PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

A PROPERTY OF PROJECTIVE IDEALS IN SEMIGROUP ALGEBRAS

WILLIAM R. NICO[1]

ABSTRACT. The condition that certain left ideals in a finite monoid generate projective left ideals in the semigroup algebra imposes a strong restriction on the intersection of principal left ideals in the semigroup.

Let S be a finite monoid, k be a commutative ring with identity, and let $I \subset S$ be a left ideal in S. We demand that kI be projective as a left kS-module and investigate the resulting restrictions on the structure of I. In particular we can look for necessary conditions on S for kS to be left hereditary. (A sufficient condition is obtained in [4] and [5].) Semigroup terminology below follows [1] and [2].

We first need the following facts, which are valid in any ring with identity.

LEMMA 1. *Let R be a ring with identity. Let $I \subset R$ be a left ideal which is projective as a left R-module. Let $e \in R$ be any idempotent. Then the left ideal $I+Re$ is projective if and only if $I \cap Re$ is a direct summand of I.*

PROOF. We observe that we have the following two short exact sequences:

$$0 \to I(1-e) \to I + Re \to Re \to 0,$$
$$0 \to I \cap Re \to I \to I(1-e) \to 0,$$

where the map on the right end of the first sequence is $x \mapsto xe$, which has kernel $(I+Re)\cap R(1-e)=I(1-e)$, and the map on the right end of the second sequence is $x \mapsto x(1-e)$. Since Re is projective, the first sequence always splits, so that $I+Re$ is projective if and only if $I(1-e)$ is. On the other hand, since I is projective, $I(1-e)$ is projective if and only if the second sequence splits.

Received by the editors June 5, 1972.

AMS (MOS) subject classifications (1970). Primary 20M10, 20M25; Secondary 16A32, 16A50, 16A60.

Key words and phrases. Semigroup, monoid, semigroup algebra, projective modules, idempotents, left ideals.

[1] This research supported in part by the National Science Foundation.

COROLLARY. *If R is a ring with identity and if e_1 and e_2 are idempotents of R, then Re_1+Re_2 is a projective left ideal of R if and only if $Re_1 \cap Re_2$ has an idempotent generator.*

PROOF. By the lemma, $Re_1 \cap Re_2$ is a direct summand of Re_1. If π: $Re_1 \to Re_1 \cap Re_2$ is a retraction of Re_1 onto $Re_1 \cap Re_2$, then $f=\pi(e_1)$ is the desired idempotent since $f^2=f\pi(e_1)=\pi(fe_1)=\pi(f)=f$ and $Rf=\pi(Re_1)=Re_1 \cap Re_2$.

Now let S be a finite monoid, $e_1, e_2 \in S$ be idempotents, and k be a commutative ring with identity. Then we have the following:

THEOREM 1. (i) *If the left ideal $kSe_1+kSe_2 \subset kS$ is projective as a left kS-module, then the semigroup left ideal $Se_1 \cap Se_2 \subset S$ is generated by idempotents of S, or $Se_1 \cap Se_2 = \varnothing$.*

(ii) *Furthermore, if L is an $\mathscr{L}$-class of S maximal in the $\mathscr{L}$-class ordering with respect to the property that $L \subset Se_1 \cap Se_2$ and if J is the $\mathscr{J}$-class ($=\mathscr{D}$-class) of S containing L, then*

(a) *J is a regular $\mathscr{J}$-class of S, and*

(b) *if $J^0 \approx \mathscr{M}^0(G; I, \Lambda, P)$ is a Rees matrix representation of J^0 with maximal subgroup G, index sets I and Λ (for the $\mathscr{R}$- and $\mathscr{L}$-classes, respectively), and sandwich matrix P, if Λ' is the subset of Λ which corresponds to the set of $\mathscr{L}$-classes of J which satisfy the above maximality condition, and if $\mathscr{M}^0(G; I, \Lambda', P')$ is the correspondingly restricted Rees matrix semigroup, then P' has a right inverse as a matrix over kG.*

PROOF. By the corollary to Lemma 1, there is some $\gamma=\gamma^2 \in kSe_1 \cap kSe_2$ such that $kSe_1 \cap kSe_2 = kS\gamma$. We observe also that $kSe_1 \cap kSe_2 = k(Se_1 \cap Se_2)$. Suppose $Se_1 \cap Se_2 \neq \varnothing$, and let L be an $\mathscr{L}$-class of S maximal with respect to $L \subset Se_1 \cap Se_2$. Let $U=\{x \in S : Sx \not\supset L\}$. Then U and $U \cup L$ are left ideals of S, $U \cap L = \varnothing$, and the maximality of L implies that $Se_1 \cap Se_2 \subset U \cup L$.

Now one can write $\gamma=\gamma_L+\gamma_U$ where γ_L is a k-linear combination of elements of L and γ_U is a k-linear combination of elements of U. If $s \in L \subset Se_1 \cap Se_2$, then $s=s\gamma=s\gamma_L+s\gamma_U$, where $s\gamma_L$ is a k-linear combination of elements of L^2 and $s\gamma_U$ is a k-linear combination of elements of U. Since the elements of S are linearly independent over k, this implies that $L \subset L^2$. If J is the $\mathscr{J}$-class containing L, then $J \cap J^2 \neq \varnothing$. Hence J is regular, and thus L contains an idempotent. Since $Se_1 \cap Se_2 = \bigcup \{SL : L \subset S$ is a maximal $\mathscr{L}$-class as above$\}$, we have that $Se_1 \cap Se_2$ is generated by idempotents.

Let $J^0 \approx \mathscr{M}^0(G; I, \Lambda, P)$; let $\{L_\lambda : \lambda \in \Lambda\}$ be the $\mathscr{L}$-classes of J; and let $\{L_\lambda : \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$ be the set of $\mathscr{L}$-classes of J maximal in the $\mathscr{L}$-class ordering with respect to $L_\lambda \subset Se_1 \cap Se_2$. Then we let P' be the corresponding

$|\Lambda'|\times|I|$ submatrix of the $|\Lambda|\times|I|$ matrix P and see that the Rees matrix semigroup $\mathscr{M}^0(G; I, \Lambda', P')$ is isomorphic to V^0 where $V=\bigcup\{L_\lambda:\lambda\in\Lambda'\}$.

Let $W=\{x\in S:Sx\cap V=\varnothing\}$. Then W and $W\cup V$ are left ideals of S, $W\cap V=\varnothing$, and $Se_1\cap Se_2\subset W\cup V$. We can write $\gamma=\gamma_V+\gamma_W$ with γ_V (respectively γ_W) a k-linear combination of elements of V (respectively W). If we write γ_V as an $|I|\times|\Lambda'|$ matrix Q over kG, then the fact that $s\gamma_V= s\gamma-s\gamma_W=s-s\gamma_W$ for all $s\in V$, and $s\gamma_W\in kW$, implies that the matrix product $P'Q$ is a $|\Lambda'|\times|\Lambda'|$ identity matrix.

Corollary 1. *In the situation of* (ii) *above, it must be that* $|I|\geqq|\Lambda'|$.

Proof. If we extend the augmentation homomorphism $kG\to k$, given by $g\mapsto 1$ for all $g\in G$, to matrices over kG, then since $P'Q$ is the identity, the rank of the image of P' must be $|\Lambda'|$, which implies that $|I|\geqq|\Lambda'|$.

Corollary 2. *No two rows of P' have zero and nonzero entries in exactly the same locations.*

Proof. If P' had two such rows, then the rank of the image of P' under the augmentation homomorphism would be less than $|\Lambda'|$, impossible.

Corollary 3. *If S is a union of groups semigroup, if e_1 and $e_2\in S$ are idempotents, and if kSe_1+kSe_2 is kS-projective, then if $Se_1\cap Se_2\neq\varnothing$, the $\mathscr{L}$-classes L of S maximal with respect to $L\subset Se_1\cap Se_2$ all lie in distinct $\mathscr{J}$-classes.*

Proof. If S is a union of groups then all the entries of P and P' above must be nonzero. Thus by Corollary 2, P' has only one row, that is, $|\Lambda'|=1$, which says that only one $\mathscr{L}$-class L of J is maximal with respect to $L\subset Se_1\cap Se_2$.

Theorem 2. *Let S be a union of groups monoid such that the $\mathscr{J}$-classes are linearly ordered. If kS is left hereditary, then the intersection of any two principal left ideals of S is either empty or itself principal.*

If, in addition, the ring k is noetherian (e.g., if k is a field), then kS is right hereditary also, so that the intersection of any two principal right ideals is either empty or again principal.

Proof. The last part of the theorem follows from the first part by observing that k noetherian and S finite imply that kS is noetherian and that right and left global dimension are equal for noetherian rings (see, e.g., [3]).

To prove the first part we see first that since S is a union of groups, principal left ideals have idempotent generators. Let e_1, $e_2\in S$ be idempotents. Then kSe_1+kSe_2 is kS-projective since kS is left hereditary. Hence there exists an idempotent $\gamma\in kSe_1\cap kSe_2$ such that $kSe_1\cap kSe_2=kS\gamma$.

Also, by Corollary 3 above, if $Se_1 \cap Se_2 \neq \varnothing$, the $\mathscr{L}$-classes L of S maximal with respect to $L \subset Se_1 \cap Se_2$ lie in distinct $\mathscr{J}$-classes of S. Then the proof is concluded by the following lemma.

LEMMA. *If S is a union of groups monoid whose $\mathscr{J}$-classes are linearly ordered and if $I \subset S$ is a left ideal such that the $\mathscr{L}$-classes L of S maximal with respect to $L \subset I$ lie in distinct $\mathscr{J}$-classes of S, then kI has an idempotent generator if and only if I is principal.*

PROOF. If I is a principal left ideal of S it has an idempotent generator in S, which will also serve as an idempotent generator for kI in kS.

Now suppose we have I and an idempotent $\gamma = \gamma^2 \in kI$ such that $kI = kS\gamma$. Suppose that L_1 and L_2 are distinct $\mathscr{L}$-classes of S maximal with respect to $L_i \subset I$. Then L_1 and L_2 lie in distinct $\mathscr{J}$-classes of S, say $L_i \subset J_i$, $i = 1, 2$, and $J_1 \geqq J_2$ in the $\mathscr{J}$-class ordering. Let $I' = I \backslash (L_1 \cup L_2)$. Then I' is a left ideal of S. We can write $\gamma = \gamma_1 + \gamma_2 + \gamma'$ where γ_i is a k-linear combination of elements of L_i for $i = 1, 2$, and γ' is a k-linear combination of elements of I'. Let $e_i \in L_i$ be idempotents for $i = 1, 2$. Now $e_1 e_2 \mathscr{J} e_2 e_1 \mathscr{J} e_2$ since S is a union of groups (see [**2**, Theorem 4.4]). Hence we know that $e_1 e_2 \mathscr{L} e_2$ so that since J_1 and J_2 are completely simple, $e_1 L_2 \subset L_2$ and $e_i L_i \subset L_i$ for $i = 1, 2$. The maximality of L_2 and the fact that $e_2 e_1 \mathscr{J} e_2$ gives $e_2 L_1 \cap L_2 = \varnothing$.

Now $e_1 = e_1\gamma = e_1\gamma_1 + e_1\gamma_2 + e_1\gamma'$ implies that $e_1\gamma_1 = e_1$, $e_1\gamma_2 = 0$, and $e_1\gamma' = 0$ since elements of S are linearly independent over k. Thus if $\gamma_2 = \sum_{y \in L_2} m_y y$, $m_y \in k$, this will imply that $\sum_{y \in L_2} m_y = 0$. But also $e_2 = e_2\gamma = e_2\gamma_1 + e_2\gamma_2 + e_2\gamma'$ implies that $e_2\gamma_2 = e_2$, since $e_2\gamma_2$ is a k-linear combination of elements of L_2 while $e_2\gamma_1 + e_2\gamma'$ is a k-linear combination of elements of $I \backslash L_2$. However, $e_2 = e_2\gamma_2$ implies that $\sum_{y \in L_2} m_y = 1$, contradiction. Thus it is impossible that there be two distinct $\mathscr{L}$-classes of S maximal with respect to containment in I. Thus there is a unique such $\mathscr{L}$-class, say L_1, which then says that $I = Se_1$.

REFERENCES

1. M. Arbib (Editor), *Algebraic theory of machines, languages, and semigroups*, Academic Press, New York, 1968. MR **38** #1198.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. 1, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1961. MR **24** #A2627.

3. J. P. Jans, *Rings and homology*, Holt, Rinehart and Winston, New York, 1964. MR **29** #1243.

4. W. R. Nico, *Homological dimension in semigroup algebras*, J. Algebra **18** (1971), 404–413. MR **43** #3359.

5. ———, *An improved upper bound for global dimension in semigroup algebras*, Proc. Amer. Math. Soc. **35** (1972), 34–36.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118