

A PURE JUMP MARKOV PROCESS WITH A RANDOM SINGULARITY SPECTRUM

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We construct a nondecreasing pure jump Markov process, whose jump measure heavily depends on the values taken by the process. We determine the singularity spectrum of this process, which turns out to be random and to depend locally on the values taken by the process. The result relies on fine properties of the distribution of Poisson point processes and on ubiquity theorems.

1. Introduction. Up to the mid-1970s, the study of the Hölder regularity of the sample paths of stochastic processes was focused on two main issues: the determination of their uniform modulus of continuity, and the existence of an almost-everywhere pointwise modulus of continuity. However, the first indications that their pointwise regularity could vary from point to point in a subtle way appeared in the works of Orey and Taylor [19] and Perkins [20], who showed that the fast and slow points of Brownian motion are located on random fractal sets. Furthermore, they determined the Hausdorff dimensions of these sets. Brownian motion, however, only displays very slight changes in its modulus of continuity (which is modified only by logarithmic corrections). This is in sharp contrast with other types of processes, such as Lévy processes, for instance, whose modulus of continuity changes completely from point to point. Let us recall the relevant definitions related with pointwise Hölder regularity in this context.

DEFINITION 1. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a locally bounded function, $t_0 \in \mathbb{R}_+$ and let $\alpha > 0$. The function f belongs to $C^\alpha(t_0)$ if there exist $C > 0$ and a polynomial P_{t_0} of degree less than α such that, for all t in a neighborhood of t_0 ,

$$|f(t) - P_{t_0}(t)| \leq C|t - t_0|^\alpha.$$

The Hölder exponent of f at t_0 is (here $\sup \emptyset = 0$)

$$h_f(t_0) = \sup\{\alpha > 0 : f \in C^\alpha(t_0)\}.$$

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The level sets of the pointwise exponent of the Hölder exponent are called the *iso-Hölder sets* of f , and defined, for $h \geq 0$, by

$$E_f(h) = \{t \geq 0 : h_f(t) = h\}.$$

The corresponding notion of “multifractal function” was put into light by Frisch and Parisi [14], who introduced the definition of the *spectrum of singularities* of a function f .

DEFINITION 2. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a locally bounded function. The spectrum of singularities of f is the function D_f defined, for $h \geq 0$, by

$$D_f(h) = \dim_H(E_f(h)).$$

We also define, for any open set $A \subset \mathbb{R}_+$, $D_f(A, h) = \dim_H(E_f(h) \cap A)$.

The definition of the Hausdorff dimension can be found in Falconer [12] for instance (by convention $\dim_H \emptyset = -\infty$). The singularity spectrum of f describes the geometric repartition of its Hölder singularities, and encapsulates a geometric information which is usually more meaningful than the Hölder exponent.

Following the way opened by Frisch and Parisi, the spectrum of singularities of large classes of stochastic processes (or random measures, in which case an appropriate notion of Hölder pointwise regularity for measures is used) have been determined. Most examples of stochastic processes f which have been studied display the following remarkable features:

- Though the iso-Hölder sets are random, the spectrum of singularities is deterministic: for some deterministic function Δ , a.s., for all $h \geq 0$, $D_f(h) = \Delta(h)$.
- The spectrum of singularities of f is *homogeneous*: a.s., for any nonempty open subset $A \subset \mathbb{R}_+$, for all $h \geq 0$, $D_f(A, h) = \Delta(h)$.

Though it is easy to construct artificial ad hoc processes that do not satisfy these properties, it is remarkable that many “natural” processes of a very different kind follow this rule: Lévy processes [17], Lévy processes in multifractal time [5], fractional Brownian motions, random self-similar measures and random Gibbs measures [9], Mandelbrot cascades [2], Poisson cascades [3], among many other examples. See, however, [10] where Durand constructed a counterexample whose (random) wavelet coefficients are defined using Markov trees.

In this paper we will investigate the regularity properties of some Markov processes. Our purpose at this stage is not to obtain results in the most general form, but rather to consider some specific examples, and check that such processes indeed display a random spectrum, which is not homogeneous. Some regularity properties of the sample paths of Markov processes (including Lévy processes) have been investigated, for instance, by Xiao in [22]. Nevertheless, until now, the only Markov processes which have been analyzed from the multifractal standpoint are the Lévy processes.

We now introduce a new notion, the *local spectrum*, which is tailored to the study of functions with nonhomogeneous spectra.

DEFINITION 3. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a locally bounded function, $t_0 \in \mathbb{R}_+$, and let $(V_n)_{n \geq 1}$ be a basis of neighborhoods of t_0 . The local spectrum of f at t_0 is the function

$$\text{for all } h \geq 0 \quad D_f(t_0, h) = \lim_{n \rightarrow \infty} D_f(V_n, h).$$

By monotonicity [if $A \subset B$, then $D_f(A, h) \leq D_f(B, h)$], the limit exists and it is independent of the particular basis chosen. Clearly, a function has a homogeneous spectrum if and only if, for all $h \geq 0$, $D_f(t, h)$ is independent of $t \geq 0$. The local spectrum allows one to recover the spectrum of all possible restrictions of f on an open interval.

LEMMA 4. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be a locally bounded function. Then for any open interval $I = (a, b) \subset \mathbb{R}_+$, for any $h \geq 0$, we have $D_f(I, h) = \sup_{t \in I} D_f(t, h)$.

PROOF. Let thus $h \geq 0$ be fixed. First, it is obvious that, for any $t \in I$, $D_f(t, h) \leq D_f(I, h)$, since for $(V_n)_{n \geq 1}$ a basis of neighborhoods of t , $V_n \subset I$ for n large enough. Next, set $\delta = D_f(I, h)$, and consider $\varepsilon > 0$. We want to find $t \in I$ such that, for all neighborhood V of t , $D_f(V, h) \geq \delta - \varepsilon$. Assume by contradiction that, for any $t \in I$, there is a neighborhood V_t of t such that $D_f(V_t, h) < \delta - \varepsilon$. One easily deduces that, for any compact $K \subset I$, $D_f(K, h) \leq \delta - \varepsilon$ (use a finite covering of K by the V_t 's). This would of course imply that $D_f(I, h) \leq \delta - \varepsilon < \delta$. □

Let us now recall the multifractal nature of Lévy processes without Brownian component. In that case, the spectrum of singularities only depends on one parameter, the *lower index* of Blumenthal and Gettoor, which quantifies the “density” of small jumps, and is defined, for any nonnegative measure ν on \mathbb{R} satisfying $\int_{-1}^1 u^2 \nu(du) < \infty$, by

$$\beta_\nu := \inf \left\{ \alpha \geq 0 : \int_{-1}^1 |u|^\alpha \nu(du) < \infty \right\}.$$

Note that the integrability condition implies that $\beta_\nu \in [0, 2]$. The following result of [17] yields the spectrum of singularities of such Lévy processes.

THEOREM 5. Let $(X_t)_{t \geq 0}$ be a Lévy process without Brownian component, with Lévy measure ν . If $\beta_\nu \in (0, 2)$, then the singularity spectrum of X is homogeneous and deterministic: a.s., for all $t \geq 0$, for all $h \geq 0$,

$$D_X(t, h) = D_X(h) = \begin{cases} h \cdot \beta_\nu, & \text{if } 0 \leq h \leq 1/\beta_\nu, \\ -\infty, & \text{if } h > 1/\beta_\nu. \end{cases}$$

Let us make a few observations. It is not stated explicitly in [17] that the spectrum of a Lévy process is homogeneous, but it is a direct consequence of the proof. Indeed, the spectrum is computed on an arbitrary interval, and the stationarity of the increments implies that it does not depend on the particular location of this interval. Although a Lévy process is random, its spectrum is almost surely deterministic. As explained above, this is the situation usually met when performing the multifractal analysis of random processes or random measures possessing either stationarity or scaling invariance properties.

The purpose of this paper is to investigate how these results are modified when the stationarity assumption is dropped. We will give examples of Markov processes whose singularity spectra are nonhomogeneous and random.

2. Statement of the main result. A quite general class of one-dimensional Markov processes consists of stochastic differential equations (SDE) with jumps; see Ethier and Kurtz [11] and Ikeda and Watanabe [15]. Since the Brownian motion is mono-fractal, the Brownian part of such a process will not be very relevant. Thus, in order to avoid technical difficulties, we consider a jumping SDE without Brownian and drift part, starting, for example, from 0, and with jump measure $\nu(y, du)$ [meaning that when located at y , the process jumps to $y + u$ at rate $\nu(y, du)$]. Again, to make the study as simple as possible, we will assume that the process has finite variations, and even that it is increasing [i.e., $\nu(y, (-\infty, 0)) = 0$ for all $y \in \mathbb{R}$]. Classically, a necessary condition for the process to be well defined is that $\int_0^\infty u \nu(y, du) < \infty$.

If ν is chosen so that the index $\beta_{\nu(y, \cdot)}$ is constant with respect to y , then we expect that $D_M(t, h)$ will be deterministic and independent of t . We thus impose that the index of the jump measure depends on the value y of the process. The most natural example of such a situation consists in choosing

$$\nu_\gamma(y, du) := \gamma(y)u^{-1-\gamma(y)}\mathbf{1}_{[0,1]}(u) du$$

for some function $\gamma : \mathbb{R} \mapsto (0, 1)$. The lower exponent of this family of measures is

$$\forall y \geq 0 \quad \beta_{\nu_\gamma(y, \cdot)} = \gamma(y).$$

We will make the following assumption:

$$(\mathcal{H}) \quad \begin{cases} \text{there exists } \varepsilon > 0 \text{ such that } \gamma : [0, \infty) \mapsto [\varepsilon, 1 - \varepsilon] \\ \text{is a Lipschitz-continuous strictly increasing function.} \end{cases}$$

We impose a monotonicity condition for simplicity, it could be dropped under additional technicalities. The global Lipschitz condition could also be slightly relaxed, as well as the uniform bounds.

PROPOSITION 6. Assume that (\mathcal{H}) holds. There exists a strong Markov process $M = (M_t)_{t \geq 0}$ starting from 0, increasing and càdlàg, and with generator \mathcal{L} defined for all $y \in [0, \infty)$ and for any function $\phi : [0, \infty) \mapsto \mathbb{R}$ Lipschitz-continuous by

$$(2.1) \quad \mathcal{L}\phi(y) = \int_0^1 [\phi(y + u) - \phi(y)] \nu_\gamma(y, du).$$

Almost surely, this process is continuous except on a countable number of jump times. We denote by \mathcal{J} the set of its jump times, that is, $\mathcal{J} = \{t > 0 : \Delta M(t) \neq 0\}$. Finally, \mathcal{J} is dense in $[0, \infty)$.

Here and below, $\Delta M_t = M_t - M_{t-}$, where $M_{t-} = \lim_{s \rightarrow t, s < t} M_s$. Proposition 6 will be checked in Section 3, by using a Poisson SDE. This representation of M will be useful for its local regularity analysis in the next sections.

The following theorem summarizes multifractal features of M .

THEOREM 7. Assume (\mathcal{H}) and consider the process M constructed in Proposition 6. Then, the following properties hold almost surely:

(i) For every $t \in (0, \infty) \setminus \mathcal{J}$, the local spectrum of M at t is given by

$$(2.2) \quad D_M(t, h) = \begin{cases} h \cdot \gamma(M_t), & \text{if } 0 \leq h \leq 1/\gamma(M_t), \\ -\infty, & \text{if } h > 1/\gamma(M_t), \end{cases}$$

while for $t \in \mathcal{J}$,

$$(2.3) \quad D_M(t, h) = \begin{cases} h \cdot \gamma(M_t), & \text{if } 0 \leq h < 1/\gamma(M_t), \\ h \cdot \gamma(M_{t-}), & \text{if } h \in [1/\gamma(M_t), 1/\gamma(M_{t-})], \\ -\infty, & \text{if } h > 1/\gamma(M_{t-}). \end{cases}$$

(ii) The spectrum of M on any interval $I = (a, b) \subset (0, +\infty)$ is, for all $h \geq 0$,

$$(2.4) \quad D_M(I, h) = \sup\{h \cdot \gamma(M_t) : t \in I, h \cdot \gamma(M_t) < 1\}$$

$$(2.5) \quad = \sup\{h \cdot \gamma(M_{s-}) : s \in \mathcal{J} \cap I, h \cdot \gamma(M_{s-}) < 1\}.$$

In (2.4) and (2.5), we adopt the convention that $\sup \emptyset = -\infty$.

In view of Theorem 5, Theorem 7 seems very natural, since the process M looks locally like a Lévy subordinator (a precise meaning will be given to this assertion later). However, the results can clearly not be inferred from Theorem 5.

REMARK 8. To prove Theorem 7, it is enough to show (2.5).

PROOF OF THEOREM 7. The equality between (2.4) and (2.5) follows from the fact that \mathcal{J} is dense in \mathbb{R}_+ , and that $t \mapsto \gamma(M_t)$ is càdlàg on I .

Next, point (i) follows from (2.4) applied to $I_n = (t - 1/n, t + 1/n)$ by taking the limit $n \rightarrow \infty$ (recall Definition 3). Let us, for example, assume that $t \in \mathcal{J}$.

- If $h > 1/\gamma(M_{t-})$, then for n large enough, $h \cdot \gamma(M_s) \geq h \cdot \gamma(M_{t-1/n}) \geq 1$ for all $s \in I_n$, whence $D_M(I_n, h) = -\infty$ [we use here that $s \mapsto \gamma(M_{s-})$ is nondecreasing]. Thus, $D_M(t, h) = -\infty$.
- If $h < 1/\gamma(M_t)$, we get from (2.4) that $h \cdot \gamma(M_t) \leq D_M(I_n, h) \leq h \times \sup_{[t-1/n, t+1/n]} \gamma(M_s) = h \cdot \gamma(M_{t+1/n})$. Since $s \mapsto \gamma(M_s)$ is right continuous, $D_M(t, h) = \lim_n D_M(I_n, h) = h \cdot \gamma(M_t)$.
- If $h \in [1/\gamma(M_t), 1/\gamma(M_{t-})]$, then for all $s \geq t$, $h \cdot \gamma(M_s) \geq 1$, while clearly $h \cdot \gamma(M_{t-1/n}) < 1$. Hence, we deduce from (2.4) that $h \cdot \gamma(M_{t-1/n}) \leq D_M(I_n, h) \leq h \cdot \sup_{[t-1/n, t]} \gamma(M_s) = h \cdot \gamma(M_{t-})$.

Finally, $D_M(t, h) = \lim_n D_M(I_n, h) = h \cdot \gamma(M_{t-})$. \square

Formula (2.5) is better understood when plotted: for every $s \in I \cap \mathcal{J}$, plot a segment whose endpoints are $(0, 0)$ and $(1/\gamma(M_{s-}), 1)$ (open on the right), and take the supremum to get $D_M(I, \cdot)$. Sample paths of the process M and their associated spectra are given in Figure 1.

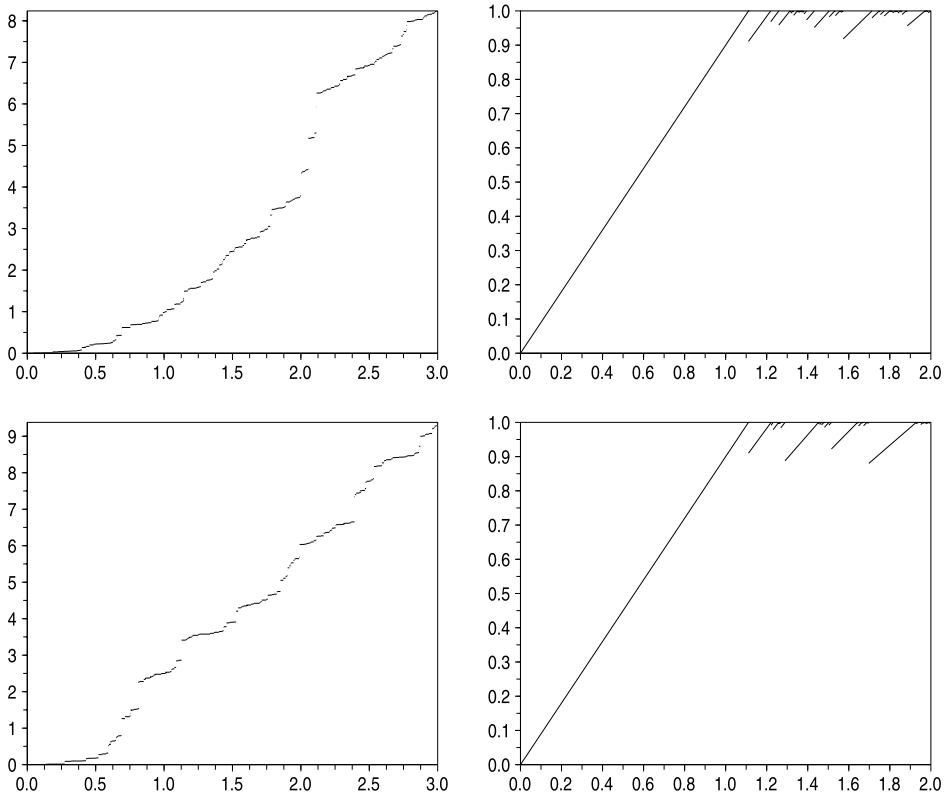


FIG. 1. Two sample paths of the stochastic process M built using the function $\gamma(y) := \min(1/2 + y/4, 0.9)$. On the right-hand side are plotted the theoretical spectra $D_M([0, 3], \cdot)$.

The formulae giving the local and global spectra are based on the computation of the pointwise Hölder exponents at all times t . We will, in particular, prove (see Theorem 19 and Proposition 23) the following properties: a.s.,

$$\begin{aligned} \text{for every } t \geq 0 & & h_M(t) &\leq 1/\gamma(M_t), \\ \text{for Lebesgue-almost every } t & & h_M(t) &= 1/\gamma(M_t), \\ \text{for every } \kappa \in (0, 1) & & \dim_H\{t \geq 0 : h_M(t) = \kappa/\gamma(M_t)\} &= \kappa. \end{aligned}$$

It is worth emphasizing that, as expected from the construction of the process M , the local spectrum (2.2) at any point $t > 0$ essentially coincides with that of a stable Lévy subordinator of index $\gamma(M_t)$. This local comparison will be strengthened in Section 7, where we prove the existence of tangent processes for M (which are Lévy stable subordinators).

The estimation of the pointwise Hölder exponents is based on a local comparison between the Markov process M and some Lévy subordinators associated with M . But of course, such a method cannot be applied to compute the singularity spectrum of M , since Hausdorff dimensions are not preserved under limits. The proof of Theorem 7 requires a so-called *ubiquity theorem*. Ubiquity theory deals with the search for lower bounds for the Hausdorff dimensions of limsup sets, and is classically required when performing the multifractal analysis of stochastic processes or (random or deterministic) measures with jumps [5, 6, 17, 18]. For our Markov process M , the ubiquity theorem needed here is the “localized ubiquity” theorem recently proved in [7]. In order to apply this result, we need to establish fine properties on the distribution of Poisson point processes (see Section 6).

REMARK 9. It follows from Theorem 7 that, for all $s \in \mathcal{J}$, all $h \in (1/\gamma(M_s), 1/\gamma(M_{s-})]$, $D_M(h) = h \cdot \gamma(M_{s-})$. Thus, the spectrum D_M is a straight line on all segments of the form $(1/\gamma(M_s), 1/\gamma(M_{s-})]$, $s \in \mathcal{J}$. Nonetheless, observe that the spectrum we obtain, when viewed as a map from \mathbb{R}_+ to \mathbb{R}_+ , is very irregular, and certainly multifractal itself. This is in sharp contrast with the spectra usually obtained, which are most of the time concave or (piecewise) real-analytic.

REMARK 10. Random processes with random singularity exponents of the most general form have already been constructed in [1], but there is no direct relationship with having a random singularity spectrum. An example of stochastic process X (built using wavelet coefficients) with a random singularity spectrum is provided by [10], but there $D_X(h)$ is random for at most two values of h .

REMARK 11. Of course we hope that Theorem 7, which concerns a specific and simple process, might have extensions to more general Markov processes $M = (M_t)_{t \in [0,1]}$. In particular, this is certainly the case if we keep the same measures ν_γ and if we drop the monotonicity assumption on the Lipschitz-continuous function γ .

The organization of the paper is the following. In Section 3 Proposition 6 is proved. We also introduce a suitable coupling of M with a family of Lévy processes. In Section 4 we introduce a family of (random) subsets of $[0, 1]$ on which we control the regularity of M . We conclude the proof of Theorem 7 in Sections 5 and 6. Finally, in Section 7, we show the existence of tangent processes for M .

In the whole paper we assume that (\mathcal{H}) is satisfied. We will restrict our study to the time interval $[0, 1]$, which of course suffices.

3. Poisson representation of the process. First of all, we observe that using the substitution $u = (1 + z)^{-1/\gamma(y)}$ in (2.1) (for each fixed y), we may rewrite, for any $y \in [0, \infty)$, for any $\phi : [0, \infty) \mapsto \mathbb{R}$ Lipschitz-continuous,

$$(3.1) \quad \mathcal{L}\phi(y) = \int_0^\infty [\phi(y + G(\gamma(y), z)) - \phi(y)] dz,$$

where

$$G(\beta, z) := (1 + z)^{-1/\beta}.$$

We recall the following representation of the Poisson measures we are going to use.

REMARK 12. Let $(Y_n)_{n \geq 1}$ be a sequence of independent exponential random variables with parameter 1. Let $(T_n)_{n \geq 1}$ be a sequence of independent $[0, 1]$ -valued uniformly-distributed random variables, and assume that $(Y_n)_{n \geq 1}$ and $(T_n)_{n \geq 1}$ are independent. For each $n \geq 1$, set $Z_n = Y_1 + \dots + Y_n$. Then the random measure

$$N(dt, dz) = \sum_{n \geq 1} \delta_{(T_n, Z_n)}(dt, dz)$$

is a Poisson measure on $[0, 1] \times [0, \infty)$ with intensity measure $dt dz$. We denote by $\mathcal{F}_t := \sigma(\{N(A), A \in \mathcal{B}([0, t] \times [0, \infty))\})$ the associated filtration.

The law of large numbers ensures us that a.s. $\lim_n Z_n/n = 1$.

This leads us to the following representation of the process M .

PROPOSITION 13. *Let N be as in Remark 12. Then there exists a unique càdlàg $(\mathcal{F}_t)_{t \in [0, 1]}$ -adapted process $M = (M_t)_{t \in [0, 1]}$ solution to*

$$(3.2) \quad M_t = \int_0^t \int_0^\infty G(\gamma(M_{s-}), z) N(ds, dz).$$

Furthermore, M is a strong Markov process with generator \mathcal{L} [see (2.1) or (3.1)], and is a.s. increasing. Finally, $\mathcal{J} = \{t \in [0, 1] : \Delta M_t \neq 0\} = \{T_n : n \geq 1\}$.

Proposition 6 is a consequence of this result.

PROOF OF PROPOSITION 13. Owing to classical arguments (see, e.g., Ikeda and Watanabe [15]), the (pathwise) existence and uniqueness of the solution to (3.2) follow from the two following properties, which are easily checked under (\mathcal{H}) :

- for all $y \in [0, \infty)$, for all $z \in [0, \infty)$, $G(\gamma(y), z) \leq (1+z)^{-1/(1-\varepsilon)} \in L^1([0, \infty), dz)$,
- for all $x, y \in [0, \infty)$, $\int_0^\infty |G(\gamma(x), z) - G(\gamma(y), z)| dz = \left| \frac{\gamma(x)}{1-\gamma(x)} - \frac{\gamma(y)}{1-\gamma(y)} \right| \leq C|x - y|$ [here we use that $G(\beta, z)$ is nondecreasing as a function of β].

Indeed, it suffices to use the Gronwall Lemma and a Picard iteration (for the norm $\|X - Y\| = \mathbb{E}[\sup_{[0,1]} |X_t - Y_t|]$). The strong Markov property follows from the pathwise uniqueness, and the monotonicity of M is obvious since G is non-negative. Finally, for $\phi : [0, \infty) \mapsto \mathbb{R}$ Lipschitz-continuous, we have

$$\begin{aligned} \phi(M_t) &= \phi(0) + \sum_{s \leq t} [\phi(M_{s-} + \Delta M_s) - \phi(M_{s-})] \\ &= \phi(0) + \int_0^t \int_0^\infty [\phi(M_{s-} + G(\gamma(M_{s-}), z)) - \phi(M_{s-})] N(ds, dz). \end{aligned}$$

Taking expectations and using (3.1), we get

$$\mathbb{E}[\phi(M_t)] = \phi(0) + \int_0^t \mathbb{E}[\mathcal{L}\phi(M_s)] ds,$$

so that the generator of M is indeed \mathcal{L} . \square

We also introduce a one-parameter family of Lévy processes, and check some regularity comparisons with M .

PROPOSITION 14. Consider the Poisson measure N and the process M introduced in Remark 12 and Proposition 13. For each fixed $\alpha \in (0, 1)$, we define

$$(3.3) \quad X_t^\alpha = \int_0^t \int_0^\infty G(\alpha, z) N(ds, dz).$$

Then $(X_t^\alpha)_{t \in [0,1]}$ is a pure-jump $(\mathcal{F}_t)_{t \in [0,1]}$ -adapted Lévy process. Its Lévy measure is $\nu^\alpha(du) = \alpha u^{-1-\alpha} \mathbf{1}_{[0,1]}(u) du$, for which $\beta_{\nu^\alpha} = \alpha$. Almost surely:

- (i) for all $0 < \alpha < \alpha' < 1$, for all $0 \leq s \leq t \leq 1$,

$$0 \leq X_t^\alpha - X_s^\alpha \leq X_t^{\alpha'} - X_s^{\alpha'};$$

- (ii) for all $0 \leq s \leq t \leq 1$,

$$0 \leq X_t^{\gamma(M_s)} - X_s^{\gamma(M_s)} \leq M_t - M_s \leq X_t^{\gamma(M_{t-})} - X_s^{\gamma(M_{t-})}.$$

PROOF. For each $\alpha \in (0, 1)$, X^α is classically a Lévy process, and its Lévy measure is the image measure of dz by $G(\alpha, \cdot)$, which is nothing but ν^α . Next, point (i) is not hard since for $0 < \alpha < \alpha'$, we have $G(\alpha, z) < G(\alpha', z)$ for all $z \in (0, \infty)$. Point (ii) is also immediate: since $u \mapsto \gamma(M_{u-})$ is nondecreasing, and since $G(\cdot, z)$ is nondecreasing for all $z \in (0, \infty)$, we have a.s., for $0 \leq s \leq t \leq 1$,

$$\begin{aligned} M_t - M_s &= \int_{(s,t]} G(\gamma(M_{u-}), z)N(du, dz) \\ &\leq \int_{(s,t]} G(\gamma(M_{t-}), z)N(du, dz) = X_t^{\gamma(M_{t-})} - X_s^{\gamma(M_{t-})}. \end{aligned}$$

Similarly, we obtain $M_t - M_s \geq X_t^{\gamma(M_s)} - X_s^{\gamma(M_s)}$, which ends the proof. \square

4. Local regularity. We consider a Poisson measure N as in Remark 12, and the associated processes M, X^α as in Propositions 13 and 14. We start with a simple observation.

LEMMA 15. *Almost surely, for all $\alpha \in (0, 1)$,*

$$(4.1) \quad \mathcal{J} = \{t \in [0, 1] : \Delta M_t \neq 0\} = \{t \in [0, 1] : \Delta X_t^\alpha \neq 0\} = \bigcup_{n \geq 1} \{T_n\},$$

and for all $n \geq 1$, all $\alpha \in (0, 1)$,

$$(\Delta M_{T_n})^{\gamma(M_{T_n-})} = (\Delta X_{T_n}^\alpha)^\alpha = (1 + Z_n)^{-1}.$$

PROOF. By construction, we have $M_t = \sum_{n \geq 1} G(\gamma(M_{T_n-}), Z_n)\mathbf{1}_{\{t \geq T_n\}}$ and $X_t^\alpha = \sum_{n \geq 1} G(\alpha, Z_n)\mathbf{1}_{\{t \geq T_n\}}$ for all $\alpha \in (0, 1)$ and all $t \in [0, 1]$. All these series converge simultaneously on the set $\{\lim_{n \rightarrow +\infty} \frac{Z_n}{n} = 1\}$, which has probability 1 as seen in Remark 12. Thus, a.s., all these processes jump at the same time, and (4.1) holds. Next, for $n \geq 1$, $\Delta M_{T_n} = G(\gamma(M_{T_n-}), Z_n) = (1 + Z_n)^{-1/\gamma(M_{T_n-})}$ and for every $\alpha \in (0, 1)$, we have $\Delta X_{T_n}^\alpha = G(\alpha, Z_n) = (1 + Z_n)^{-1/\alpha}$. \square

REMARK 16. It might seem surprising that the previous result holds a.s. simultaneously for all $\alpha \in (0, 1)$. But for all $0 \leq s \leq t$, the map $\alpha \mapsto X_t^\alpha - X_s^\alpha$ is continuous (and actually C^∞) on $(0, 1)$. Indeed, we can write $X_t^\alpha - X_s^\alpha = \sum_{n \geq 1} G(\alpha, Z_n)\mathbf{1}_{\{s < T_n \leq t\}}$, and this series converges a.s. normally for $\alpha \in [\varepsilon, 1 - \varepsilon]$ for any $\varepsilon \in (0, 1/2)$. To prove this last assertion, it suffices to note that $\sup_{\alpha \in [\varepsilon, 1-\varepsilon]} |G(\alpha, Z_n)| = G(1 - \varepsilon, Z_n)$.

Next, we introduce some (random) sets of times, more or less well-approximated by the times of jumps of our process M . The main idea is that at times well-approximated by the jump times of M , the pointwise regularity of M will be poor, while at times which are far from the jump times of M , M will have greater pointwise exponents.

We thus set, for all $\delta \geq 1$,

$$(4.2) \quad A_\delta = \bigcap_{p \geq 1} \bigcup_{n \geq p} B(T_n, (\Delta M_{T_n})^{\delta \gamma(M_{T_n}^-)}) = \bigcap_{p \geq 1} \bigcup_{n \geq p} B(T_n, (\Delta X_{T_n}^\alpha)^{\delta \alpha})$$

$$(4.3) \quad = \bigcap_{p \geq 1} \bigcup_{n \geq p} B(T_n, (1 + Z_n)^{-\delta}).$$

Here, $B(t, r) = (t - r, t + r)$. Let us observe at once that

$$\text{for all } \delta_1 \leq \delta_2 \quad A_{\delta_2} \subset A_{\delta_1}.$$

PROPOSITION 17. *Almost surely, $A_1 \supset [0, 1]$.*

PROOF. Observe that $\sum_{n \geq 1} \delta_{(T_n, (1+Z_n)^{-1})}$ is a Poisson measure on $[0, 1] \times (0, 1)$ with intensity measure $dt \mu(du)$ where $\mu(du) = du/u^2$ [because $\mu(du)$ is the image measure of $\mathbf{1}_{\{z>0\}} dz$ by the application $(1 + z)^{-1}$]. Due to Shepp's Theorem [21] (we use here the version used in the papers of Bertoin [8] and Jaffard [17], Lemma 3), it suffices to prove that

$$S = \int_0^1 \exp\left(2 \int_t^1 \mu((u, 1)) du\right) dt = \infty.$$

But $\mu((u, 1)) = u^{-1} - 1$, so that $S = \int_0^1 e^{2(t-1-\log t)} dt \geq e^{-2} \int_0^1 dt/t^2 = \infty$. \square

In order to characterize the pointwise exponent of M at every time t , we need to introduce the notion of approximation rate by a Poisson process.

DEFINITION 18. Recall that a.s. $\delta \mapsto A_\delta$ is nonincreasing and $A_1 = [0, 1]$. We introduce, for any $t \in [0, 1]$, the (random) index of approximation of t ,

$$(4.4) \quad \delta_t := \sup\{\delta \geq 1 : t \in A_\delta\} \geq 1.$$

We now are able to give the value of $h_M(t)$.

THEOREM 19. *Almost surely, for all $t \in [0, 1] \setminus \mathcal{J}$,*

$$(4.5) \quad h_M(t) = \frac{1}{\delta_t \cdot \gamma(M_t)}.$$

In particular, this implies that, for every $t \in [0, 1]$, $h_M(t) \leq 1/\gamma(M_t)$.

We introduce, for $f : \mathbb{R}_+ \mapsto \mathbb{R}$ a locally bounded function and $t_0 \in \mathbb{R}_+$,

$$\tilde{h}_f(t_0) := \sup\{\alpha > 0 : \exists C, |f(t) - f(t_0)| \leq C|t - t_0|^\alpha \text{ in a neighborhood of } t_0\}.$$

This notion of the Hölder exponent of f at t_0 is slightly different of that introduced in Definition 1 (which may involve a polynomial). Note that we always

have $\tilde{h}_f(t_0) \leq h_f(t_0)$. In the case where f is purely discontinuous and increasing, one might expect that $h_f(t_0) = \tilde{h}_f(t_0)$ in many cases. This is the case when f is a Lévy subordinator without drift. Indeed, from [17], Lemma 4 and Proposition 2, we have a.s.

$$\text{for all } t \in [0, 1] \setminus \mathcal{J}, \text{ for all } \alpha \in (0, 1) \quad \tilde{h}_{X^\alpha}(t) = h_{X^\alpha}(t) = (\delta_t \cdot \alpha)^{-1}$$

(here we use that X^α is a pure jump Lévy process without drift with Lévy measure ν^α satisfying $\beta_{\nu^\alpha} = \alpha$).

PROOF OF THEOREM 19 (Lower bound). Let $t \in [0, 1] \setminus \mathcal{J}$ and $\varepsilon > 0$ small enough be fixed. By construction, M is continuous at t . Since γ is also continuous, there exists $\eta > 0$ such that, for all $s \in (t - \eta, t + \eta)$, $\gamma(M_s) \in (\gamma(M_t) - \varepsilon, \gamma(M_t) + \varepsilon)$. We deduce from Proposition 14(ii) that, for all $s \in (t - \eta, t)$,

$$0 \leq M_t - M_s \leq X_t^{\alpha_\varepsilon^+} - X_s^{\alpha_\varepsilon^+},$$

where $\alpha_\varepsilon^+ := \gamma(M_t) + \varepsilon$. Similarly, when $s \in (t, t + \eta)$,

$$0 \leq M_s - M_t \leq X_s^{\alpha_\varepsilon^+} - X_t^{\alpha_\varepsilon^+}.$$

Thus, applying the definition of \tilde{h} , we conclude that $h_M(t) \geq \tilde{h}_M(t) \geq \tilde{h}_{X^{\alpha_\varepsilon^+}}(t) = (\delta_t \cdot \alpha_\varepsilon^+)^{-1}$.

Letting ε go to zero, we deduce that $h_M(t) \geq (\delta_t \cdot \gamma(M_t))^{-1}$. \square

To prove an upper bound for $h_M(t)$, we use the following result of [16], Lemma 1.

LEMMA 20. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function discontinuous on a dense set of points, and let $t \in \mathbb{R}$. Let $(t_n)_{n \geq 1}$ be a real sequence converging to t such that, at each t_n , f has right and left limits at t_n and $|f(t_n+) - f(t_n-)| = s_n$. Then

$$h_f(t) \leq \liminf_{n \rightarrow +\infty} \frac{\log s_n}{\log |t_n - t|}.$$

PROOF OF THEOREM 19 (Upper bound). By (4.2) and (4.4), for every $\varepsilon > 0$, $t \in A_{\delta_t - \varepsilon}$, so that there exists a sequence of jump instants $(T_{n_k})_{k \geq 1}$ converging to t such that $|t - T_{n_k}| \leq (\Delta M_{T_{n_k}})^{(\delta_t - \varepsilon)\gamma(M_{T_{n_k}-})}$. Hence, by Lemma 20, we get

$$\begin{aligned} h_M(t) &\leq \liminf_{k \rightarrow +\infty} \frac{\log \Delta M_{T_{n_k}}}{\log |T_{n_k} - t|} \leq \liminf_{k \rightarrow +\infty} \frac{\log \Delta M_{T_{n_k}}}{\log (\Delta M_{T_{n_k}})^{(\delta_t - \varepsilon)\gamma(M_{T_{n_k}-})}} \\ &\leq \liminf_{k \rightarrow +\infty} \frac{1}{(\delta_t - \varepsilon)\gamma(M_{T_{n_k}-})} = \frac{1}{(\delta_t - \varepsilon)\gamma(M_t)}. \end{aligned}$$

The last point comes from the fact that M is continuous at t . Letting ε tend to zero yields that $h_M(t) \leq (\delta_t \cdot \gamma(M_t))^{-1}$.

To finish the proof of Theorem 19, it is clear from (4.5) and (4.4) that, for all $t \in [0, 1] \setminus \mathcal{J}$, $h_M(t) \leq 1/\gamma(M_t)$. Finally, $h_M(t) = 0$ for all $t \in \mathcal{J}$, since M jumps at t . \square

5. Computation of the spectrum: A localized ubiquity theorem. We are now in a position to prove Theorem 7. Recall that the Poisson measure $N = \sum_n \delta_{(T_n, Z_n)}$ has been introduced in Remark 12, that the process M has been built in Proposition 13, and that δ_t has been introduced in Definition 18.

We will use here the *localized Jarnik–Besicovitch theorem* of [7], that we explain now. We introduced in (4.2) and (4.4) the approximation rate of any real number $t \in [0, 1]$ by our Poisson point process. In fact, such an approximation rate can be defined for any system of points.

DEFINITION 21. (i) A *system of points* $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ is a $[0, 1] \times (0, \infty)$ -valued sequence such that l_n decreases to 0 when n tends to infinity.

(ii) $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ is said to satisfy the covering property if

$$(5.1) \quad \bigcap_{p \geq 1} \bigcup_{n \geq p} B(t_n, l_n) \supset [0, 1].$$

(iii) For $t \in [0, 1]$, the approximation rate of t by \mathcal{S} is defined as

$$(5.2) \quad \delta_t = \sup\{\delta \geq 1 : t \text{ belongs to an infinite number of balls } B(t_n, l_n^\delta)\}.$$

Set $\lambda_n := (1 + Z_n)^{-1}$. Then $\mathcal{P} = \{(T_n, \lambda_n)\}_{n \geq 1}$ is a system of points. Of course, formula (5.2) coincides with formula (4.4) when the system of points is \mathcal{P} . This system \mathcal{P} is a Poisson point process with intensity measure

$$(5.3) \quad \Lambda(s, \lambda) = \mathbf{1}_{[0,1] \times (0,1)}(s, \lambda) ds \frac{d\lambda}{\lambda^2}.$$

Let us state the result of [7], Theorem 1.3. The definitions of a *weakly redundant system* and the *condition (C)* are recalled in next section. There, the Poisson system \mathcal{P} is shown to enjoy these properties almost surely.

THEOREM 22. Consider a weakly redundant system \mathcal{S} satisfying the covering property (5.1) and condition (C). Let $I = (a, b) \subset [0, 1]$ and $f : I \rightarrow [1, +\infty)$ be continuous at every $t \in I \setminus \mathcal{Z}$, for some countable $\mathcal{Z} \subset [0, 1]$. Consider

$$S(I, f) = \{t \in I : \delta_t \geq f(t)\} \quad \text{and} \quad \tilde{S}(I, f) = \{t \in I : \delta_t = f(t)\}.$$

Then

$$\dim_H S(I, f) = \dim_H \tilde{S}(I, f) = \sup\{1/f(t) : t \in I \setminus \mathcal{Z}\}.$$

Observe that \mathcal{P} satisfies the covering property due to Proposition 17. We assume for a while that a.s. the Poisson system \mathcal{P} is weakly redundant and fulfills (C).

PROPOSITION 23. *Consider the process M built in Proposition 13. Almost surely:*

- (i) for all $I = (a, b) \subset [0, 1]$, all $\kappa \in (0, 1)$,

$$\dim_H\{t \in I : h_M(t) = \kappa/\gamma(M_t)\} = \dim_H\{t \in I : h_M(t) \leq \kappa/\gamma(M_t)\} = \kappa;$$
- (ii) for Lebesgue-almost every $t \in [0, 1]$, $h_M(t) = 1/\gamma(M_t)$.

PROOF. By Theorem 19 and since \mathcal{J} is countable, we observe that, for $I = (a, b) \subset [0, 1]$,

$$\dim_H\{t \in I : h_M(t) = \kappa/\gamma(M_t)\} = \dim_H\{t \in I : \delta_t = 1/\kappa\}.$$

Let $\kappa \in (0, 1)$. Since the Poisson system $\mathcal{P} = \{(T_n, \lambda_n)\}_{n \geq 1}$ satisfies all the required conditions, we may apply Theorem 22 with the constant function $f \equiv 1/\kappa$ and get $\dim_H\{t \in I : h_M(t) = \kappa/\gamma(M_t)\} = \kappa$. The same arguments hold for $\dim_H\{t \in I : h_M(t) \leq \kappa/\gamma(M_t)\}$, which concludes the proof of (i). Next, we write, for $I = (a, b) \subset [0, 1]$,

$$(5.4) \quad I = \{t \in I : h_M(t) = 1/\gamma(M(t))\} \cup \left(\bigcup_{n \geq 1} S_n \right),$$

where $S_n := \{t \in I : h_M(t) \leq (1 - 1/n)/\gamma(M(t))\}$. By (i), for every $n \geq 1$, the Lebesgue measure of the set S_n is zero since it is of Hausdorff dimension strictly less than 1. We deduce from (5.4) that, for Lebesgue-a.e. $t \in I$, $h_M(t) = 1/\gamma(M(t))$. Since this holds for any $I = (a, b) \subset [0, 1]$, the conclusion follows. \square

PROOF OF THEOREM 7. By Remark 8, it suffices to prove (2.5). Let $I = (a, b) \subset [0, 1]$. By Theorem 19, for all $h \geq 0$,

$$\begin{aligned} D_M(I, h) &= \dim_H\{t \in (a, b) : h_M(t) = h\} \\ &= \dim_H\{t \in (a, b) : \delta_t = (h \cdot \gamma(M_t))^{-1}\}. \end{aligned}$$

The jump times \mathcal{J} are countable and of exponents zero for M , so they do not interfere in the computation of Hausdorff dimensions.

For $s \in \mathcal{J} \cap (a, b)$ a jump time of M and $h \in (0, 1/\gamma(M_{s-}))$, consider the function f_s defined on the interval $I_s = (a, s)$ by $f_s(t) = (h \cdot \gamma(M_t))^{-1}$. This function is continuous on $I_s \setminus \mathcal{J}$, and satisfies, for every $t \in I_s$, $f_s(t) \geq (h \cdot \gamma(M_{s-}))^{-1} \geq 1$. Applying Theorem 22 to the Poisson system $\mathcal{P} = \{(T_n, \lambda_n)\}_{n \geq 1}$ (which satisfies all the required conditions), we obtain

$$\dim_H\{t \in I_s : \delta_t = (h \cdot \gamma(M_t))^{-1}\} = \sup\{h \cdot \gamma(M_t) : t \in I_s \setminus \mathcal{J}\} = h \cdot \gamma(M_{s-}),$$

since $\gamma(M_t)$ increases to $\gamma(M_{s-})$ as t increases to s . Hence, for every $s \in \mathcal{J} \cap (a, b)$, for every h such that $0 < h \leq 1/\gamma(M_{s-})$, we have

$$\dim_H\{E_M(h) \cap I_s\} = h \cdot \gamma(M_{s-}).$$

Furthermore, for $s \in \mathcal{J} \cap (a, b)$, when $h \geq 1/\gamma(M_{s-})$, we have $E_M(h) \cap [s, b] = \emptyset$. Indeed, by Theorem 19, for $t \geq s$, $h_M(t) \leq 1/\gamma(M_t) \leq 1/\gamma(M_s) < 1/\gamma(M_{s-})$.

Let now $h \geq 0$ be fixed. Then using the density of \mathcal{J} ,

$$E_M(h) \cap I = \left(\bigcup_{s \in \mathcal{J} \cap (a,b), \gamma(M_{s-}) < 1/h} (E_M(h) \cap (a, s)) \right) \cup \left(\bigcup_{s \in \mathcal{J} \cap (a,b), \gamma(M_{s-}) \geq 1/h} (E_M(h) \cap [s, b]) \right).$$

As noted previously, the second term of the right-hand side is empty. Thus, we get, since \mathcal{J} is countable,

$$\begin{aligned} D_M(I, h) &= \sup\{\dim_H(E_M(h) \cap I_s) : s \in \mathcal{J} \cap I, \gamma(M_{s-}) < 1/h\} \\ &= \sup\{h \cdot \gamma(M_{s-}) : s \in \mathcal{J} \cap I \text{ and } h \cdot \gamma(M_{s-}) < 1\}, \end{aligned}$$

which was our aim.

Observe that if $h \geq 1/\gamma(M_a)$, this formula gives $D_M(I, h) = -\infty$. \square

6. Study of the distribution of the Poisson point process. To conclude the proof, we only have to check that \mathcal{P} is a weakly redundant system satisfying (C). Recall that $\mathcal{P} = \{(T_n, \lambda_n)\}_{n \geq 1}$ is a Poisson point process with intensity measure (5.3).

6.1. *Weak redundancy and condition (C).* The weak redundancy property asserts that the balls $B(t_n, l_n)$ naturally associated with a system of points \mathcal{S} do not overlap excessively. The precise definition is the following.

DEFINITION 24. Let $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ be a system of points. For any integer $j \geq 0$, we set

$$\mathcal{T}_j = \{n : 2^{-(j+1)} < l_n \leq 2^{-j}\}.$$

Then \mathcal{S} is said to be *weakly redundant* if $t_n \neq t_{n'}$ for all $n \neq n'$ and if there exists a nondecreasing sequence of positive integers $(N_j)_{j \geq 0}$ such that:

- (i) $\lim_{j \rightarrow \infty} (\log_2 N_j) / j = 0$.
- (ii) for every $j \geq 1$, \mathcal{T}_j can be decomposed into N_j pairwise disjoint subsets (denoted $\mathcal{T}_{j,1}, \dots, \mathcal{T}_{j,N_j}$) such that, for each $1 \leq i \leq N_j$, the balls $B(t_n, l_n)$, $n \in \mathcal{T}_{j,i}$, are pairwise disjoint.

In other words, for every $x \in [0, 1]$, x cannot belong to more than N_j balls $B(t_n, l_n)$ with $2^{-j-1} < l_n \leq 2^j$.

As shown in [4], Proposition 6.2, \mathcal{P} is weakly redundant.

A weak redundant system does not necessarily satisfy the conclusion of Theorem 22. This is the reason why condition (C), which imposes finer properties on the distribution of the system \mathcal{S} , has to be introduced.

We denote by Φ the set of functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

- φ is increasing, continuous and $\varphi(0) = 0$,
- $r^{-\varphi(r)}$ is decreasing and tends to ∞ as r tends to 0,
- $r^{\alpha-\beta\varphi(r)}$ is increasing near 0 for all $\alpha, \beta > 0$.

For example, the function $\varphi(r) = \frac{\log|\log(r)|}{|\log r|}$, defined for $r \geq 0$ close enough to 0, has the required behavior around 0.

DEFINITION 25. Suppose that a system of points $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ is weakly redundant and adopt the notation of Definition 24. For every $\varphi \in \Phi$ and for any $j \geq 1$, we define

$$\psi(j, \varphi) = \max\{m \in \mathbb{N} : N_m \cdot 2^m \leq 2^{j(1-\varphi(2^{-j}))}\}.$$

Of course, the sequence of integers $(N_j)_{j \geq 1}$ is the one appearing in Definition 24 of the weak redundancy.

Obviously $\psi(j, \varphi) \leq j$, for every $\varphi, (N_j)$ and j .

For any dyadic interval $U = [k2^{-j}, (k+1)2^{-j}]$, we set $g(U) = j$, that is, $g(U)$ is the dyadic generation of U . We denote by \mathcal{G}_j the set of all dyadic intervals of generation j . Finally, we denote by $\mathcal{G}_* := \bigcup_{j \geq 0} \mathcal{G}_j$ the set of all dyadic intervals.

DEFINITION 26. Suppose that a system of points $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ is weakly redundant and adopt the notation of Definitions 24 and 25. Let $\varphi \in \Phi$. For every dyadic interval $V \in \mathcal{G}_j$ and every $\delta > 1$, the property $\mathcal{P}(V, \delta)$ is satisfied when there exists an integer $n \in \mathcal{T}_j$ such that $t_n \in V$ and

$$B(t_n, (l_n)^\delta) \cap \left\{ t_p : p \in \mathcal{T}_k, \text{ where } k \text{ satisfies } \psi(j, \varphi) \leq k \leq -\log_2(l_n)^\delta + 4 \right\} = \{t_n\}.$$

Let us try to give an intuition of the meaning of $\mathcal{P}(V, \delta)$ for a dyadic interval V . $\mathcal{P}(V, \delta)$ holds true when, except t_n , the family of points $(t_p)_{p \geq 1}$ ‘‘avoids’’ the ball $B(t_n, (l_n)^\delta)$ when p describes all the sets \mathcal{T}_k , for k ranging between $g(V)$ and $-\log_2(l_n)^\delta + 4$, that is, between the dyadic generations of $B(t_n, l_n)$ and $B(t_n, (l_n)^\delta)$.

This condition seems to be reasonable, maybe not for all dyadic intervals, but at least for a large number among them. Condition (C) is meant to ensure the validity of $\mathcal{P}(V, \delta)$ for a sufficient set of intervals V and approximation degrees δ .

DEFINITION 27. Suppose that a system of points $\mathcal{S} = \{(t_n, l_n)\}_{n \geq 1}$ is weakly redundant and adopt the notation of Definitions 24 and 25. The system \mathcal{S} is said to satisfy condition (C) if there exist:

- a function $\varphi \in \Phi$,
- a continuous function $\kappa : (1, +\infty) \rightarrow (0, 1]$,
- a dense subset Δ of $(1, \infty)$,

such that for every $\delta \in \Delta$, for every dyadic interval $U \in \mathcal{G}_*$, there are infinitely many integers $j \geq g(U)$ satisfying

$$\#Q(U, j, \delta) \geq \kappa(\delta) \cdot 2^{j-g(U)},$$

where

$$Q(U, j, \delta) = \{V \in \mathcal{G}_j : V \subset U \text{ and } \mathcal{P}(V, \delta) \text{ holds}\}.$$

6.2. *Proof of (C) for the Poisson process \mathcal{P} .* We only need to find a function $\varphi \in \Phi$ and a continuous function $\kappa : (1, +\infty) \rightarrow (0, 1]$ such that for every $\delta > 1$, with probability 1, for every $U \in \mathcal{G}_*$, there are infinitely many integers $j \geq g(U)$ satisfying $\#Q(U, j, \delta) \geq \kappa(\delta) \cdot 2^{j-g(U)}$. Then, for any countable and dense subset Δ of $(1, \infty)$, with probability 1, for every $\delta \in \Delta$, for every $U \in \mathcal{G}_*$, there are infinitely many integers $j \geq g(U)$ satisfying $\#Q(U, j, \delta) \geq \kappa(\delta) \cdot 2^{j-g(U)}$.

In fact, any $\varphi \in \Phi$ is suitable.

Let $\varphi \in \Phi$ and $\delta > 1$. For $U \in \mathcal{G}_*$ and $V \subset U$ such that $V \in \bigcup_{j>g(U)} \mathcal{G}_j$, let us introduce the event

$$\mathcal{A}(U, V, \delta) = \left\{ \begin{array}{l} \exists n \in \mathcal{T}_{g(V)} \text{ such that } T_n \in V \text{ and} \\ B(T_n, (\lambda_n)^\delta) \cap \left(\bigcup_{\psi(g(V), \varphi) \leq k \leq h(V)} \mathcal{T}_j \right) = \{T_n\} \end{array} \right\},$$

where $h(V) = [\delta(g(V) + 1)] + 4$. Recall that $n \in \mathcal{T}_{g(V)}$ means that $2^{-g(V)-1} < \lambda_n \leq 2^{-g(V)}$. By construction, the inclusion $\mathcal{A}(U, V, \delta) \subset \{\mathcal{P}(V, \delta) \text{ holds}\}$ is satisfied.

For every $j \geq 1$, let $\tilde{\mathcal{G}}_j = \{[2k \cdot 2^{-j}, (2k + 1) \cdot 2^{-j}] : 0 \leq 2k \leq 2^j - 1\}$. The restrictions of the Poisson point process to the strips $V \times (0, 1)$, where V describes $\tilde{\mathcal{G}}_j$, are independent. Moreover, the events $\mathcal{A}(U, V, \delta)$, when $V \in \tilde{\mathcal{G}}_j$ and $V \subset U$, are independent [we must separate the intervals in $\tilde{\mathcal{G}}_j$ because if $V \in \mathcal{G}_j$, $T_n \in V$ and $\lambda_n \leq 2^{-j}$, then $B(T_n, (\lambda_n)^\delta)$ may overlap with the neighbors of V].

We denote by $X(U, V, \delta)$ the random variable $\mathbf{1}_{\mathcal{A}(U, V, \delta)}$. For a given generation $j > g(U)$, the random variables $(X(U, V, \delta))_{V \in \tilde{\mathcal{G}}_j}$ are i.i.d. Bernoulli variables, whose common parameter is denoted by $p_j(\delta)$. The following lemma holds.

LEMMA 28. *There exists a continuous function $\kappa_1 : (1, +\infty) \rightarrow (0, 1)$ such that, for every $j \geq 1$, $p_j(\delta) \geq \kappa_1(\delta)$.*

Let us assume Lemma 28 for the moment. By definition, we have

$$\#Q(U, j, \delta) \geq \sum_{V \in \mathcal{G}_j: V \subset U} X(U, V, \delta).$$

The right-hand side of this inequality is a binomial variable of parameters $(2^{j-g(U)}, p_j(\delta))$, with $p_j(\delta) \geq \kappa_1(\delta) > 0$. Consequently, choosing $\kappa(\delta) = \kappa_1(\delta)/2 > 0$, we have

$$(6.1) \quad \mathbb{P}\left(\sum_{V \in \mathcal{G}_j, V \subset U} X(U, V, \delta) \geq \kappa(\delta) \cdot 2^{j-g(U)}\right) \geq 1/2,$$

provided that j is large enough (depending on δ). Indeed, for θ fixed in $(0, 1)$, if $X_n \sim \mathcal{B}(n, \theta)$ (a binomial distribution), one easily checks that $\mathbb{P}(X_n > n\theta/2)$ tends to 1 when n tends to infinity.

Let $(j_n)_{n \geq 1}$ be the sequence defined inductively by $j_1 = g(U) + 1$ and $j_{n+1} = (j_n + 1)\delta + 5$. We notice that the events E_n defined for $n \geq 1$ by

$$E_n = \{\#Q(U, j_n, \delta) \geq \kappa(\delta) \cdot 2^{j_n-g(U)}\}$$

are independent. Moreover, (6.1) implies that $\sum_{n \geq 1} \mathbb{P}(E_n) = +\infty$. The Borel–Cantelli Lemma yields that, with probability 1, there is an infinite number of generations j_n satisfying $\#Q(U, j_n, \delta) \geq \kappa(\delta) \cdot 2^{j_n-g(U)}$. This holds true for every $U \in \mathcal{G}_*$ almost surely, hence almost surely for every $U \in \mathcal{G}_*$. Condition (C) is proved.

We prove Lemma 28. For every $V \in \mathcal{G}_*$, let us introduce the sets

$$S_V = V \times [2^{-(g(V)+1)}, 2^{-g(V)}] \quad \text{and} \quad \tilde{S}_V = V \times [2^{-h(V)}, 2^{-\psi(g(V), \varphi)}].$$

We denote by N_V and \tilde{N}_V , respectively, the cardinality of $\mathcal{P} \cap S_V$ and $\mathcal{P} \cap (\tilde{S}_V \setminus S_V)$. These random variables N_V and \tilde{N}_V are independent. We set $l_V = \Lambda(S_V)$ and $\tilde{l}_V = \Lambda(\tilde{S}_V)$ [Λ is the intensity of the Poisson point process (5.3)]. Due to the form of the intensity Λ , N_V and \tilde{N}_V are Poisson random variables of parameter $l_V = 1$ and $\tilde{l}_V = 2^{-g(V)}(2^{h(V)} - 2^{g(V)+1} + 2^{g(V)} - 2^{\psi(g(V), \varphi)})$, respectively. Observe that $\tilde{l}_V \leq 2^{h(V)-g(V)}$ since, by definition, $\psi(g(V), \varphi) \leq g(V)$.

We also consider two sequences of random variables in \mathbb{R}^2 ($\xi_p = (X_p, Y_p)_{p \geq 1}$ and $(\tilde{\xi}_q = (\tilde{X}_q, \tilde{Y}_q)_{q \geq 1}$) such that

$$\begin{aligned} \mathcal{P} \cap S_V &= \{\xi_p : 1 \leq p \leq N_V\}, \\ \mathcal{P} \cap (\tilde{S}_V \setminus S_V) &= \{\tilde{\xi}_q : 1 \leq q \leq \tilde{N}_V\}. \end{aligned}$$

The event $\mathcal{A}(U, V, \delta)$ contains the event $\tilde{\mathcal{A}}(U, V, \delta)$ defined as

$$\{N_V = 1 \text{ and } B(X_1, Y_1^\delta) \cap \{\tilde{X}_q : 1 \leq q \leq \tilde{N}_V\} = \{X_1\}\},$$

where $\xi_1 = (X_1, Y_1)$. The difference between $\mathcal{A}(U, V, \delta)$ and $\tilde{\mathcal{A}}(U, V, \delta)$ is that the latter one imposes that there is one, and only one, Poisson point in S_V . We

have

$$\begin{aligned} \mathbb{P}(\tilde{\mathcal{A}}(U, V, \delta)) &= \mathbb{P}(\{B(X_1, Y_1^\delta) \cap \{\tilde{X}_q : 1 \leq q \leq \tilde{N}_V\} = \emptyset\} | \{N_V = 1\}) \\ &\quad \times \mathbb{P}(\{N_V = 1\}) \\ &= \mathbb{P}(\{\forall 1 \leq q \leq \tilde{N}_V, \tilde{X}_q \notin B(X_1, Y_1^\delta)\} | \{N_V = 1\}) \times e^{-1}, \end{aligned}$$

where $\mathbb{P}(\{N_V = 1\}) = e^{-1}$ since N_V is a Poisson random variable of parameter 1. The random variables \tilde{X}_q are independent and uniformly distributed in V . Thus,

$$\begin{aligned} \mathbb{P}(\{\forall 1 \leq q \leq \tilde{N}_V, \tilde{X}_q \notin B(X_1, Y_1^\delta)\} | \{N_V = 1\}) \\ \geq \mathbb{E}\left(\left[1 - \frac{\ell(B(X_1, Y_1^\delta))}{2^{-g(V)}}\right]^{\tilde{N}_V}\right). \end{aligned}$$

Observe that, since $\delta > 1$, provided that $g(V)$ is large enough, conditionally on $\{N_V \geq 1\}$, $\ell(B(X_1, Y_1^\delta)) \leq 2^{-g(V)\delta}$. This implies that

$$(6.2) \quad \mathbb{P}(\tilde{\mathcal{A}}(U, V, \delta)) \geq e^{-1} \times \mathbb{E}([1 - 2^{-g(V)(\delta-1)}]^{\tilde{N}_V}).$$

Let us define $\eta_{g(V)} = 2^{-g(V)(\delta-1)}$. Using that \tilde{N}_V is a Poisson random variable of parameter \tilde{l}_V , a classical calculus shows that (6.2) can be rewritten as

$$\mathbb{P}(\tilde{\mathcal{A}}(U, V, \delta)) \geq e^{-1} e^{-\tilde{l}_V \cdot \eta_{g(V)}}.$$

By using the definition of $h(V) = [(g(V) + 1)\delta] + 4$, we can get

$$\tilde{l}_V \cdot \eta_{g(V)} \leq 2^{h(V)-g(V)} 2^{-g(V)(\delta-1)} \leq 16 \cdot 2^\delta.$$

Thus, $\tilde{l}_V \eta_{g(V)}$ is bounded from above independently of V by a continuous function of δ . Consequently, $\mathbb{P}(\tilde{\mathcal{A}}(U, V, \delta))$ and, thus, $\mathbb{P}(\mathcal{A}(U, V, \delta))$, is bounded from below by some quantity $\kappa_1(\delta)$ which is strictly positive and continuously dependent on $\delta > 1$. Lemma 28 is proved.

7. Some tangent stable Lévy processes. Let $t_0 \geq 0$ be fixed. Then one observes (recall Theorems 5 and 7) that the local multifractal spectrum $D_M(t_0, \cdot)$ of our process M essentially coincides with the multifractal spectrum of a stable Lévy process with Lévy measure $\gamma(M_{t_0})u^{-1-\gamma(M_{t_0})} du$. A possible explanation for this is that such a stable Lévy process is *tangent* to our process. The notion of tangent processes was, for instance, studied in [13].

PROPOSITION 29. *Let $t_0 \geq 0$ be fixed. Conditionally on \mathcal{F}_{t_0} , the family of processes $(\frac{M_{t_0+\alpha t} - M_{t_0}}{\alpha^{1/\gamma(M_{t_0})}})_{t \in [0,1]}$ converges in law, as $\alpha \rightarrow 0^+$, to a stable Lévy subordinator with Lévy measure $\gamma(M_{t_0})u^{-1-\gamma(M_{t_0})} du$. Here the Skorokhod space of càdlàg functions on $[0, 1]$ is endowed with the uniform convergence topology (which is stronger than the Skorokhod topology).*

One might conjecture that under many restrictive conditions, a result of the following type might hold: if a process $(M_t)_{t \in [0,1]}$ has a tangent process $(Y_t^{t_0})_{t \in [0,1]}$ at time t_0 , then $D_M(t_0, \cdot)$ coincides with the multifractal spectrum of Y^{t_0} . This would allow, for example, to generalize our results to the study of the multifractal spectrum of any reasonable jumping SDE (which is always tangent, in some sense, to a Lévy process).

PROOF OF PROPOSITION 29. Using the Markov property, it suffices to treat the case $t_0 = 0$. Let thus $N(ds, dz)$ be a Poisson measure on $[0, 1] \times [0, \infty)$ with intensity measure $ds dz$. Recall that

$$M_t = \int_0^t \int_0^\infty (1+z)^{-1/\gamma(M_{s-})} N(ds, dz)$$

and introduce the Lévy processes

$$L_t = \int_0^t \int_0^\infty (1+z)^{-1/\gamma(0)} N(ds, dz), \quad \tilde{L}_t = \int_0^t \int_1^\infty z^{-1/\gamma(0)} N(ds, dz),$$

$$S_t = \int_0^t \int_0^\infty z^{-1/\gamma(0)} N(ds, dz).$$

One immediately checks that $(L_t)_{t \in [0,1]}$ and $(\tilde{L}_t)_{t \in [0,1]}$ have the same law, and that $(S_t)_{t \in [0,1]}$ is a stable Lévy process with Lévy measure $\gamma(0)u^{-1-\gamma(0)} du$. Thus, our aim is to prove that $(\alpha^{-1/\gamma(0)} M_{\alpha t})_{t \in [0,1]}$ tends in law to $(S_t)_{t \in [0,1]}$. First, $(\alpha^{-1/\gamma(0)} S_{\alpha t})_{t \in [0,1]}$ has the same law as $(S_t)_{t \in [0,1]}$ for each $\alpha > 0$. Next, it is clear that

$$\mathbb{P}[(\alpha^{-1/\gamma(0)} \tilde{L}_{\alpha t})_{t \in [0,1]} = (\alpha^{-1/\gamma(0)} S_{\alpha t})_{t \in [0,1]}] \geq \mathbb{P}[N([0, \alpha] \times [0, 1]) = 0] = e^{-\alpha},$$

which tends to 1 as α tends to 0.

We will now show that $\alpha^{-1/\gamma(0)} \Delta_\alpha$ tends to 0 in probability, where

$$\Delta_t := \sup_{[0,t]} |M_s - L_s| = \int_0^t \int_0^\infty [(1+z)^{-1/\gamma(M_{s-})} - (1+z)^{-1/\gamma(0)}] N(ds, dz),$$

and this will conclude the proof. A first computation, using that $\gamma(y) \leq 1 - \varepsilon < 1$ by assumption, shows that, for all $t \geq 0$,

$$\mathbb{E}[M_t] = \int_0^t \int_0^\infty \mathbb{E}[(1+z)^{-1/\gamma(M_s)}] dz ds \leq \int_0^t \int_0^\infty (1+z)^{-1/(1-\varepsilon)} dz ds \leq Ct$$

for some constant C . Next, we introduce, for $\eta > 0$, the stopping time $\tau_\eta = \inf\{t \geq 0 : \gamma(M_t) > \gamma(0) + \eta\}$. Denoting by A the Lipschitz constant of γ , one easily gets

$$\mathbb{P}[\tau_\eta < \alpha] \leq \mathbb{P}[M_\alpha \geq \eta/A] \leq (A/\eta)\mathbb{E}[M_\alpha] \leq CA\alpha/\eta = C_\eta\alpha.$$

Now for $\beta \in (\gamma(0) + \eta, 1]$, by subadditivity, one obtains, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}[M_{t \wedge \tau_\eta}^\beta] &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_0^\infty (1+z)^{-\beta/\gamma(M_{s^-})} N(ds, dz)\right] \\ &= \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_0^\infty (1+z)^{-\beta/\gamma(M_s)} dz ds\right] \\ &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_0^\infty (1+z)^{-\beta/(\gamma(0)+\eta)} dz ds\right] \leq C_{\eta, \beta} t. \end{aligned}$$

Let us introduce for $m \geq 0$ the quantity $\kappa(m) := 1/\gamma(0) - 1/\gamma(m) \geq 0$. Still by subadditivity, for $\beta \in (\gamma(0) + \eta, 1]$, for $t \geq 0$, we have

$$\begin{aligned} \mathbb{E}[\Delta_{t \wedge \tau_\eta}^\beta] &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_0^\infty [(1+z)^{-1/\gamma(M_s)} - (1+z)^{-1/\gamma(0)}]^\beta dz ds\right] \\ &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_0^{2^{1/\kappa(M_s)}-1} (1+z)^{-\beta/\gamma(0)} [(1+z)^{\kappa(M_s)} - 1]^\beta dz ds\right] \\ &\quad + \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \int_{2^{1/\kappa(M_s)}-1}^\infty (1+z)^{-\beta/\gamma(M_s)} dz ds\right] =: I_t^{\beta, \eta} + J_t^{\beta, \eta}. \end{aligned}$$

But for $z \leq 2^{1/\kappa(m)} - 1$, there holds $(1+z)^{\kappa(m)} - 1 \leq C\kappa(m) \log(1+z) \leq Cm \log(1+z)$, the last inequality following from the facts that $\kappa(0) = 0$ and κ is Lipschitz-continuous. Hence,

$$\begin{aligned} I_t^{\beta, \eta} &\leq \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} C M_s^\beta \int_0^\infty (1+z)^{-\beta/\gamma(0)} (\log(1+z))^\beta dz ds\right] \\ &\leq C_\beta \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} M_s^\beta ds\right] \leq C_\beta \int_0^t \mathbb{E}[M_{s \wedge \tau_\eta}^\beta] ds \leq C_{\eta, \beta} t^2. \end{aligned}$$

Next, $\beta/\gamma(M_s) \geq \beta/(\gamma(0) + \eta) > 1$ on $[0, \tau_\eta)$, whence (since $2^{-ax} \leq C_a x$ for all $x > 0$),

$$\begin{aligned} J_t^{\beta, \eta} &\leq C_{\eta, \beta} \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} 2^{[1-\beta/(\gamma(0)+\eta)]/\kappa(M_s)} ds\right] \leq C_{\eta, \beta} \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} \kappa(M_s) ds\right] \\ &\leq C_{\eta, \beta} \mathbb{E}\left[\int_0^{t \wedge \tau_\eta} M_s ds\right] \leq C_{\eta, \beta} \int_0^t \mathbb{E}[M_{s \wedge \tau_\eta}] ds \leq C_{\eta, \beta} t^2. \end{aligned}$$

Again, we used here that $\kappa(0) = 0$ and that κ is Lipschitz-continuous. As a conclusion, $\mathbb{E}[\Delta_{t \wedge \tau_\eta}^\beta] \leq C_{\eta, \beta} t^2$.

We may now conclude that, for all $\delta > 0$, all $\alpha > 0$, all $\eta > 0$, all $\beta \in (\gamma(0) + \eta, 1]$,

$$\begin{aligned} \mathbb{P}[\alpha^{-1/\gamma(0)} \Delta_\alpha \geq \delta] &\leq \mathbb{P}[\tau_\eta \geq \alpha] + \mathbb{P}[\Delta_{\alpha \wedge \tau_\eta} \geq \delta \alpha^{1/\gamma(0)}] \\ &\leq C_\eta \alpha + (\delta \alpha^{1/\gamma(0)})^{-\beta} E[\Delta_{\alpha \wedge \tau_\eta}^\beta] \leq C_\eta \alpha + C_{\eta, \delta, \beta} \alpha^{2-\beta/\gamma(0)}. \end{aligned}$$

Choosing $\eta = \min(\gamma(0), 1 - \gamma(0))/2$ and then $\beta \in (\gamma(0) + \eta, 1 \wedge 2\gamma(0))$, we deduce that $2 - \beta/\gamma(0) > 0$, so that $\alpha^{-1/\gamma(0)} \Delta_{\alpha T}$ tends to 0 in probability when α goes to 0, which was our aim. \square

REFERENCES

- [1] AYACHE, A., JAFFARD, S. and TAQQU, M. S. (2007). Wavelet construction of generalized multifractional processes. *Rev. Mat. Iberoamericana* **23** 327–370. [MR2351137](#)
- [2] BARRAL, J. (2000). Continuity of the multifractal spectrum of a random statistically self-similar measure. *J. Theoret. Probab.* **13** 1027–1060. [MR1820501](#)
- [3] BARRAL, J. and MANDELBROT, B. B. (2004). Random multiplicative multifractal measures. In *Fractal Geometry and Applications. Proc. Symp. Pure Math.* AMS, Providence, RI.
- [4] BARRAL, J. and SEURET, S. (2007). Continuity of the multifractal spectrum of a random statistically self-similar measure. *Bull. Braz. Math. Soc.* **38** 467–515.
- [5] BARRAL, J. and SEURET, S. (2007). The singularity spectrum of Lévy processes in multifractal time. *Adv. Math.* **214** 437–468. [MR2348038](#)
- [6] BARRAL, J. and SEURET, S. (2008). The multifractal nature of heterogeneous sums of Dirac masses. *Math. Proc. Cambridge Philos. Soc.* **144** 707–727. [MR2418713](#)
- [7] BARRAL, J. and SEURET, S. (2008). A localized Jarník–Besicovitch theorem. Preprint. Available at [arXiv:0903.2215v1](#).
- [8] BERTOIN, J. (1994). On nowhere differentiability for Lévy processes. *Stochastics Stochastics Rep.* **50** 205–210. [MR1786116](#)
- [9] BROWN, G., MICHON, G. and PEYRIÈRE, J. (1992). On the multifractal analysis of measures. *J. Statist. Phys.* **66** 775–790. [MR1151978](#)
- [10] DURAND, A. (2008). Random wavelet series based on a tree-indexed Markov chain. *Comm. Math. Phys.* **283** 451–477. [MR2430640](#)
- [11] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes. Characterization and Convergence*. Wiley, New York. [MR838085](#)
- [12] FALCONER, K. (2003). *Fractal Geometry. Mathematical Foundations and Applications*, 2nd ed. Wiley, Hoboken, NJ. [MR2118797](#)
- [13] FALCONER, K. J. (2003). The local structure of random processes. *J. London Math. Soc.* **67** 657–672.
- [14] FRISCH, U. and PARISI, G. (1985). Fully developed turbulence and intermittency. In *Proc. International Summer School Phys., Enrico Fermi* 84–88. North Holland, Amsterdam.
- [15] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. North-Holland Mathematical Library **24**. North-Holland, Amsterdam. [MR637061](#)
- [16] JAFFARD, S. (1997). Old friends revisited: The multifractal nature of some classical functions. *J. Fourier Anal. Appl.* **3** 1–22. [MR1428813](#)
- [17] JAFFARD, S. (1999). The multifractal nature of Lévy processes. *Probab. Theory Related Fields* **114** 207–227. [MR1701520](#)
- [18] JAFFARD, S. (2000). On lacunary wavelet series. *Ann. Appl. Probab.* **10** 313–329. [MR1765214](#)
- [19] OREY, S. and TAYLOR, S. J. (1974). How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc.* (3) **28** 174–192. [MR0359031](#)
- [20] PERKINS, E. (1983). On the Hausdorff dimension of the Brownian slow points. *Z. Wahrsch. Verw. Gebiete* **64** 369–399. [MR716493](#)
- [21] SHEPP, L. A. (1972). Covering the line with random intervals. *Z. Wahrsch. Verw. Gebiete* **23** 163–170. [MR0322923](#)

- [22] XIAO, Y. (2004). Random fractals and Markov processes. In *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Part 2. Proc. Sympos. Pure Math.* **72** 261–338. Amer. Math. Soc., Providence, RI. [MR2112126](#)

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