'']$ is the result of the billiard (optical reflection) of PP' in $\mathcal{E}$ at P', then the chord $[P', P'']$ also does not intersect $[-P_0, P_0]$ and both chords are tangent to the same ellipse $\mathcal{E}^*$ confocal with $\mathcal{E}$. Iterating, we infer that the entire billiard trajectory $P, P', P'', P''', \dots$ obtained by successive reflections in $\mathcal{E}$ consists of segments tangent to $\mathcal{E}^*$. More conceptually, we have a phase space $\mathcal{S}$ consisting of oriented chords of $\mathcal{E}$ not intersecting $[-P_0, P_0]$ and a map $T : \mathcal{S} \to \mathcal{S}$ sending a chord to its reflection at an end-point (billiard map), and $\mathcal{S}$ is foliated by T-invariant circles consisting of the chords tangent to a given confocal ellipse of $\mathcal{E}$. On any such circle, in a suitable coordinate, T is just rotation by a fixed angle. In particular, if one trajectory of such a circle is n-periodic for some n (we are concerned here with the case $n = 4$), then all are [**4**, Corollary 4.5]. On the other hand, n-periodic billiard trajectories are inscribed n-gons of extremal length. On the invariant circle of $\mathcal{S}$ corresponding to $n = 4$ (or any other fixed value of n), the perimeter length of the corresponding quadilateral (or n-gon) is constant, since this circle is a critical manifold for the perimeter length function. It follows that, for any point P of $\mathcal{E}$, the greatest perimeter of an inscribed quadilateral with one vertex at P is independent of P. But it is also easy to see that a 4-periodic (or n-periodic with n even) billiard trajectory is necessarily centrally symmetric, so this is just the statement of Theorem 1.

This argument requires more sophisticated mathematics and more knowledge than the proofs presented in Sections 2 or 3, but it also yields a more general result:

for any even n the maximal length of a centrally symmetric inscribed n-gon in $\mathcal{E}$ having a given point P of $\mathcal{E}$ as a vertex is independent of P. Theorem 2 can be proved and generalized along the same lines. The generalization reads as follows: if $\mathcal{E}_1, \dots, \mathcal{E}_k$ are confocal ellipses, then the maximal length of a centrally symmetric $2k$-gon $(P_1, \dots, P_k, -P_1, \dots, -P_k)$ with P_i in $\mathcal{E}_i$, as $P_2, \dots, P_k$ vary with P_1 fixed, is independent of P_1.

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An Elementary Proof of the Wallis Product Formula for pi

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1. THE WALLIS PRODUCT FORMULA. In 1655, John Wallis wrote down the celebrated formula

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots = \frac{\pi}{2}. \tag{1}$$

Most textbook proofs of (1) rely on evaluation of some definite integral like

$$\int_0^{\pi/2} (\sin x)^n \, dx$$

by repeated partial integration. The topic is usually reserved for more advanced calculus courses. The purpose of this note is to show that (1) can be derived using only the mathematics taught in elementary school, that is, basic algebra, the Pythagorean theorem, and the formula $\pi \cdot r^2$ for the area of a circle of radius r.

Viggo Brun gives an account of Wallis's method in [**1**] (in Norwegian). Yaglom and Yaglom [**2**] give a beautiful proof of (1) which avoids integration but uses some quite sophisticated trigonometric identities.

2. A NUMBER SEQUENCE. We define a sequence of numbers by $s_1 = 1$, and for $n \geq 2$,

$$s_n = \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2}.$$