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À PROPOS DE CANARDS (APROPOS CANARDS)

MARTIN WECHSELBERGER

ABSTRACT. We extend canard theory of singularly perturbed systems to the general case of k slow and m fast dimensions, with $k \geq 2$ and $m \geq 1$ arbitrary. A folded critical manifold of a singularly perturbed system, a generic requirement for canards to exist, implies that there exists a local $(k+1)$-dimensional center manifold spanned by the k slow variables and the critical eigendirection of the fast variables. If one further assumes that the $m-1$ nonzero eigenvalues of the $m \times m$ Jacobian matrix of the fast equation have all negative real part, then the $(k+m)$-dimensional singularly perturbed problem is locally governed by the flow on the $(k+1)$-dimensional center manifold. By using the blow-up technique (a desingularization procedure for folded singularities) we then show that the local flow near a folded singularity of a k-dimensional folded critical manifold is, to leading order, governed by a three-dimensional canonical system for any $k \geq 2$. Consequently, results on generic canards from the well-known case $k = 2$ can be extended to the general case $k \geq 2$.

1. A brief history of canards

Canards were discovered by a group of French mathematicians [3] by means of nonstandard analysis [15] who studied the van der Pol relaxation oscillator [44] with constant forcing. In this slow-fast vector field in $\mathbb{R}^2$, the *canard phenomenon* explains the transition upon variation of a system parameter from small limit cycles via *canard cycles* to large amplitude relaxation cycles. To be more precise, this transition happens in an exponentially small interval of the system parameter and is thus called a *canard explosion*. The term 'canard' indicates that these 'creatures' are degenerate since they only exist in one-parameter families of slow-fast vector fields in $\mathbb{R}^2$. Furthermore, they are always associated with a nearby singular Hopf bifurcation which creates the small limit cycles. In practice, only the small limit cycles or the large relaxation cycles are observed but no canard cycles due to the exponential sensitivity in the parameter. Even if the exponentially small parameter interval is known, numerical observation of all canard cycles with standard initial value solvers is impossible due to extreme sensitivity to numerical errors. Thus, their discovery seemed more like a hoax in a newspaper – a 'canard'. Furthermore, certain canard cycles in $\mathbb{R}^2$ resemble (with a little help of imagination) the shape of a 'duck'. Voilà, the notion of canard was born and since then the chase

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of these creatures was pursued not only by nonstandard techniques [3, 16] but also by matched asymptotic expansion techniques [19, 37] and geometric singular perturbation theory [18, 33].

Besides being the missing link to explain a smooth transition from small to large amplitude relaxation oscillations in $\mathbb{R}^2$, the relevance of canards in applications was considered 'spurious' in the sciences due to its degenerate nature in $\mathbb{R}^2$. For example, in the chemical literature, the abrupt transition from small to large amplitude relaxation oscillations is referred to as 'hard', i.e. making an explicit statement that canard cycles are not observed.

Fortunately, this degenerate situation does not occur anymore in systems with two slow variables where canards are generic, i.e. their existence is insensitive to small parameter perturbations. Benoît [1, 2] was the first to study generic canards in $\mathbb{R}^3$. He also observed how a certain class of generic canards (known as canards of folded node type) cause unexpected rotational properties of nearby solutions. Extending geometric singular perturbation theory [21, 29] to canard problems in $\mathbb{R}^3$, Szmolyan and Wechselberger [41] provided a detailed geometric study of generic canards. In particular, Wechselberger [47] showed that rotational properties of folded node type canards are related to a complex local geometry of invariant manifolds near these canards and associated bifurcations of these canards.

Thus contrary to their original discovery in $\mathbb{R}^2$, canards are actually not spurious but relevant 'creatures' and their influence on slow-fast dynamics is readily seen in applications. For example, coupling this local canard structure with a global return mechanism can explain complex oscillatory patterns known as mixed-mode oscillations (MMOs); see, e.g., [6, 10, 35, 47] for details on the theory and see, e.g., [10, 12, 14] for numerical aspects. There now exists a substantial amount of literature on applications of canard theory in the sciences including neuroscience [11, 17, 20, 30, 31, 35, 38, 39, 40, 47, 49, 50], chemistry [13, 35, 36], calcium signalling [26, 27, 45] and even in high speed machining [7]. This list is far from being complete, and we refer to detailed tables on relevant literature provided in a review on MMOs [10] as well as to a focus issue on mixed-mode oscillations [5] and to a special issue on bifurcation delay [9]. Complex temporal pattern generation is not the only application area of canard theory. Spatio-temporal patterns such as travelling and shock waves in advection-reaction-diffusion models with a slow-fast structure might also be related to generic canards as shown in [48].

1.1. **Outline of the paper.** In this paper, we show that the generic canard theory originally developed for systems in $\mathbb{R}^3$ with two slow variables and one fast variable can be extended to systems in $\mathbb{R}^n$ with k slow and m fast variables for any finite $n = k+m \geq 3$ and $k \geq 2$. In section 2 we introduce geometric singular perturbation theory and present a basic center manifold reduction of the singularly perturbed system with a folded critical manifold (a generic geometric requirement for the existence of canards) which leads to a $(k+1)$-dimensional system. In section 3 we define folded singularities and derive a canonical form for singularly perturbed systems near such folded singularities. We then give a classification of folded singularities and introduce their corresponding singular canards. In section 4 we provide the main results about the existence of canards in arbitrary dimensions. The key observation is that a cylindrical blow-up transforms the $(k+1)$-dimensional singularly perturbed system to a problem with three fast and $k-2$ slow variables. The limiting three-dimensional system recovers the canonical form of canard theory in problems with two slow variables and one fast variable. Thus, the local $(k+1)$-dimensional dynamics near generic canards are completely described (to leading order) by the three-dimensional canonical system *for any* $k \geq 2$. We finally conclude in section 5 and remark on degenerate canards and on canard induced mixed-mode oscillations.

2. Geometric singular perturbation theory

We consider a system of ordinary differential equations that has an explicit slow-fast splitting of the form

$$\begin{aligned} \dot{x} &= g(x,z,\varepsilon), \\ \varepsilon\,\dot{z} &= f(x,z,\varepsilon), \end{aligned} \tag{1}$$

where $(x,z) \in \mathbb{R}^k \times \mathbb{R}^m$ are state space variables. The variables $z = (z_1, \ldots, z_m)$ are denoted fast and the variables $x = (x_1, \ldots, x_k)$ are denoted as slow, the overdot denotes the time derivative $d/d\tau$ and $\varepsilon \ll 1$ is a small positive parameter encoding the time scale separation between the slow and fast variables. The functions $f : \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^k$ are assumed to be sufficiently smooth. By switching from the slow time scale τ to the fast time scale $t = \tau/\varepsilon$, System (1) transforms to

$$\begin{aligned} x' &= \varepsilon\, g(x,z,\varepsilon), \\ z' &= f(x,z,\varepsilon)\,. \end{aligned} \tag{2}$$

System (1) respectively (2) is called a *singularly perturbed system*. Solutions of such a system frequently consist of a mix of slow and fast segments, i.e. long periods of small changes interspersed by short periods of dramatic changes. As $\varepsilon \to 0$, the trajectories of (1) converge during fast segments to solutions of the *layer problem*

$$\begin{aligned} x' &= 0, \\ z' &= f(x,z,0), \end{aligned} \tag{3}$$

while during slow segments, trajectories of (2) converge to solutions of

$$\begin{aligned} \dot{x} &= g(x,z,0), \\ 0 &= f(x,z,0), \end{aligned} \tag{4}$$

which is a differential-algebraic equation (DAE) called the *reduced problem*. One major goal of geometric singular perturbation theory [21, 29] is to use these lower-dimensional subsystems (3) and (4) to understand the dynamics of the full system (1) or (2) for $\varepsilon > 0$.

2.1. **The critical manifold.**

The algebraic equation in (4) defines the interface between the two subsystems, called the *critical manifold*

$$S := \{(x,z) \in \mathbb{R}^k \times \mathbb{R}^m \mid f(x,z,0) = 0\}\,, \tag{5}$$

which is the phase space of the reduced problem (4) as well as the set of equilibrium points for the layer problem (3). The basic classification of singularly perturbed systems is given by properties of the layer problem (3). A subset $S_h \subseteq S$ is called *normally hyperbolic* if all $(x,z) \in S_h$ are hyperbolic equilibria of the layer problem, that is, the Jacobian $(D_z f)(x,z,0)$ has no eigenvalues with zero real part. We call a normally hyperbolic subset $S_a \subset S$ *attracting* if all eigenvalues of $(D_z f)(x,z,0)$ have negative real parts for $(x,z) \in S_a$; similarly $S_r \subset S$ is called *repelling* if all eigenvalues of $(D_z f)(x,z,0)$ have positive real parts for $(x,z) \in S_r$. If $S_s \subset S$ is normally hyperbolic and neither attracting nor repelling we say it is of *saddle type*.

Normal hyperbolicity fails at points on S where the Jacobian $D_z f$ has (at least) one eigenvalue with zero real part. Generically, such points are folds in the sense of singularity theory. We refer to Takens [43] who provided a classification of possible singularities[1] in constrained differential equations. A geometric singular perturbation analysis of a slow-fast system with a cusp-like critical manifold is given (in the case $k = 2$ and $m = 1$) in [4]. In this paper, the focus is on the generic case of a folded critical manifold.

2.2. **The reduced flow.** By definition, the main requirement on the reduced vector field (4) is that it has to be in the tangent bundle TS of the critical manifold S. Taking the total time derivative of the constraint $f(x, z, 0) = 0$ defining S, i.e. $D_z f \cdot \dot{z} + D_x f \cdot \dot{x} = 0$, defines a vector field for the z variables that is constrained to the tangent bundle TS. This leads to the following representation of the reduced problem (4):

$$\begin{aligned} \dot{x} &= g(x, z, 0), \\ -D_z f \cdot \dot{z} &= (D_x f \cdot g)(x, z, 0), \end{aligned} \tag{6}$$

where $(x, z) \in S$. Let $\operatorname{adj}(D_z f)$ denote the adjoint of the matrix $D_z f$, which is the transpose of the co-factor matrix of $D_z f$.[2] We apply $\operatorname{adj}(D_z f)$ to both sides of the second equation in (6) to obtain

$$\begin{aligned} \dot{x} &= g(x, z, 0), \\ -\det(D_z f)\,\dot{z} &= (\operatorname{adj}(D_z f) \cdot D_x f \cdot g)(x, z, 0), \end{aligned} \tag{7}$$

where $(x, z) \in S$. Suppose that the critical manifold S is normally hyperbolic, i.e. the Jacobian of the fast subsystem, $D_z f$, has full rank for all $(x, z) \in S$. The implicit function theorem then implies that S has a graph representation over the slow variable base x, i.e. $z = h(x)$, and the reduced flow is simply given by $\dot{x} = g(x, h(x), 0)$. Fenichel theory [21, 29] guarantees the persistence of a normally hyperbolic manifold close to $S_h \subset S$ and the persistence of a corresponding slow flow on this manifold close to the reduced flow of S_h in the following way:

Theorem 2.1 (Fenichel's Theorem [21])**.** *Suppose S_h is a compact normally hyperbolic (sub)manifold (possibly with boundary) of the critical manifold S of* (2) *and that $f, g \in C^r$, $1 \le r < \infty$. Then for $\varepsilon > 0$ sufficiently small and some $K > 0$ the following holds:*

- (i) *There exists a C^r-smooth, locally invariant manifold $S_{h,\varepsilon}$ diffeomorphic to S_h which has a Hausdorff distance $O(\varepsilon)$ from S_h.*
- (ii) *The slow flow on $S_{h,\varepsilon}$ converges to the reduced flow on S_h as $\varepsilon \to 0$.*
- (iii) *$S_{h,\varepsilon}$ is, in general, not unique but all representations of $S_{h,\varepsilon}$ lie within a Hausdorff distance $O(e^{-K/\varepsilon})$ from each other, i.e. all r-jets are uniquely determined.*

From this theorem it follows that singularities of the reduced flow, i.e. $(x, h(x))$ such that $g(x, h(x), 0) = 0$, that are generically hyperbolic equilibria of the reduced problem persist as hyperbolic equilibria of the full problem for $\varepsilon \ll 1$.

If we assume that $S_h = S_a$ is an attracting normally hyperbolic manifold, then Fenichel theory implies that the reduced dynamics of system (1) are completely

[1] folds, cusps, swallow tails, hyperbolic umbilics or elliptic umbilics
[2] In the case $m = 1$, $D_z f = \det(D_z f) = \frac{\partial f}{\partial z}$ is a scalar and $\operatorname{adj}(D_z f) = 1$.

described[3] by the dynamics on the k-dimensional slow manifold $S_{a,\varepsilon}$, which to leading order can be completely determined by the reduced flow on S_a.

On the other hand, if S is not normally hyperbolic, i.e. generically a folded critical manifold, Fenichel theory does not hold in the neighbourhood of the fold. Thus an extension of geometric singular perturbation theory to nonhyperbolic problems is needed to understand the local dynamics near such folded critical manifolds.

Assumption 1. *The critical manifold S of the singularly perturbed system* (1) *is locally folded, i.e. the set of fold points F forms a $(k-1)$-dimensional manifold in the k-dimensional critical manifold S defined by*

$$F \;:=\; \{(x,z)\in\mathbb{R}^k\times\mathbb{R}^m|\; f(x,z,0)=0\,, \mathsf{rank}\,(D_zf)(x,z,0)=m-1\,,\; w\cdot[(D^2_{zz}f)(x,z,0)\,(v,v)]\neq 0,\; w\cdot[(D_xf)(x,z,0)]\neq 0\} \tag{8}$$

with corresponding left and right null vectors w and v of the Jacobian $(D_zf)(x,z,0)$.

Looking at the reduced system (7), we observe that $\det(D_zf)=0$ along F, i.e. that system (7) is singular along F. Since $w\cdot[(D^2_{zz}f)(x,z,0)\,(v,v)]\neq 0$ along F, this implies that $\det(D_zf)$ has different signs on adjacent subsets (branches) of the critical manifold S bounded by F. We rescale time by setting $\tau=-\det(D_zf)\,\tau_1$ in system (7) to obtain the *desingularized problem*

$$\begin{aligned} \dot{x} &= (-\det(D_zf)\cdot g)\;(x,z,0),\\ \dot{z} &= (\mathrm{adj}\,(D_zf)\cdot D_xf\cdot g)\,(x,z,0), \end{aligned} \tag{9}$$

where $(x,z)\in S$ and the overdot now denotes $d/d\tau_1$. From the time rescaling it follows that the direction of the flow in (9) has to be reversed on branches where $\det(D_zf)>0$ to obtain the corresponding reduced flow (7). Otherwise, the flows of (7) and (9) are equivalent. Obviously, the analysis of the desingularized problem (9) is preferable.

Similar to the normally hyperbolic case, a (local) graph representation of the critical manifold S is used to analyse the k-dimensional desingularised flow (9). From the definition (8) of the folded critical manifold it follows that there exists at least one slow variable $x_j,\ j\in\{1,\dots,k\}$ with $w\cdot[(D_{x_j}f)(x,z,0)]\neq 0$. Without loss of generality, let x_1 be this slow variable. It follows that one column in D_zf (we assume, without loss of generality, that this column is $D_{z_1}f$) can be replaced by the column of $D_{x_1}f$ such that $\mathsf{rank}\,D_{(x_1,z_2,\dots,z_m)}f(x_*,z_*,0)=m$. The implicit function theorem then implies that S is locally the graph of a function $h:\mathbb{R}^k\to\mathbb{R}^m$ over the base $U\subset\{(x_2,\dots,x_k,z_1)\in\mathbb{R}^k\}$, i.e. $y=h(x_2,\dots,x_k,z_1)$, where $y=(x_1,z_2,\dots,z_m)$. Incorporating this graph representation of S leads to the projection of the desingularized vector field (9) onto the base U constrained to the base variables $(x_2,\dots,x_k,z_1)\in\mathbb{R}^k$.

2.3. **A center manifold reduction.** In system (1) near a fold point $(x_*,z_*,0)\in F$, there exist locally invariant manifolds $\tilde{W}_{cs}$ (center-stable) and $\tilde{W}_{cu}$ (center-unstable) where $\tilde{W}_{cu}\cup\tilde{W}_{cs}$ spans the whole phase space and $\tilde{W}_c=\tilde{W}_{cu}\cap\tilde{W}_{cs}$ corresponds to a $(k+1)$-dimensional center manifold. The following result gives a center manifold reduction of system (1) that captures the local dynamics on $\tilde{W}_c$ near $(x_*,z_*,0)\in F$ and hence the reduced dynamics of system (1).

[3]up to an exponentially small error and possibly after some initial transient time

Theorem 2.2. *Given system* (1) *under Assumption* 1*, then there exists a* $(k+1)$*-dimensional center manifold* $\tilde{W}_c$ *in a neighbourhood of* $(x_*, z_*, 0) \in F$*. System* (1) *reduced to* $\tilde{W}_c$ *has the form:*

$$\begin{aligned} \dot{x} &= g(x, z_1, \varepsilon), \\ \varepsilon \dot{z}_1 &= f_1(x, z_1, \varepsilon) \\ &= x_1(d_1 + O(x, z_1)) + z_1^2(d_2 + O(x, z_1)) + \varepsilon\, O(x, z_1, \varepsilon), \end{aligned} \tag{10}$$

where d_1 *and* d_2 *are nonzero constants. The point* $(\hat{x}, 0, 0)$ *with* $\hat{x} = (0, x_{2*}, \ldots, x_{k*})$ *in* (10) *corresponds to the fold point* $(x_*, z_*, 0) \in F$ *in* (1)*. The corresponding critical manifold* S *is given in its canonical form*

$$x_1 = \xi_1(x_2, \ldots, x_k, z_1) = -z_1^2\left(\frac{d_2}{d_1} + O(x_2, \ldots, x_k, z_1)\right). \tag{11}$$

Proof. The first part of the proof is a variant of invariant manifold theory [8, 28] for singularly perturbed systems [21] which we adapt to the nonhyperbolic case. First, we translate $(x_*, z_*, 0)$ to $(x_*, 0, 0)$. Then we make a linear coordinate transformation $\tilde{z} = Lz$, where L is the left eigenvector matrix of $D_z f$ such that the critical eigendirection spanned by the right null-vector v becomes the (new) $\tilde{z}_1$ direction. This gives

$$\begin{aligned} \dot{x} &= \tilde{g}(x, \tilde{z}, \varepsilon), \\ \varepsilon \dot{\tilde{z}} &= \tilde{f}(x, \tilde{z}, \varepsilon), \end{aligned} \tag{12}$$

with $x \in \mathbb{R}^k$, $\tilde{z} \in \mathbb{R}^m$, $\tilde{f} : \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, and function $\tilde{f}(x, \tilde{z}, \varepsilon) = L \cdot f(x, \tilde{z}, \varepsilon)$.

In this new coordinate frame, the first row and the first column in $D_{\tilde{z}} \tilde{f}(x_*, 0, 0)$ have only zero entries. We also have $D_x \tilde{f}_1(x_*, 0, 0) \neq 0$ and we assume, without loss of generality, that $D_{x_1} \tilde{f}_1(x_*, 0, 0) = \partial \tilde{f}_1 / \partial x_1(x_*, 0, 0) = d_1 \neq 0$. Thus $\mathsf{rank}\, D_{(x_1, \tilde{z}_2, \ldots, \tilde{z}_m)} \tilde{f}(x_*, 0, 0) = m$ and S is given as a graph $y = h(x_2, \ldots, x_k, \tilde{z}_1)$, where $y = (x_1, \tilde{z}_2, \ldots, \tilde{z}_m)$.

The coordinate transformation $\bar{z}_j = \tilde{z}_j - h_j(x_2, \ldots, x_k, \tilde{z}_1)$, $j = 2, \ldots, m$, rectifies the fast $\tilde{z}_j$ components of S to $\bar{z}_2 = \ldots = \bar{z}_m = 0$, i.e. $\bar{f}_j(x_1, \ldots, x_k, z_1, 0, \ldots, 0) = 0$ for $j = 2, \ldots, m$.[4] This implies that all partial derivatives of the functions $\bar{f}_j$, $j = 2, \ldots, m$, with respect to $(x_1, \ldots, x_k, \tilde{z}_1)$ evaluated at $(x_*, 0) \in F$ vanish. Hence, the $(k+1)$-dimensional center manifold $\tilde{W}_c$ is spanned by the $k+1$ slow directions $(x_1, \ldots, x_k, \tilde{z}_1)$ and the flow on $\tilde{W}_c$ is described by

$$\begin{aligned} \dot{x} &= \tilde{g}(x, \tilde{z}_1, \varepsilon), \\ \varepsilon \dot{\tilde{z}}_1 &= \tilde{f}_1(x, \tilde{z}_1, \varepsilon), \end{aligned} \tag{13}$$

with $\tilde{f}_1(x, \tilde{z}_1, \varepsilon) = w \cdot \tilde{f}(x, \tilde{z}_1, 0, \ldots, 0, \varepsilon)$, where w is the left null-vector of $D_z f$. Following from Assumption 1, we have the following condition on the function $\tilde{f}_1$ evaluated at a fold point $(x_*, 0) \in F$:

$$\tilde{f}_1 = 0, \quad \frac{\partial \tilde{f}_1}{\partial x_1} = d_1 \neq 0, \quad \frac{\partial \tilde{f}}{\partial \tilde{z}_1} = 0, \quad \frac{\partial^2 \tilde{f}}{\partial \tilde{z}_1^2} = 2d_2 \neq 0\,.$$

The implicit function theorem now gives a parametrization of the $(k-1)$-dimensional manifold F by $\{(\xi_1(y, \zeta_1(y)), y, \zeta_1(y)),\ y = (x_2, \ldots, x_m) \in B_r(0)\}$ for a suitable

[4] Note that the same coordinate transformation for x_1 is not possible because it would change the singularly perturbed structure of the system.

ball $B_r(0) \subset \mathbb{R}^{k-1}$ centered at the origin. The final part of the proof uses coordinate transformations $\bar{x}_1 = x_1 - \xi_1(y)$ and $\bar{z}_1 = \tilde{z}_1 - \zeta_1(y)$ to rectify the set F to $\bar{x}_1 = \bar{z}_1 = 0$. This gives the result (after dropping bars and tilde for convenience). □

Assumption 2. *In Assumption* 1*, the* $m-1$ *nonzero eigenvalues of* $D_z f(x_*, z_*, 0)$ *have all negative real part. This implies that* $\tilde{W}_{cs} = \mathbb{R}^{k+m}$*.*

Proposition 2.1. *Given system* (1) *under Assumptions* 1 *and* 2*, then the* $(k+1)$*-dimensional center manifold* $\tilde{W}_c$ *described in Theorem* 2.2 *is exponentially attracting, i.e. system* (10) *describes the local dynamics of system* (1) *near* $(x_*, z_*, 0) \in F$*.*

In the following we focus on such exponentially attracting manifolds W_c and the corresponding flow described by system (10).

3. Folded singularities and singular canards

The corresponding desingularized problem of system (10) is given by

$$\begin{aligned} \dot{x}_j &= \left(-\tfrac{\partial f_1}{\partial z_1}\, g_j\right)(x_2, \dots, x_k, z_1, 0), \quad j = 2, \dots, k, \\ \dot{z}_1 &= (D_x f_1 \cdot g)\,(x_2, \dots, x_k, z_1, 0), \end{aligned} \tag{14}$$

where the graph representation of S given in (11) is substituted. Ordinary singularities of (14) away from the fold F are defined by $g = 0$ and correspond to equilibria of the reduced problem. Of special interest are singularities $(\hat{x}, 0, 0)$ of (14) that are constrained to the fold F given by $z_1 = 0$.

Definition 1. Singularities of system (14) defined by

$$\frac{\partial f_1}{\partial z_1}(\hat{x}, 0, 0) = 0 \quad \text{and} \quad (D_x f_1 \cdot g)\,(\hat{x}, 0, 0) = 0 \tag{15}$$

are called *folded singularities*.

The set of these folded singularities M_f defined by condition (15) forms a submanifold of codimension one in the $(k-1)$-dimensional set of fold points F. Thus, this set M_f viewed as a set of equilibria of the desingularized system (14) has generically $k-2$ zero eigenvalues and two eigenvalues $\lambda_{1/2}$ with nonzero real part. The classification of folded singularities is based on these two nonzero eigenvalues, and it follows the classification of singularities in two-dimensional vector fields. In the case that $\lambda_{1/2}$ are complex conjugates and $\mathrm{Re}\,\lambda_{1/2} \neq 0$, then the corresponding singularity is called a *folded focus*. In the case that $\lambda_{1/2}$ are real, then the corresponding singularity is either a *folded saddle* if $\lambda_1\lambda_2 < 0$, or a *folded node* if $\lambda_1\lambda_2 > 0$. Necessary conditions for generic folded singularities are

$$g_j(\hat{x}, 0, 0) \neq 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}(D_x f_1 \cdot g)(\hat{x}, 0, 0) \neq 0 \tag{16}$$

for (at least) one common index $j \in \{2, \dots, k\}$. The second condition is necessary for $\mathrm{Re}\,\lambda_{1/2} \neq 0$ but not sufficient. With the additional condition

$$\frac{\partial}{\partial z_1}(D_x f_1 \cdot g)(\hat{x}, 0, 0) \neq 0\,, \tag{17}$$

a generic folded singularity is guaranteed.

Remark 3.1. The first condition in (16) corresponds to $g \neq 0$ in the original system (9) and implies that generic folded singularities do not constitute equilibria of the reduced problem. The second condition in (16) does not lead, in general, to a 'simple' structure in (9), i.e. the full Jacobian has to be evaluated.

For a generic folded singularity, the algebraic multiplicity of the corresponding singularities on both sides of the second equation in the reduced problem (7) is the same (i.e. one). This leads in the case of a folded saddle or a folded node to a nonzero but finite speed of the reduced flow through a folded singularity.[5] Hence, folded saddles and folded nodes create possibilities for the reduced flow to cross to different (normally hyperbolic) branches of the critical manifold S via such folded singularities $(\hat{x},0,0) \in M_f$. This is the hallmark of singular canards in systems with two or more slow variables, explained in detail in section 3.2, and makes them an important generic feature of DAEs.

Remark 3.2. In the numerical literature on DAEs (see, e.g., [25]), folds are not considered. There, critical manifolds S are considered to be either normally hyperbolic or degenerate, i.e. $\det(D_z f)$ is either not equal or equal to zero on the whole domain of S.

3.1. **A canonical form near folded singularities.** System (10) encodes the geometry of the critical manifold S given by Assumption 1. In the next transformation step, we will encode the existence of a folded singularity in system (10).

Assumption 3. *In system* (10)*, there exists a generic folded singularity* $(\hat{x},0,0) \in M_f \subset F$ *defined by conditions* (16) *and* (17)*. We further assume, without loss of generality, that* $(\hat{x},0,0) = (0,0,0)$ *is located at the origin and that condition* (16) *is fulfilled for* $j = 2$.

This assumption implies the existence of a $(k-2)$-dimensional set of generic folded singularities, $M_f \subset F$, locally near the origin.

Theorem 3.1. *Given system* (10) *under Assumptions* 1-3*, then there exists a smooth change of coordinates which transforms system* (10) *to*

$$
\begin{aligned}
\dot{x}_1 &= B_2(x_3,\dots,x_k)x_2 + C(x_3,\dots,x_k)z_1 \\
&\quad + O(x_1,x_2^2,x_2z_1,z_1^2) + \varepsilon\, O(x_1,\dots,x_k,z_1), \\
\dot{x}_j &= A_j(x_3,\dots,x_k) + O(x_1,x_2,z_1,\varepsilon)\,, \qquad j = 2,\dots,k, \\
\varepsilon\dot{z}_1 &= x_1(1 + z_1\, O(x_2,\dots,x_k)) + z_1^2(1 + O(x_1,z_1)) \\
&\quad + \varepsilon\, O(x_1,x_2,z_1,\varepsilon),
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
A_j = A_j(x_3,\dots,x_k) &= a_j + g_{j,1}(x_3,\dots,x_k)\,, \quad j = 2,\dots,k, \\
B_2 = B_2(x_3,\dots,x_k) &= b_2 + g_{1,1}(x_3,\dots,x_k), \\
C = C(x_3,\dots,x_k) &= c + g_{1,2}(x_3,\dots,x_k)
\end{aligned}
\tag{19}
$$

with $g_{j,1}(0,\dots,0) = 0$*,* $g_{1,i}(0,\dots,0) = 0$*,* $i = 1,2$*, and computable constants* a_j*,* b_2 *and* c *where* a_2*,* b_2 *and* c *are generically nonzero.*

Proof. The implicit function theorem gives a parametrization of the $(k-2)$-dimensional manifold M_f by $\{(0,\xi_2(y),y,0),\ y = (x_3,\dots,x_k) \in B_r(0)\}$ for a suitable ball $B_r(0) \subset \mathbb{R}^{k-2}$. The coordinate transformation $\bar{x}_2 = x_2 - \xi_2(y)$ rectifies the set M_f to $x_1 = \bar{x}_2 = z_1 = 0$. A sequence of linear and near identity transformations

[5]This does not apply to a folded focus; see section 3.2.3.

then gives system (18), where we drop for convenience the bars from the variables. The nonzero conditions on the parameters a_2, b_2, c follow from (16), $j = 2$, and (17). □

Assumption 4. *The folded singularity in system* (10) *is a folded saddle or a folded node, i.e. the two nonzero eigenvalues* $\lambda_{1/2}$ *of the corresponding desingularized problem of system* (10) *are real and nonzero.*

Theorem 3.2. *Given system* (10) *under Assumptions* 1-4*, then there exist a smooth change of time and a smooth change of coordinates that brings system* (10) *to the canonical form*

$$\begin{aligned} \dot{x}_1 &= \tfrac{1}{2}\mu(x_3,\ldots,x_k)x_2 - (1+\mu(x_3,\ldots,x_k))z_1 \\ &\quad + O(x_1, x_2^2, x_2 z_1, z_1^2) + \varepsilon\, O(x_1,\ldots,x_k,z_1), \\ \dot{x}_2 &= 1 + O(x_1,x_2,z_1,\varepsilon), \\ \dot{x}_j &= a_j + g_{j,1}(x_3,\ldots,x_k) + O(x_1,x_2,z_1,\varepsilon)\,, \qquad j = 3,\ldots,k, \\ \varepsilon\dot{z}_1 &= x_1(1 + z_1\, O(x_2,\ldots,x_k)) + z_1^2(1+O(x_1,z_1)) \\ &\quad + \varepsilon\, O(x_1,x_2,z_1,\varepsilon) \end{aligned} \tag{20}$$

with

$$\mu = \mu(x_3,\ldots,x_k) := \frac{\lambda_1\lambda_2}{\lambda^2}\,. \tag{21}$$

Proof. Theorem 3.1 gives system (18). The two nonzero eigenvalues $\lambda_{1/2}$ of the corresponding desingularized problem of system (18) satisfy the characteristic equation $\lambda^2 - C\lambda + 2A_2B_2 = 0$. These eigenvalues $\lambda_{1/2}$ are assumed to be real and nonzero, i.e. they correspond to a folded saddle or a folded node. A change of time by $d\hat{\tau} = d\tau A_2(x_3,\ldots,x_k)$ and a sequence of linear and near identity transformations gives the equations for the variables $(x_2,\ldots,x_k,z_1)$ in system (20). Then, the coordinate transformation

$$\tilde{x}_1 = \lambda^2 x_1, \quad \tilde{x}_2 = \lambda^2 x_2, \quad \tilde{z}_1 = -\lambda z_1, \quad \tilde{\varepsilon} = -\lambda^3\varepsilon, \quad \tilde{t} = \lambda^2 t\,, \tag{22}$$

gives system (20), after dropping for convenience the tilde from the variables, where $\lambda < 0$ is one of the eigenvalues $\lambda_{1/2}$ and μ is defined by (21). Note, in the case of a folded saddle, there is always one negative eigenvalue. In the folded node case, two negative eigenvalues imply that the reduced flow is towards the fold F on S_a which is the case we are interested in. □

Remark 3.3. In the case $k = 2$, system (20) is equivalent to the normal form derived in [47], Proposition 2.1.

3.2. **Singular canards.** Let us first focus on the properties of the reduced flow of system (18) in Theorem 3.1. We have a k-dimensional folded critical manifold S defined by $x_1 = \xi_1(x_2,\ldots,x_m,z_1) = -z_1^2(1+O(x_2,\ldots,x_k,z_1))$, a $(k-1)$-dimensional manifold of fold points $F \subset S$ defined by $z_1 = \tilde{\zeta}_1(x_2,\ldots,x_k) = 0$ and a $(k-2)$-dimensional manifold of folded singularities $M_f \subset F$ defined by $x_2 = \tilde{\xi}_2(x_3,\ldots,x_k) = 0$. The desingularized flow on S, projected onto the base $(x_2,\ldots,x_m,z_1)$, is given by

$$\begin{aligned} \dot{x}_j &= -2z_1(A_j(x_3,\ldots,x_k) + O(x_2,\ldots,x_k,z_1)), \quad j = 2,\ldots,k, \\ \dot{z}_1 &= B_2(x_3,\ldots,x_k)x_2 + C(x_3,\ldots,x_k)z_1 + O(x_2^2, x_2 z_1, z_1^2)\,. \end{aligned} \tag{23}$$

The Jacobian of the vector field (23) evaluated along M_f in a neighbourhood $B_r(0)$ of the origin is given by

$$\left(\begin{array}{ccccc} 0 & 0 & \cdots & 0 & -2A_2(x_3,\dots,x_k) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -2A_k(x_3,\dots,x_k) \\ B_2(x_3,\dots,x_k) & 0 & \cdots & 0 & C(x_3,\dots,x_k) \end{array}\right), \tag{24}$$

which has $k-2$ zero eigenvalues and the corresponding eigenvectors span the tangent bundle TM_f of the set of folded singularities M_f.[6] Additionally, there are (generically) two nonzero eigenvalues $\lambda_{1/2}$. The corresponding eigenvectors span the (x_2, z_1)-space. This leads to the following classification of folded singularities which depends only on A_2, B_2 and C for $(x_3,\dots,x_k) \in B_r(0)$:

$$\begin{array}{cclc} A_2B_2 < 0 & \Rightarrow & \lambda_{1/2} \in \mathbb{R},\ \lambda_1\lambda_2 < 0 & \text{folded saddle,} \\ C^2/8 \geq A_2B_2 > 0 & \Rightarrow & \lambda_{1/2} \in \mathbb{R},\ \lambda_1\lambda_2 > 0 & \text{folded node,} \\ C^2/8 < A_2B_2 & \Rightarrow & \lambda_{1/2} \in \mathbb{C},\ \bar{\lambda}_1 = \lambda_2 & \text{folded focus,} \end{array} \tag{25}$$

where $\bar{\lambda}_1 = \lambda_2$ denotes the complex conjugate eigenvalue for $\lambda_1 \in \mathbb{C}$. In the case of generic folded singularities (25), M_f represents a normally hyperbolic manifold in system (23) and $(x_3,\dots,x_k)$ are slow variables in a neighbourhood of M_f while (x_2, z_1) are fast variables. The main difference of this desingularized system (23) to a 'classical' singularly perturbed system (1) is that there exists no uniform timescale separation (in the form of a singular perturbation parameter $\varepsilon \ll 1$) and that there exists no (nontrivial) reduced flow on the normally hyperbolic manifold M_f.

In the case of M_f consisting of folded saddles or folded nodes, we have derived the canonical form (20) in Theorem 3.2. Its corresponding desingularized flow is given by

$$\begin{aligned} \dot{x}_2 &= -2z_1(1 + O(x_2,\dots,x_k,z_1)), \\ \dot{x}_j &= -2z_1(A_j(x_3,\dots,x_k) + O(x_2,\dots,x_k,z_1)), \quad j = 3,\dots,k, \\ \dot{z}_1 &= \tfrac{1}{2}\mu(x_3,\dots,x_k)x_2 - (1 + \mu(x_3,\dots,x_k))z_1 + O(x_2^2, x_2z_1, z_1^2)\,, \end{aligned} \tag{26}$$

and the Jacobian along M_f is given by

$$\left(\begin{array}{ccccc} 0 & 0 & \cdots & 0 & -2 \\ 0 & 0 & \cdots & 0 & -2A_3(x_3,\dots,x_k) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -2A_k(x_3,\dots,x_k) \\ \frac{1}{2}\mu(x_3,\dots,x_k) & 0 & \cdots & 0 & -(1+\mu(x_3,\dots,x_k)) \end{array}\right). \tag{27}$$

Thus, the two nonzero eigenvalues are given by $\lambda_1 = -\mu$ and $\lambda_2 = -1$. If we fix $\lambda = \lambda_2$ in definition (21), then $\mu = \lambda_1/\lambda_2$ represents the eigenvalue ratio which is positive for folded nodes and negative for folded saddles.

3.2.1. *Folded saddles.* In the case $\mu < 0$, there exists a $(k-1)$-dimensional center-stable manifold W_{cs} and a $(k-1)$-dimensional center-unstable manifold W_{cu} along the $(k-2)$-dimensional normally hyperbolic manifold $W_c = W_{cs} \cap W_{cu} = M_f$. Both manifolds, W_{cs} and W_{cu}, are uniquely foliated by one-dimensional fast fibers W_s respectively W_u over the base M_f, where the fibers are tangent to the stable respectively unstable eigenvector of the corresponding folded singularity on the base M_f.

[6]Here, we can identify TM_f to M_f.

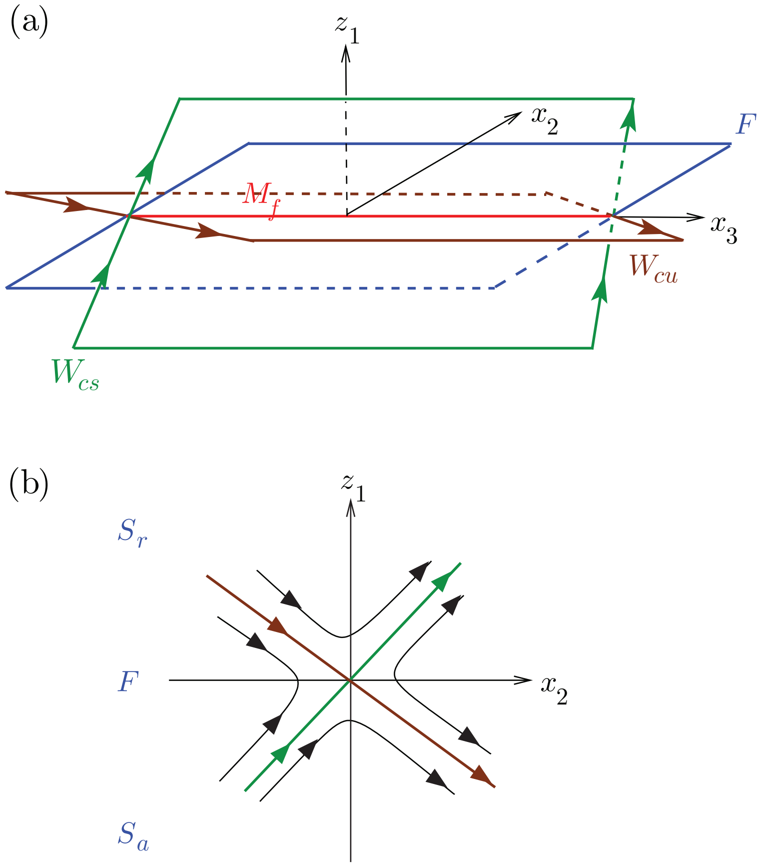

FIGURE 1. (Color online.) Sketch of the reduced flow of system (20) near the manifold M_f of folded saddles, case $k = 3$: (a) two-dimensional manifold of fold points F (blue), two-dimensional manifold W_{cs} (green) of singular canards through M_f and two-dimensional manifold W_{cu} (brown) of faux canards through M_f; view is from below and the x_2-axis is pointing towards the observer; (b) projection of reduced flow on a section x_3 = constant. The attracting part S_a of the critical manifold, $z_1 < 0$, is below the fold F while the repelling part S_a, $z_1 > 0$, is above F.

Recall that the reduced flow is obtained from the desingularized flow by changing the direction of the flow on S_r. Thus, trajectories that start in a stable fiber $W_s \subset W_{cs} \subset S_a$ approach M_f in finite time and cross M_f tangent to the stable eigenvector of the corresponding folded singularity on M_f to the unstable branch $W_{cu} \subset S_r$; see Figure 1. This leads to the following.

Definition 2. Given a singularly perturbed system (1) with a folded critical manifold S, a trajectory of the reduced problem that has the ability to cross in finite time from one branch of the critical manifold to the other via a folded singularity $M_f \subset F$ is called a *singular canard*.

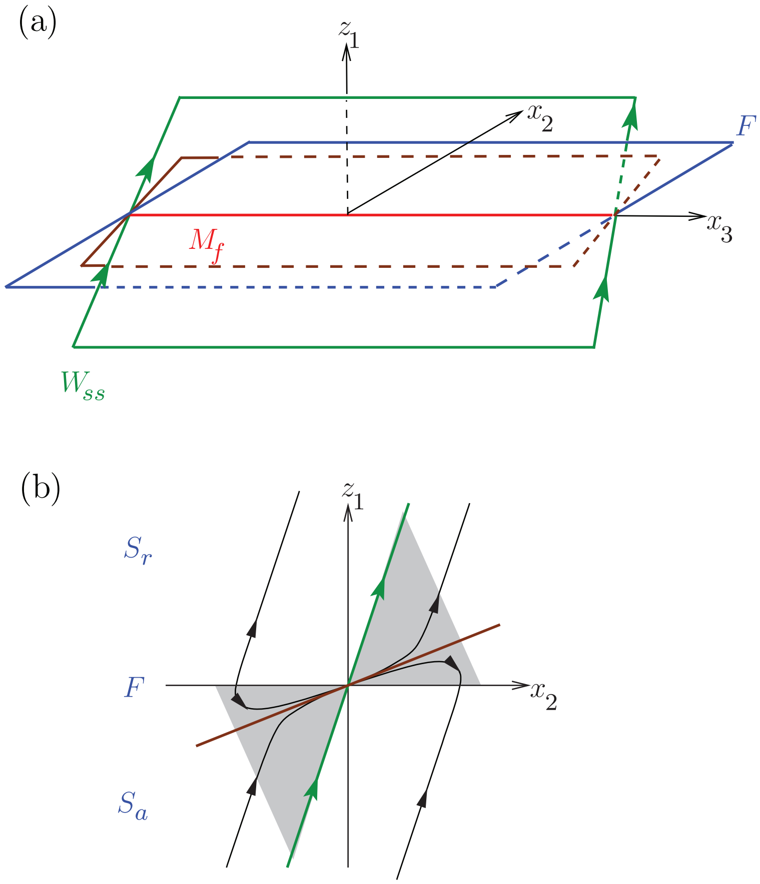

FIGURE 2. (Color online.) Sketch of the reduced flow near the manifold M_f of folded nodes, case $k = 3$: (a) two-dimensional manifold of fold points F (blue), two-dimensional manifold W_{ss} (green) of singular strong canards through M_f; view is from below and the x_2-axis is pointing towards the observer; (b) projection of reduced flow on a section $x_3 =$ constant. There exists a sector of singular canards within the singular funnel (shaded grey) bounded by W_{ss} (green) and F (blue) which cross tangent to the weak eigendirection (brown line) of a folded node.

All other trajectories of the reduced flow starting in S_a (close to F) reach the set of fold points F/M_f in either finite forward or backward time, where they cease to exist due to finite time blow-up or they do not reach the set F/M_f at all.

Remark 3.4. Trajectories starting on an unstable fiber $W_u \subset W_{cu} \subset S_r$ approach M_f in finite time and cross it tangent to the unstable eigenvector of the corresponding folded singularity on M_f to the stable branch $W_{cu} \subset S_a$. Such solutions are called *singular faux canards*.

3.2.2. *Folded nodes.* In the case $\mu > 0$, the whole phase space S is equivalent to W_{cs} (since $\lambda_{1/2}$ are negative). Let us define $W_{ss} \subset W_{cs}$ as the $(k-1)$-dimensional subset of unique fast fibers corresponding to the span of the strong stable eigenvectors along the base M_f. Again, the reduced flow is obtained from the desingularized flow by changing the direction of the flow on S_r.

Definition 3. The set W_{ss} together with the $(k-1)$-dimensional set of fold points F bounds a sector in S_a, called the *singular funnel*, with the property that every trajectory starting in the singular funnel reaches the set of folded node singularities M_f in finite time and subsequently crosses the set F transversely to the unstable branch S_r in the direction that is tangent to the weak stable eigenvector of the corresponding folded node singularity on M_f.

Thus, every trajectory within a singular funnel is a singular canard. Trajectories that start on the boundary set $W_{ss} \subset S_a$ also reach the set M_f in finite time but cross it tangent to the strong stable eigenvector of the corresponding folded node singularity (by definition); see Figure 2.

All other trajectories of the reduced flow starting in S_a (close to F) reach the set of fold points F/M_f in finite forward or backward time where they cease to exist due to finite time blow-up.

Remark 3.5. In the proof of Theorem 3.1 we assumed that a folded node corresponds to two negative eigenvalues. Theoretically, we can also assume the case where both eigenvalues are positive. Thus the whole phase space S is equivalent to W_{cu} and we define $W_{uu} \subset W_{cu}$ as the $(k-1)$-dimensional subset of unique fast fibers that corresponds to the span of the strong unstable eigenvectors along the base M_f. This set W_{uu} together with the $(k-1)$-dimensional set of fold points F bounds a sector in S_r with the property that every trajectory starting in this sector reaches the set of folded node singularities in finite time and subsequently crosses the set F transversely to the stable branch S_a in the direction that is tangent to the weak unstable eigenvector of the corresponding folded node singularity on M_f. Thus, every trajectory within this sector is a singular faux canard. Trajectories that start on the boundary set $W_{uu} \subset S_r$ also reach the set M_f in finite time but cross it tangent to the strong unstable eigenvector of the corresponding folded node singularity.

3.2.3. *Folded foci.* In this case, all solutions starting in S_a (close to F) reach the set of fold points F/M_f in finite forward or backward time where they cease to exist due to finite time blow-up.[7]

Corollary 3.1. *There exist no singular canards near a folded focus.*

4. Maximal canards

In the following we analyse the canonical form (20) by means of geometric singular perturbation theory to derive results for $0 < \varepsilon \ll 1$. In particular, we want to ask the question if singular canards persist as canards of the full system (20). To answer this question, we first provide a geometric definition of canards for $\varepsilon > 0$. Recall that the branches S_a and S_r are normally hyperbolic away from the fold F. Thus, Fenichel theory implies the existence of an (nonunique but exponentially close) invariant slow manifold $S_{a,\varepsilon}$ up to a section $\Sigma_{1,a} : z_1 = -\delta_1$ and an

[7] The analysis presented in [42] applies to these fold points F/M_f (i.e. they are 'jump points').

(nonunique but exponentially close) invariant slow manifold $S_{r,\varepsilon}$ up to a section $\Sigma_{1,r} : z_1 = \delta_1$, i.e. $O(1)$ away from F. Fix a representative for each of these manifolds.

Definition 4. A maximal canard corresponds to the intersection of the manifolds $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ extended by the flow of (20) into the neighbourhood of the set $M_f \subset F$.

Such a maximal canard defines a family of canards nearby which are exponentially close to the maximal canard, i.e. a family of solutions of (20) that follow an attracting branch $S_{a,\varepsilon}$ of the slow manifold towards the neighbourhood of the set $M_f \subset F$, pass close to $M_f \subset F$ and then follow, rather surprisingly, a repelling branch $S_{r,\varepsilon}$ of the slow manifold for a considerable amount of slow time. This follows from the nonuniqueness of $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$. In the singular limit $\varepsilon \to 0$, such a family of canards is represented by a unique singular canard.

4.1. **'Blow-up' analysis.** The key to understanding the local dynamics near the set $M_f \subset F$ by means of geometric singular perturbation theory is the 'blow-up' technique. The 'blow-up' desingularizes degenerate singularities such as the set of folded singularities M_f. It is a coordinate transformation applied to the extended system $\{(20), \dot{\varepsilon} = 0\}$ by which the set of folded singularities M_f is blown up to a cylinder $B = S^3 \times \mathbb{R}^{k-2}$. With this procedure, one gains enough hyperbolicity on the blown-up locus B to apply standard tools from dynamical system theory. For a detailed description of the blow-up technique and its application to the cases $k = 1, 2$, we refer to [18, 32, 33, 34, 41, 47]. In the following, we will present the results of this blow-up analysis of the general case $k \geq 2$ and point out the differences to the case $k = 2$ wherever necessary.

In a first step, the blow-up analysis shows that Fenichel theory and hence the invariant slow manifolds $S_{a,\varepsilon}$, respectively $S_{r,\varepsilon}$, can be extended up to sections $\Sigma_{2,a}$: $z_1 = -\sqrt{\varepsilon}\delta_2$, respectively $\Sigma_{2,r} : z_1 = \sqrt{\varepsilon}\delta_2$, i.e. up to an $O(\sqrt{\varepsilon})$ neighbourhood of the fold F. We denote these extended manifolds by $S_{a,\sqrt{\varepsilon}}$, respectively $S_{r,\sqrt{\varepsilon}}$.

Proposition 4.1. *For system* (20), *the sets $S_{a,\sqrt{\varepsilon}}$, respectively $S_{r,\sqrt{\varepsilon}}$, are C^r-smooth, locally invariant, normally hyperbolic manifolds and $O(\sqrt{\varepsilon})$ perturbations of S_a, respectively S_r. The flow on $S_{a,\sqrt{\varepsilon}}$, respectively $S_{r,\sqrt{\varepsilon}}$, is an $O(\sqrt{\varepsilon})$ perturbation of the reduced flow on S_a, respectively S_r.*

Proof. This follows directly from the proof presented in Szmolyan, Wechselberger [41], case $k = 2$, since the same blow-up analysis can be used for any $k \geq 2$. It is an application of the center manifold theorem in the extended blown-up system $\{(20), \dot{\varepsilon} = 0\}$; see also [47, 6]. □

Remark 4.1. The section $\Sigma_{2,a} : z_1 = -\sqrt{\varepsilon}\delta_2$ is only a cross section of the slow flow for a certain subset of $S_{a,\sqrt{\varepsilon}}$. In the case of a set M_f of folded nodes, it covers the essential region of the funnel. In the case of a set M_f of folded saddles, it covers a neighbourhood of the set of singular canards.

To understand the flow past the set $M_f \subset F$ we rescale system (20) by

$$x_1 = \varepsilon y_1, \quad x_2 = \sqrt{\varepsilon} y_2, \quad z_1 = \sqrt{\varepsilon} z_2, \quad t = \sqrt{\varepsilon} t_2, \tag{28}$$

which gives

$$
(29)\qquad
\begin{array}{rcll}
y_1' &=& \frac{1}{2}\mu(x_3,\dots,x_k)y_2-(1+\mu(x_3,\dots,x_k))z_2+O(\sqrt{\varepsilon}), & \\
y_2' &=& 1+O(\sqrt{\varepsilon}), & \\
x_j' &=& \sqrt{\varepsilon}(A_j(x_3,\dots,x_k)+O(\sqrt{\varepsilon}))\,, & j=3,\dots,k,\\
z_2' &=& y_1+z_2^2+O(\sqrt{\varepsilon}), &
\end{array}
$$

with μ given by (21). This system represents a zoom (or microscope) of the vector field (20) in a neighbourhood of the set of folded singularities M_f.

Remark 4.2. The rescaling (28) is only part of the full blow-up analysis; see e.g. [41] for details. Here, we are using a 'cylindrical blow-up' that zooms only into the (x_1, x_2, z_1)-space transverse to M_f. This reflects the fact that we are dealing with a manifold M_f of folded singularities. In the case $k = 2$, the same 'blow-up' (28) is used to zoom into a single (and hence isolated) folded singularity; see [41]. Similarly, the blow-up analysis of a generic fold F (no folded singularities, only jump points) in a system with $k \geq 2$ is also done by a cylindrical blow-up; see [42].

In system (29) after rescaling, the sections $\Sigma_{2,a}$ respectively $\Sigma_{2,r}$ are now given by $z_2 = -\delta_2$ respectively $z_2 = \delta_2$. From Proposition 4.1 it follows that there exist k-dimensional invariant manifolds $S_{a,\sqrt{\varepsilon}}$ respectively $S_{r,\sqrt{\varepsilon}}$ that extend from infinity to these sections $\Sigma_{2,a}$ respectively $\Sigma_{2,r}$. This also holds in the limit $\sqrt{\varepsilon} \to 0$, i.e. there exist invariant manifolds S_a respectively S_r.[8]

System (29) still has a slow/fast splitting of variables with time scale separation of $O(\sqrt{\varepsilon})$, but now with three fast variables (y_1, y_2, z_2) and $k-2$ slow variables $(x_3, \dots, x_k)$. Note further that system (29) possesses no $(k-2)$-dimensional critical manifold since y_2' in (29) does not equilibrate (y_2 is to leading order the intermediate timescale t_2), but it possesses k-dimensional invariant manifolds $S_{a,\sqrt{\varepsilon}}$ and $S_{r,\sqrt{\varepsilon}}$ that extend from 'infinity' as shown above. It follows that the slow/fast splitting of the variables persists for the flow of (29) on compact domains and this system is governed to leading order solely by the $\sqrt{\varepsilon} \to 0$ limiting system,[9] the three-dimensional 'layer problem'

$$
(30)\qquad
\begin{array}{rcl}
y_1' &=& \frac{1}{2}\mu(x_3,\dots,x_k)y_2-(1+\mu(x_3,\dots,x_k))z_2,\\
y_2' &=& 1,\\
z_2' &=& y_1+z_2^2\,,
\end{array}
$$

where $(x_3, \dots, x_k)$ are considered as parameters and hence μ is a parameter as well. System (30) was extensively studied in the folded saddle and folded node case for singularly perturbed systems with $k = 2$; we refer to [41, 47, 6, 1] for details. The most important insight is that there exist two explicitly known algebraic solutions given by $(y_1, y_2, z_2) = (-\lambda_{1/2}^2 t_2^2 + \lambda_{1/2}, t_2, \lambda_{1/2} t_2)$, where $\lambda_{1/2} = -1, -\mu$ are the nonzero real eigenvalues of the folded (saddle or node) singularities M_f. These special solutions are the 'blown-up' extensions of the sets of singular canards and lie within the invariant manifolds S_a respectively S_r. Thus they provide a connection between these two distinct invariant manifolds. This enables us to extend these manifolds into the vicinity of the set M_f. A transverse intersection of S_a and S_r along a set of singular canards implies persistence of this intersection for $S_{a,\sqrt{\varepsilon}}$ and

[8] We abuse notation and use the same symbol for manifolds $S_{a,\sqrt{\varepsilon}}$ respectively $S_{r,\sqrt{\varepsilon}}$ in (29) and in (20) where one is the blown-up version of the other.

[9] In other words, system (29) is a regular perturbation problem.

$S_{r,\sqrt{\varepsilon}}$ for sufficiently small $\sqrt{\varepsilon} > 0$ and hence the existence of a set of maximal canards. We have the following results:

Theorem 4.1. *In the folded saddle case* $(\lambda_1 < 0 < \lambda_2)$ *of system* (20)*, the* $(k-1)$*-dimensional set* W_{cs} *of singular canards perturbs to a* $(k-1)$*-dimensional set of maximal canards for sufficiently small* $\varepsilon \ll 1$*.*

Proof. This follows directly from Szmolyan and Wechselberger [41], case $k = 2$. □

Theorem 4.2. *In the folded node case* $\lambda_1 < \lambda_2 < 0$ *of system* (20)*, let* $\mu := \lambda_2/\lambda_1 < 1$*.*

(i) *The* $(k-1)$*-dimensional set* W_{ss} *of singular strong canards perturbs to a* $(k-1)$*-dimensional set of maximal strong canards called* primary strong canards *for sufficiently small* $\varepsilon \ll 1$*.*

(ii) *If* $1/\mu \notin \mathbb{N}$*, then the* $(k-1)$*-dimensional set of singular weak canards perturbs to a* $(k-1)$*-dimensional set of maximal weak canards called* primary weak canards *for sufficiently small* $\varepsilon \ll 1$*.*

(iii) *If* $2l + 1 < \mu^{-1} < 2l + 3$*,* $l \in \mathbb{N}$ *and* $\mu^{-1} \neq 2l + 2$*, then there exist* l *additional sets of maximal canards, all* $(k-1)$*-dimensional, called* secondary canards *for sufficiently small* $\varepsilon \ll 1$*. These* l *sets of secondary canards are* $O(\varepsilon^{(1-\mu)/2})$ *close to the set of primary strong canards in* $\Sigma_{1,a}$ *respectively* $\Sigma_{1,r}$*.*

Proof. Statements about the primary canards follow from Szmolyan and Wechselberger [41], case $k = 2$. Statements about secondary canards follow from Wechselberger [47] and Brøns, Krupa and Wechselberger [6], case $k = 2$. □

Geometrically, the $(k-1)$-dimensional set of primary weak canards forms locally an 'axis of rotation' for the k-dimensional sets $S_{a,\varepsilon}$ and $S_{r,\varepsilon}$ and hence also for the set of primary strong canards and the set of secondary canards; this follows from [47], case $k = 2$. These rotations happen in a compact domain of the blown-up system (29) which corresponds to an $O(\sqrt{\varepsilon})$ neighbourhood of F in system (20). Furthermore, the rotations are confined to the (x_1, x_2, z_1)-subspace of system (20). The rotational properties of maximal canards are summarized in the following result.

Theorem 4.3. *In the folded node case of system* (20) *with* $2l + 1 < \mu^{-1} < 2l + 3$*,* $l \in \mathbb{N}$ *and* $\mu^{-1} \neq 2l + 2$*,*

(i) *the set of primary strong canards twists once around the set of primary weak canards in an* $O(\sqrt{\varepsilon})$ *neighbourhood of* F*,*

(ii) *the* j*-th set of secondary canards,* $1 \leq j \leq l$*, twists* $2j + 1$ *times around the set of primary weak canards in an* $O(\sqrt{\varepsilon})$ *neighbourhood of* F*,*

where a twist corresponds to a half rotation. Thus each set of maximal canards has a distinct rotation number.

Proof. This follows directly from Wechselberger [47], case $k = 2$. □

As a geometric consequence, the funnel region of the set of folded nodes M_f in S_a is split by the secondary canards into $l + 1$ subsectors I_j, $j = 1, \ldots, l + 1$, with distinct rotational properties. I_1 is the subsector bounded by the primary strong canard and the first secondary canard, I_2 is the subsector bounded by the first and second secondary canard, I_l is the subsector bounded by the $(l-1)$-th and the l-th secondary canard and finally, I_{l+1} is bounded by the l-th secondary canard and the set of fold points F. Trajectories with initial conditions in the interior of I_j, $1 \le j < l+1$, make $2j+1/2$ twists around the set of primary weak canards, while trajectories with initial conditions in the interior of I_{l+1} make at least $[2(l+1)-1/2]$ twists around the set of primary weak canards. All these solutions are forced to follow the *funnel* created by the manifolds $S_{a,\sqrt{\varepsilon}}$ and $S_{r,\sqrt{\varepsilon}}$. After solutions leave the funnel in an $O(\sqrt{\varepsilon})$-neighbourhood of F they get repelled by the manifold $S_{r,\sqrt{\varepsilon}}$ and will follow close to a fast fiber of system (20). Hence, folded node type canards form separatrix sets in the phase space for different rotational properties near folded critical manifolds.

These results on the dynamics near folded nodes have been numerically confirmed (in the case $k = 2$) in [22, 24, 47] by using sweeping methods and in [12] by using boundary value solvers and continuation methods; see [10] for applications and for an overview on numerical methods for singular perturbation problems.

5. Conclusion

In this paper, we have shown that the canard theory developed in [1, 41, 47] for the case of $k = 2$ slow variables and $m = 1$ fast variables can be extended to the case of arbitrary slow dimensions $k \ge 2$ and arbitrary fast dimensions $m \ge 1$. The existence of a folded critical manifold, a necessary generic condition for canards to exist, implies immediately that the additional $m - 1$ fast hyperbolic (generalized) eigendirections do not alter the dynamics significantly, which follows from the application of the center manifold theorem. This result was already shown in [6] for the case $k = 2$. Thus the local dynamics near a folded critical manifold (independent of the existence of canards) are governed by a $(k + 1)$-dimensional system.

The remarkable insight obtained in this paper is that the local $(k+1)$-dimensional dynamics near generic canards can be understood by understanding the three-dimensional system (30); this is true for any $k \ge 2$. System (30) can be written as a second-order inhomogeneous differential equation

$$z_2'' - 2z_2z_2' + (1 + \mu(x_3, \dots, x_k))z_2 = \frac{1}{2}\mu(x_3, \dots, x_k)t_2 \tag{31}$$

with parameter $\mu(x_3, \dots, x_k)$ under the convention that $\mu(x_3, \dots, x_k) = \mu$ in the case $k = 2$. This equation (31) serves as a local canonical form for generic canard problems in singularly perturbed systems with arbitrary slow dimensions $k \ge 2$. This is similar to the Riccati equation $z_2'' - 2z_2z_2' = 1$ which serves as a local canonical form in the case of a k-dimensional folded critical manifold, $k \ge 1$, without folded singularities [42].

The evolution of the $(k + 1)$-dimensional flow in the canard problem (20) splits into three distinct phases reflecting three distinct time scales of the problem, the slow time scale of $O(1)$, the fast time scale of $O(1/\varepsilon)$ and an intermediate time scale of $O(1/\sqrt{\varepsilon})$, in the following way:

(i) Away from the fold F the flow gets quickly attracted on the fast time scale $O(1/\varepsilon)$ along one-dimensional fast fibers to the k-dimensional critical manifold S as described by Proposition 2.1. Then solutions follow the k-dimensional slow flow on the slow time scale $O(1)$ towards the fold F.

(ii) As the slow flow reaches the vicinity of the $(k-1)$-dimensional fold F and hence that of the $(k-2)$-dimensional set of folded singularities M_f, the original slow-fast time scale separation in (20) breaks down and is replaced by a slow-intermediate time scale separation with three variables evolving on the intermediate time scale $O(1/\sqrt{\varepsilon})$ which includes two of the original slow variables and the original fast variable, and the other $k-2$ original slow variables are evolving still on the slow time scale $O(1)$. By the flow box theorem, the dynamics are governed by the three-dimensional system (30) on the intermediate time scale $O(1/\sqrt{\varepsilon})$ to leading order which describes the transition in finite (intermediate) time near the set of folded singularities M_f. All rotations happen in this region, which is an $O(\sqrt{\varepsilon})$ neighbourhood of the set M_f in system (20).

(iii) Finally, after passing the $O(\sqrt{\varepsilon})$ vicinity of the set of folded singularitites M_f, solutions either follow the slow manifold on the repelling branch S_r on the slow time scale $O(1)$, i.e. they are canards, or they jump along a fast fiber away from the fold F on the fast time scale $O(1/\varepsilon)$ of system (20).

5.1. **Remarks on degenerate canards.** A folded singularity $(\hat{x}, \hat{z}, 0)$ in a singularly perturbed system (1) with $k=1$ is necessarily degenerate since $(\hat{x}, \hat{z}, 0)$ has to fulfill $g(\hat{x}, \hat{z}, 0) = 0$, which violates the nondegeneracy condition (16). In this case, an equilibrium of the reduced flow crosses the fold F as a system parameter is varied. Recall that this degenerate case relates to two phenomena, a *singular Hopf bifurcation* and an associated *canard explosion* as described in section 1. For details, we refer to the literature on planar singularly perturbed systems ($k=m=1$); see, e.g., [16, 18, 33].

In the case $k=2$, there exist two different types of degenerate folded singularities. Suppose that the first nondegeneracy condition in (16) is violated. Then we are dealing with a *folded saddle-node (type II)* [35, 41, 34] and an associated singular Hopf bifurcation [23]. This case corresponds to the unfolding of the planar canard case and was first studied by Milik and Szmolyan [35] in a three-dimensional autocatalator model. Closely related to this case are also singularly perturbed problems that have three distinct time scales; see e.g. Krupa, Popovic and Kopell [30].

On the other hand, suppose that the first nondegeneracy condition in (16) is satisfied but the second is violated. Then this leads to the case of a folded saddle-node (type I) [34, 41, 46]. Here we have a true saddle-node bifurcation of folded singularities and this type I case does *not* involve a Hopf bifurcation. The most prominent example is the periodically forced van der Pol relaxation oscillator where the forcing period evolves on the slow time scale; see e.g. [41].

Degenerate canards in the case $k>2$ have not been studied so far, and this will be part of future work.

5.2. **Remarks on canard induced mixed-mode oscillations.** Mixed-mode oscillations, a mix of small and large amplitude oscillations in a periodic pattern, are frequently observed in applications such as chemical reaction systems, neuronal dynamics and cell signalling. In recent years, canard induced mixed-mode oscillations, i.e. a folded node structure coupled with a global return mechanism, was identified as one likely explanation for systems with a local slow-fast structure with two slow variables and one fast variable; see e.g. [6, 47] for details. Closely related to this phenomenon is the folded saddle-node type II structure coupled with a global return mechanism and an associated singular Hopf bifurcation [23, 34]. A comprehensive review of these and other related mechanisms can be found in [10] and the references therein.

We have shown that the canard theory developed in the case $k = 2$ can be extended to arbitrary slow dimensions $k \geq 2$. This implies that the theory of canard induced MMOs developed in the case $k = 2$ can also be extended to arbitrary slow-fast systems. Thus, a (local) model reduction to two slow variables, a task usually hard to establish, is, in principle, not necessary to identify canard induced mixed-mode oscillations. Hence, we will focus in future work on slow-fast models where a (local) reduction to two slow variables is not possible or rigorously justified. A first application of this general canard theory deals with intracellular calcium dynamics [26] where $k = 3$. We are certain that there are many applications that can benefit from the results presented in this paper.

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School of Mathematics & Statistics, University of Sydney, Sydney NSW 2006, Australia

E-mail address: wm@maths.usyd.edu.au

URL: www.maths.usyd.edu.au/u/wm/