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A PTAS for minimum connected dominating set in 3-dimensional Wireless sensor networks

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Abstract When homogeneous sensors are deployed into a three-dimensional space instead of a plane, the mathematical model for the sensor network is a unit ball graph instead of a unit disk graph. It is known that for the minimum connected dominating set in unit disk graph, there is a polynomial time approximation scheme (PTAS). However, that construction cannot be extended to obtain a PTAS for unit ball graph. In this paper, we will introduce a new construction, which gives not only a PTAS for the minimum connected dominating set in unit ball graph, but also improves running time of PTAS for unit disk graph.

Keywords Wireless sensor network · Connected dominating set · Unit ball graph · PTAS

1 Introduction

Virtual backbone in wireless sensor network has a wide range of applications (cf [3] and references there). A virtual backbone is a subset of nodes D such that non-adjacent nodes can communicate with each other through the nodes in D. Modeling the wireless sensor network as a graph, the virtual backbone is exactly a connected dominating set. A *dominating set* of a graph G is a subset D of vertices such that every vertex x in $V(G)\setminus D$ is adjacent to a vertex y in D. Vertex x is said to be *dominated* by y, or y is said to *dominate* x. A vertex

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$y \in D$ dominates itself. A *connected dominating set* is a dominating set D such that the subgraph of G induced by D, denoted by $G[D]$, is connected. In the real world, one often expects the size of the virtual backbone as small as possible, in order to save the energy of transmissions, lessen the cost of construction, avoid broadcast storm (that is, unbearable interferences caused by too many transmissions at the same time) etc. Hence we are faced with a *minimum connected dominating set problem* (MCDS): to find a connected dominating set with the minimum cardinality. The MCDS has been studied extensively in the literatures [2,11,12,14–16,18].

In practice, the sensors are often assumed to be homogeneous, that is, they have omni-directional antennas with the same transmission range. In this case, the topology of the three-dimensional wireless sensor network can be modeled as a unit ball graph. In an *unit ball graph* (UBG), each vertex corresponds to a point in the space, two vertices are adjacent if and only if the Euclidean distance between their corresponding points is less than or equal to one. In other words, a vertex u is adjacent with a vertex v if and only if u is within the transmission range of v, which has been scaled to one. When restricted to the plane, a unit ball graph degenerates to a *unit disk graph* (UDG). Compared with the large number of studies on UDGs, the studies on UBGs are relatively much less. There are cases in which three-dimensional models are needed, such as under-water sensor systems, outer-space sensor systems, notebooks in a multi-layered buildings, etc.

For MCDS in general graphs, it was proved in [8] that for any $0 < \rho < 1$, there is no polynomial time $\rho \ln n$-approximation unless $NP \subseteq DTIME(n^{O(\ln n)})$, where n is the number of vertices. A greedy $(\ln \Delta + 3)$-approximation [13] and a greedy $(\ln \Delta + 2)$-approximation [8,13] were given, where Δ is the maximum degree of the graph. When restricted to UDG, the MCDS problem is still NP-hard [7]. Hence computing an MCDS in a UBG is also NP-hard. Distributed constant-approximations for MCDS in UDG were studied in [1,5,10,17], etc. Also by distributed strategy, Butenko and Ursulenko [4] gave a 22-approximation for MCDS in UBG. As to centralized algorithm for CDS in UDG, Cheng et al. [6] gave a polynomial time approximation scheme (PTAS), that is, for any $\varepsilon > 0$, there exists a polynomial-time $(1 + \varepsilon)$-approximation. The question is: can their method be generalized to obtain a PTAS for MCDS in UBG? The answer is 'no', since their proof depends on a geometrical property which holds in the plane but is no longer true in the space.

In this paper, we present a PTAS for UBG. The method of analyzing the performance ratio is new. In fact, this method can be used to compute CDS for any n-dimensional unit ball graph. Furthermore, when our method is applied to UDG, the running time can be improved, compared with the algorithm presented in [6].

In Sect. 2, the algorithm is presented, the correctness is proved, the time complexity is analyzed. In Sect. 2.1, we prove that this algorithm is a PTAS. A conclusion is given in Sect. 3.

2 The algorithm

In this section, we present an algorithm for MCDS in UBG. The algorithm uses partition technique combined with a shifting strategy (which was introduced by Hochbaum and Maass [9]).

Let $Q = \{(x, y, z) \mid 0 \le x \le q, 0 \le y \le q, 0 \le z \le q\}$ be a minimal three-dimensional cube containing all the unit balls. For a given positive real number $\varepsilon < 1$, let m be an integer with $m = \lceil 300\rho/\varepsilon \rceil$, where ρ is the performance ratio of a constant-approximation for MCDS in UBG, for example $\rho = 22$ by the algorithm given by Butenko and Ursulenko [4].

Set $p = \lfloor q/m \rfloor + 1$, and $\tilde{Q} = \{(x, y, z) \mid -m \leq x \leq mp, -m \leq y \leq mp, -m \leq z \leq mp\}$. Divide $\tilde{Q}$ into $(p+1) \times (p+1) \times (p+1)$ grid such that each cell is an $m \times m \times m$ cube (each cube is half cloesed and half open, including the back, left, and bottom sides, excluding the front, right, and top sides). Denote this partition as $P(0)$. For $a = 0, 1, \ldots, m-1$, $P(a)$ is the partition obtained by shifting $P(0)$ such that the left-bottom-hind corner of $P(a)$ is at the coordinate $(a - m, a - m, a - m)$. For each cell e, the *boundary region* B_e of e is the region contained in e such that each point in this region is at most distance 3 from the boundary of e. The *central region* C_e of e is the region of e such that each point is at least distance 2 away from the boundary of e. Note that B_e and C_e have an overlap.

Algorithm Input: The geometric representation of a connected unit ball graph G and a positive real number $\varepsilon < 1$.

Output: A connected dominating set D of G.

1. Let $m = \lceil 300\rho/\varepsilon \rceil$.
2. Use the ρ-approximation algorithm to compute a connected dominating set D_0 of G. For each $a \in \{0, 1, \ldots, m-1\}$, denote by $D_0(a)$ the set of vertices of D_0 lying in the boundary region of $P(a)$. Choose a^* with the minimum $|D_0(a)|$.
3. For each cell e of $P(a^*)$, denote by G_e the subgraph of G induced by the vertices in the central region C_e. Compute a minimum subset D_e of vertices in e, such that

for each component H of G_e, $G[D_e]$ has a connected component dominating H. (1)

4. Let $D = D_0(a^*) \cup \bigcup_{e \in P(a^*)} D_e$.

The following lemma shows the correctness of the algorithm.

Lemma 1 *The output D of the algorithm is a CDS of G.*

Proof We first show that D is a dominating set. For each vertex $x \in V(G)$, suppose x is in cell e. If $x \in C_e$, then x is dominated by D_e. If $x \in e \backslash C_e$, then x is in the region of e at distance less than two from the boundary of e. If $x \in D_0$, then $x \in D_0(a^*)$. If $x \notin D_0$, then the vertex $y \in D_0$ which dominates x is in $D_0(a^*)$. By the arbitrariness of x, D is a dominating set of G.

Next, we show that $G[D]$ is connected.

Suppose F_1, F_2 are two components of $G[D_0(a^*)]$ which can be connected by D_0 through the central region of some cell e. Then there exist two vertices $x_1 \in V(F_1) \cap B_e \cap C_e$ and $x_2 \in V(F_2) \cap B_e \cap C_e$ such that x_1, x_2 are in a same component H of G_e. By step 3 of the algorithm, x_1 and x_2 are connected through D_e, and thus F_1 and F_2 are also connected through $D_e \subseteq D$. We have shown that any components of $G[D_0(a^*)]$ are connected in $G[D]$.

Let $\tilde{G}$ be the component of $G[D]$ containing all vertices in $D_0(a^*)$. If $\tilde{G} \neq G[D]$, then there exists a cell e and a component R of $G[D_e]$ such that $V(R) \cap D_0(a^*) = \emptyset$ and R is not adjacent with any vertex in $D_0(a^*)$. Let x be a vertex in D_0 such that x dominates some vertex $y \in V(R)$ (y may coincide with x). Since $x \notin D_0(a^*)$, we have $x \in e \backslash B_e$. Hence $y \in C_e$. Let H be the connected component of G_e containing y. By step 3 of the algorithm, we see that R dominates H. Since $G[D_0]$ is connected, there is a path in $G[D_0]$ connecting x to the other parts of G outside of cell e. Such a path must contain a vertex $z \in D_0 \cap B_e \cap C_e \subseteq D_0(a^*)$. Note that z is also in H. Hence there is a vertex w in $V(R)$ dominating z, contradicting that R is not adjacent with any vertex in $D_0(a^*)$. Hence $\tilde{G} = G[D]$, and thus $G[D]$ is connected. □

The next lemma follows from the well-known fact that any dominating set always has two connected components which are at most three hops away from each other (see for example [8]).

Lemma 2 *For any dominating set D in a connected graph, at most $2(|D|-1)$ vertices are needed to connect D. In particular, $|D_2| \leq 3|D_1| - 2$, where D_1, D_2 are, respectively, a minimum dominating set and a minimum CDS.*

The next lemma shows that the time complexity of the algorithm is polynomial in n and ε.

Lemma 3 *The above algorithm runs in time $n^{O(1/\varepsilon^3)}$.*

Proof Clearly, the most time-consuming part is the third step. Since any vertex in a $\sqrt{3}/3 \times \sqrt{3}/3 \times \sqrt{3}/3$ cube dominates any other vertices in the same cube, we see that a minimum dominating set of e uses at most $(\sqrt{3}m)^3$ vertices. By Lemma 2, $|D_e| \leq 3(\sqrt{3}m)^3$. Hence the exhaust search time takes at most $\sum_{k=0}^{(3\sqrt{3}m)^3} \binom{n_e}{k} = n_e^{O(m^3)}$ to compute D_e, where n_e is the number of vertices in e. It follows that the total time complexity is bounded by $\sum_{e \in P(a^*)} n_e^{O(m^3)} = n^{O(m^3)} = n^{O(1/\varepsilon^3)}$. □

2.1 The performance ratio

In this section, we show that our algorithm is a PTAS for CDS in UBG. For this purpose, we need the following two lemmas.

For a path P in G, the length of P, denoted by $len(P)$, is the number of edges in P. Let H be a subgraph of G. For two subgraphs H_1 and H_2 of G, the distance between H_1 and H_2 in H is $dist_H(H_1, H_2) = \{len(P) \mid P$ is the shortest path connecting H_1 and H_2 in $H\}$. In another word, if $dist_H(H_1, H_2) = k$, then H_1 and H_2 can be connected through at most $k-1$ vertices of H.

Lemma 4 *Let H be a connected subgraph of G, and D be a subset of $V(G)$ dominating H. If $G[D]$ does not contain a connected component dominating H, then there exist two components R and K of $G[D]$ such that $dist_H(R, K) \leq 3$.*

Proof Let $H_1, H_2, \ldots, H_k$ $(k \geq 2)$ be a minimum set of components of $G[D]$ the union of which dominates H. The 'minimum' ensures that every H_i is adjacent with H. Since H is connected, we see that $\bigcup_{i=1}^{k} H_i$ can be connected through vertices in H. Choose H_i and H_j such that $dist_H(H_i, H_j)$ is minimum. Let $P = x_0 x_1 \ldots x_t$ be the shortest path connecting H_i and H_j such that $x_1, \ldots, x_{t-1} \in V(H)$ and $x_0 \in V(H_i)$, $x_t \in V(H_j)$. Suppose $t \geq 4$. Let H_ℓ be a component dominating x_2. Then $\ell \neq i$ and $dist_H(H_\ell, H_j) < dist_H(H_i, H_j)$, a contradiction. □

Lemma 5 *For any vertex u in a unit ball graph G, the neighborhood $N_G(u)$ contains at most 12 independent vertices.*

Proof The result can be obtained by transforming the problem into the famous Gregory–Newton Problem concerning about kissing number [19]. The *kissing number* is the maximum number of unit balls that can simultaneously touch the surface of a unit ball ('touch' means two balls have exactly one point in common). Let $S(u)$ be the unit ball with center u, and $\{x_1, \ldots, x_t\}$ be a maximum set of independent vertices in $S(u)$. For each $i = 1, \ldots, t$, draw a radial r_i with origin u which goes through x_i. Suppose r_i intersects the surface of $S(u)$ at point $\tilde{x}_i$. Let S_i be a unit ball touching $S(u)$ at $\tilde{x}_i$. Since x_i's are independent, the angle between any two radials is at least $\pi/3$. Hence S_i's are non-intersecting. It follows that t is at most the kissing number, which is 12. □

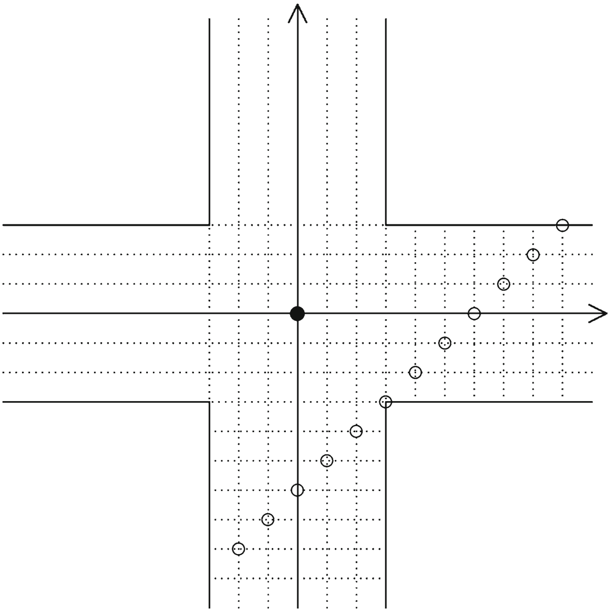

Fig. 1 When the partition shifts, each vertex falls into at most 12 boundary regions

Next, we analyze the performance ratio of the algorithm.

Theorem 1 *The algorithm is a* $(1+\varepsilon)$*-approximation for CDS in UBG.*

Proof Let D^* be an optimal CDS of G.

Note that when a runs over $0, 1, \ldots, m-1$, each vertex belongs to at most 12 boundary regions of $P(a)$'s (see Fig. 1). Hence

$$|D_0(0)| + |D_0(1)| + \cdots + |D_0(m-1)| \leq 12|D_0|,$$

and thus

$$|D_0(a^*)| \leq \frac{12}{m}|D_0| \leq \frac{12\rho}{m}|D^*| \leq \frac{\varepsilon}{25}|D^*|. \tag{2}$$

In the following, we are to add some vertices to D^* such that the resulting vertex set $\tilde{D}$ satisfies:

(i) $|\tilde{D}| \leq |D^*| + 24|D_0(a^*)|$, and
(ii) for each cell e and each connected component H of G_e, $G[\tilde{D} \cap e]$ contains a connected component dominating H.

Before showing how to construct $\tilde{D}$, we first show that as long as this can be done, then the theorem is proved. In fact, since D_e is a minimum subset of e satisfying the requirement (1) and $\tilde{D} \cap e$ satisfies (ii), we have

$$|D_e| \leq |\tilde{D} \cap e|.$$

Then it follows from condition (i) and inequality (2) that

$$\left|\bigcup_{e\in P(a^*)} D_e\right| = \sum_{e\in P(a^*)} |D_e| \le \sum_{e\in P(a^*)} |\tilde{D}\cap e| \\ = |\tilde{D}| \le |D^*| + 24|D_0(a^*)| \le \left(1+\frac{24\varepsilon}{25}\right)|D^*|. \tag{3}$$

Combining inequalities (2) and (3), we have

$$|D| \le |\bigcup_{e\in P(a^*)} D(e)| + |D_0(a^*)| \le (1+\varepsilon)|D^*|,$$

where D is the output of the algorithm. This proves the theorem.

In the following we show how to construct $\tilde{D}$ satisfying conditions (i) and (ii).

We first claim that for any cell e and any component H of G_e, H is dominated by $D^*\cap e$. In fact, any vertex $x\in V(H)$ is dominated by some vertex $y\in D^*$. Since $x\in C_e$, we have $y\in e$.

Set $\tilde{D}^*_e = D^*\cap e$. Suppose $\tilde{D}^*_e$ does not satisfy condition (ii). Then there is a component H of G_e such that H is not dominated by one connected component of $G[\tilde{D}^*_e]$. By Lemma 4, there are two components R and K of $G[\tilde{D}^*_e]$ such that $dist_H(R,K)\le 3$. That is, R and K can be connected through at most two vertices in $V(H)\setminus\tilde{D}^*_e$. Add these vertices into $\tilde{D}^*_e$ to merge R and K. Continue this procedure until $\tilde{D}^*_e$ satisfies condition (ii). Suppose k mergences are executed. Then the resulting $\tilde{D}^*_e$ satisfies

$$|\tilde{D}^*_e| \le |D^*\cap e| + 2k. \tag{4}$$

Next, we use vertices in $D_0(a^*)\cap e$ to compensate for the $2k$ term of inequality (4). Suppose the components are merged in the order that: H_1 is merged with H_2, H_3 is merged with $H_4,\cdots,H_{2k-1}$ is merged with H_{2k}. To simplify the presentation of the idea, we first assume that the H_i's are all distinct components of the original $G[\tilde{D}^*_e]$. Denote by I_e the region of e between distance 1 and 2 from the boundary of e. For each $i=1,2,\cdots,k$, let x_i be a vertex in $V(H_{2i-1})\cap I_e$. Such x_i exists since H_{2i-1} dominates some vertex in H which is a component in the central region of e (hence H_{2i-1} is within distance 1 from the central region), and $G[D^*]$ is connected (hence H_{2i-1} is accessible from the outer side of e). Because D_0 is a dominating set of G, there is a vertex $z_i\in D_0$ dominating x_i. Since $x_i\in I_e$, we have $z_i\in B_e$, and thus $z_i\in D_0(a^*)\cap e$. Note that for $i\ne j$, it is possible that $z_i=z_j$. However, in this case, x_i and x_j are independent since they are in different components of $G[\tilde{D}^*_e]$. Hence by Lemma 5, a vertex serves at most 12 times as z_i's. Thus we have shown that

$$k \le 12|D_0(a^*)\cap e|. \tag{5}$$

Next, consider the case that there are some repetitions among the H_i's. For example, suppose H_3 is the component of the new $G[\tilde{D}^*_e]$ obtained by merging H_1 and H_2. Since x_1 is chosen to be in $V(H_1)\cap I_e$, we can choose $x_3\in V(H_2)\cap I_e$. In general, we are always able to choose x_i's such that they are in different components of the original $G[\tilde{D}^*_e]$. Hence (5) holds in any case. Combining (5) with (4), we have

$$|\tilde{D}^*_e| \le |D^*\cap e| + 24|D_0(a^*)\cap e|. \tag{6}$$

Let $\tilde{D}$ be the union of the modified $\tilde{D}^*_e$'s, where e runs over all cells of $P(a^*)$. Then

$$|\tilde{D}| = \sum_{e\in P(a^*)} |\tilde{D}^*_e| \le \sum_{e\in P(a^*)} \left(|D^*\cap e| + 24|D_0(a^*)\cap e|\right) = |D^*| + 24|D_0(a^*)|. \tag{7}$$

Hence $\tilde{D}$ satisfies requirements (i) and (ii). This completes the proof. □

3 Conclusion

We presented a construction and an analysis of PTAS for the minimum connected dominating set in unit ball graphs. This construction is different from that in [6] for the minimum connected dominating set in unit disk graphs. In fact, the construction in [6] cannot be extended to three-dimensional space since a process of merging many parts of connected components into one in boundary area cannot work. Actually, our construction can be applied to unit ball graphs in n-dimensional space for any $n \geq 1$. An important observation is that the number of independent vertices in an n-dimensional unit ball is upper bounded by a constant (depending only on n). In addition, when applied to unit disk graph, the $(1+\varepsilon)$-approximation constructed in this paper runs in time $n^{O(1/\varepsilon^2)}$ while the $(1+\varepsilon)$-approximation constructed in [6] runs in time $n^{O((1/\varepsilon^2)\ln(1/\varepsilon))}$. Therefore, our construction also improves the running time of PTAS for unit disk graph.

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