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A PUTNAM AREA INEQUALITY FOR THE SPECTRUM OF n-TUPLES OF p-HYPONORMAL OPERATORS

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Abstract. We prove an n-tuple analogue of the Putnam area inequality for the spectrum of a single p-hyponormal operator.

Let $B(H)$ denote the algebra of operators (i.e. bounded linear transformations) on a separable Hilbert space H. The operator $A \in B(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $|A^*|^{2p} \leq |A|^{2p}$. Let $\mathcal{H}(p)$ denote the class of p-hyponormal operators. Then $\mathcal{H}(1)$ consists of the class of p-hyponormal operators and $\mathcal{H}(\frac{1}{2})$ consists of the class of semi-hyponormal operators introduced by D. Xia. (See [**11**, p. 238] for the appropriate reference.) $\mathcal{H}(p)$ operators for a general p with $0 < p < 1$ have been studied by a number of authors in the recent past; (see [**3**, **4**, **5**] for further references). Generally speaking, $\mathcal{H}(p)$ operators $(0 < p < 1)$ have spectral properties very similar to those of hyponormal operators. In particular, a Putnam inequality relating the norm of the commutator $D_p = |A|^{2p} - |A^*|^{2p}$ of $A \in \mathcal{H}(p)$ to the area of the spectrum $\sigma(A)$ of A holds; indeed

$$||D_p|| \leq \frac{p}{\pi} \int_{\sigma(A)} r^{2p-1} \mathrm{d}r \, \mathrm{d}\theta \tag{1}$$

(See [**4**, Theorem 3]; also see [**7**,**8**] for the case $p = 1$.)

Let $\mathcal{U} = (U_1, U_2, \ldots U_n)$ be a commuting n-tuple of unitaries, and let E$(\cdot)$ denote the spectral measure of $\mathcal{U}$. Let $\partial\mathbb{D}$ denote the boundary of the unit disc in the complex plane $\mathbb{C}$, and let $\Gamma(\mathbf{z})$, $\mathbf{z} = (z_1, z_2 \ldots z_n) \in \sigma(\mathcal{U})$ the Taylor joint spectrum of $\mathcal{U}$. Denote the set of (all) products $\Delta = \delta_1 \times \delta_2 \times \ldots \times \delta_n$ of open arcs $\delta_i \in \partial\mathbb{D}$ containing z_i $(i = 1, 2, \ldots, n)$. Let $\mathcal{A} \in B(H)$. The Xia spectrum of the non-commuting $(n+1)$-tuple $(\mathcal{U}, \mathcal{A})$, denoted $\sigma_x(\mathcal{U}, \mathcal{A})$, is defined to be the set

$$\{(\mathbf{z}, r) : \mathbf{z} \in \sigma(\mathcal{U}), r \in \bigcap_{\Delta \in \Gamma(\mathbf{z})} \sigma(E(\Delta)\mathcal{A}E(\Delta))\}.$$

(see [**10**]). The concept of Xia spectrum has proved to be a very useful one: it has been used by Xia [**10**] to study the spectra of semi-hyponormal n-tuples, by Chen and Huang [**2**] to describe the Taylor spectrum of (and prove a Putnam area inequality for) n-tuples of hyponormal operators, and (recently) by Chō and Huruya [**3**] in their consideration of p-hyponormal tuples. Let $Q_i : B(H) \to B(H)$ be the operator $Q_i L = L - \mathcal{U}_i L \mathcal{U}_i^*$; $\mathcal{U}_i s$, as above. Let $\mathcal{A} \geq 0$. Then $(\mathcal{U}, \mathcal{A})$ is said to be a *p-hyponormal tuple* if $Q_{i_1} Q_{i_2} \ldots Q_{i_k} \mathcal{A}^{2p} \geq 0$, for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

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Extending Xia's result on semi-hyponormal tuples [**10**], Chō and Huruya [**3**] have shown that if $(\mathcal{U}, \mathcal{A})$ is a p-hyponormal tuple, then

$$||Q_1Q_2\ldots Q_n\mathcal{A}^{2p}|| \leq \frac{2p}{(2\pi)^n} \int_{\sigma_x(\mathcal{U},\mathcal{A})} \cdots \int r^{2p-1}\, \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \mathrm{d}r. \tag{2}$$

In this note we prove an analogue of inequality (1) for n-tuples of doubly commuting $\mathcal{HU}(p)$ operators (notation as below).

It is an immediate consequence of the Löwner inequality [**11**, p. 5] that an $\mathcal{H}(p)$ operator is an $\mathcal{H}(q)$ operator, for all $0 < q \leq p$; hence we may assume that $0 < p < \frac{1}{2}$. If an $A \in \mathcal{H}(p)$, $0 < p < \frac{1}{2}$, has equal defect and nullity, then the partial isometry U in the polar decomposition $A = U|A|$ may be taken to be a unitary. Let $\mathcal{HU}(p)$ denote those $A \in \mathcal{H}(p)$ for which the partial isometry U (in $A = U|A|$) is unitary. Given an $A_i \in \mathcal{HU}(p)$, $A_i = U_i|A_i|$, define $\hat{A}_i = V_i|\hat{A}_i|$ and $\tilde{A}_i = W_i|\tilde{A}_i|$ by $\hat{A}_i = |A_i|^{\frac{1}{2}}U_i|A_i|^{\frac{1}{2}}$ and $\tilde{A}_i = |\hat{A}_i|^{\frac{1}{2}}V_i|\hat{A}_i|^{\frac{1}{2}}$; $\hat{A}_i$ then $\in \mathcal{HU}(p+\frac{1}{2})$ and $\tilde{A}_i \in \mathcal{HU}(1)$. Let $\mathbb{A}$ denote the n-tuple $\mathbb{A} = (A_1, A_2, \ldots, A_n)$, $A_i \in \mathcal{HU}(p)$ for all $1 \leq i \leq n$, and let $\breve{\mathbb{A}}$ denote the n-tuple $\breve{\mathbb{A}} = (U_1V_1|\tilde{A}_i|^p, \ldots, U_nV_n|\tilde{A}_n|^p)$. Define the commutators D_{pi}, $\tilde{D}_{pi}$, $\tilde{D}_i$ and $\breve{D}_{pi}$ as follows:

$$D_{pi} = |A_i|^{2p} - |A_i^*|^{2p} (\geq 0),\ \tilde{D}_{pi} = |\tilde{A}_i|^{2p} - |\tilde{A}_i^*|^{2p} (\geq 0),$$
$$\tilde{D}_i = |\tilde{A}_i|^2 - |\tilde{A}_i^*|^2 (\geq 0) \text{ and } \breve{D}_{pi} = |\tilde{A}_i|^{2p} - U_iV_i|\tilde{A}_i^*|^{2p}V_i^*U_i^*.$$

Let

$$D_p = \prod_{i=1}^n D_{pi}, \quad \tilde{D} = \prod_{i=1}^n \tilde{D}_i \quad \text{and} \quad \breve{D}_p = \prod_{i=1}^n \breve{D}_{pi}.$$

Let $\mathrm{d}\nu$ denote the Lebesgue volume measure in $\mathbb{C}^n$, let m denote the (normalized) Haar measure on $\partial\mathbb{D}$ and let μ denote the linear Lebesgue measure. For a given $A_i \in \mathcal{HU}(p)$, let P_i denote the pure part (=completely non-normal part) of A_i. We prove the following result.

THEOREM. *If $\mathbb{A}$ is doubly commuting, then*

$$||D_p|| \leq \min\left\{ \frac{2p}{(2\pi)^n} \int_{\sigma_x(\mathcal{U},\mathcal{A}_n)} \cdots \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \mathrm{d}r,\ \frac{1}{\pi^n} \int_{\sigma(\breve{\mathbb{A}})} \int \mathrm{d}\nu \right\}, \tag{3}$$

where $\mathcal{A}_n = \prod_{i=1}^n |\tilde{A}_i|$ and $\mathcal{U}$ is as defined in Lemma 3 (below). If also either

(i) $m(\sigma(U_iV_i)) = 0$ *or (ii)* $\mu(\sigma|P_i|) = 0$, *for all* $1 \leq i \leq n$, *then*

$$||D_p|| \leq \frac{1}{\pi^{np}} \left(\int\limits_{\sigma(\mathbb{A})} \int d\nu \right)^p. \tag{4}$$

REMARK. The hypothesis that $\mu(\sigma(|P_i|)) = 0$ implies that there exists a finite or countably infinite number of pairwise disjoint annuli $0_n = \{\lambda : a_n < |\lambda| < b_n\}$, $n = 1, 2, \ldots,$ such that $\sigma(P_i) = U\, 0_n$ (see [**9**, Theorem 9]).

The proof of the theorem proceeds through a number of steps, stated below as lemmas.

LEMMA 1. *$0 \leq D_{pi} \leq \breve{D}_{pi}$, for all $1 \leq i \leq n$.*

Proof. Let $E_i^{1/2p} = U_i^*|A_i|^{2p}U_i$, $F_i = |A_i|^{2p}$ and $G_i = U_i|A_i|^{2p}U_i^*$; then, since $A_i \in \mathcal{H}U(p)$, $E_i = U_i^*|A_i|^{2p}U_i \geq F_i \geq G_i$. It follows from an application of the Furuta inequality [**6**] that

$$|A_i^*|^{2\left(p+\frac{1}{2}\right)} \leq |A_i|^{2\left(p+\frac{1}{2}\right)} \leq |\hat{A}_i|^{2\left(p+\frac{1}{2}\right)}.$$

The operator A_i being $\mathcal{H}U(p+\frac{1}{2})$ is $\mathcal{H}U(\frac{1}{2})$, and so $V_i|A_i|V_i^* \leq |\hat{A}_i| \leq V_i^*|\hat{A}_i|V_i$. This (together with an additional argument, similar to the one above) implies that

$$|\tilde{A}_i^*|^{2p} \leq |\hat{A}_i|^{2p} \leq |\tilde{A}_i|^{2p}.$$

Hence

$$\begin{aligned}
\breve{D}_{pi} &= |\tilde{A}_i|^{2p} - U_iV_i|\tilde{A}_i^*|^{2p}V_i^*U_i^* \\
&\geq |\hat{A}_i|^{2p} - U_iV_i|\hat{A}_i|^{2p}V_i^*U_i^* = |\hat{A}_i|^{2p} - U_i|\hat{A}_i^*|^{2p}U_i^* \\
&\geq |A_i|^{2p} - U_i|A_i|^{2p}U_i^* \\
&= |A_i|^{2p} - |A_i^*|^{2p} = D_{pi} \geq 0.
\end{aligned}$$

Given $A, B, \in B(H)$, let $[A, B] = AB - BA$. Recall that the n-tuple A is said to be *doubly commuting* if

$$[A_i, A_j] = 0 = [A_i, A_j^*],$$

for all $1 \leq i \neq j \leq n$.

LEMMA 2. *If $\mathbb{A}$ is doubly commuting, then $[D_{pi}, D_{pj}] = 0 = [\breve{D}_{pi}, \breve{D}_{pj}]$, for all $1 \leq i$, $j \leq n$, and $0 \leq D_p \leq \breve{D}_p$.*

Proof. The doubly commuting property of $\mathbb{A}$ implies that $[S_i, T_j] = 0$, for all $1 \leq i \neq j \leq n$, where S_i is either of $U_i, V_i, W_i, |A_i|, |\hat{A}_i|$ and $|\tilde{A}_i|$, and (similarly) T_j is either $U_j, V_j, W_j, |A_j|, |\hat{A}_j|$ and $|\tilde{A}_j|$. (See [**5**, Lemma 1].) This implies that

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$[D_{pi}, D_{pj}] = 0 = [\breve{D}_{pi}, \breve{D}_{pj}]$, for all $1 \le i, j \le n$. The commutativity of D_{pi} with D_{pj} taken together with the positivity of D_{pi}, for all $1 \le i \le n$, implies that $D_p \ge 0$. Since $\breve{D}_{pi} \ge D_{pi}$, for all i, and $\breve{D}_{pi}$ commutes with $\breve{D}_{pj}$, for all $1 \le i \ne j \le n$, $\breve{D}_p \ge D_p$.

Let $\mathbb{A}$ be doubly commuting; let $\mathcal{A}_n$ and $U_i\,(1 \le i \le n)$ be the operators $\mathcal{A}_n = \prod_{i=1}^{n} |\tilde{A}_i|$ and $\mathcal{U}_i =$ sum of the $\binom{n}{i}$ combinations of $U_1V_1W_1,\ U_2V_2W_2,\ \ldots,$ $U_nV_nW_n$ taken i at a time. Let

$$\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n);$$

then the n-tuple $\mathcal{U}$ consists of mutually commuting unitaries and the Xia spectrum $\sigma_x(\mathcal{U}, \mathcal{A}_n)$ is well defined.

Lemma 3. *If $\mathbb{A}$ is doubly commuting, then*

$$||D_p|| \le \frac{2p}{(2\pi)^n} \int\limits_{\sigma_x(\mathcal{U},\mathcal{A}_n)} \cdots \int r^{2p-1} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \mathrm{d}r.$$

Proof. Let $Q_i : B(H) \to B(H)$, $1 \le i \le n$, be defined as before. A straight forward computation (using the commutativity relations $[S_i, T_j]$ of Lemma 2) shows that

$$\begin{aligned} 0 \le \prod_{j=1}^{k} D_{pi_j} &\le \prod_{j=1}^{k} \breve{D}_{pi_j} \\ &= Q_{i_1} Q_{i_2} \ldots Q_{i_k} \left(\prod_{j=1}^{k} |\tilde{A}_{i_j}| \right)^{2p} \\ &= Q_{i_1} Q_{i_2} \ldots Q_{i_k} \mathcal{A}_k^{2p}, \end{aligned}$$

for all $1 \le i_1 < i_2 < \ldots < i_k \le n$. Hence the $(n+1)$-tuple $(\mathcal{U}, \mathcal{A}_n)$ is p-hyponormal (equivalently, $(\mathcal{U}, \mathcal{A}_n^{2p})$ is semi-hyponormal). It follows from [**3**, Theorem 2] and [**10**, Theorem 5]) that

$$||D_p|| \le ||\breve{D}_p|| \le \frac{2p}{(2\pi)^n} \int\limits_{\sigma_x(\mathcal{U},\mathcal{A}_n)} \cdots \int r^{2p-1} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \mathrm{d}r.$$

Given an $A_i \in \mathcal{H}U(p)$, let $A_i = N_i \oplus P_i$ denote the direct sum decomposition of A_i into its normal and pure parts.

Lemma 4. *Given $A_i \in \mathcal{H}U(p)$, we have*

$$\begin{aligned} || \, |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*|| \\ \le \min\{|| \, |A_i|^{2p}||m(\sigma(U_iV_i)),\, \mu(\sigma(|A_i|^{2p}))\}. \end{aligned} \tag{5}$$

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Proof. Let $A_i = U_i|A_i| \in \mathcal{H}U(p)$; then

$$\begin{aligned}
|A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^* &= U_i\{U_1^*|A_i|^{2p}U_i - V_i|A_i|^{2p}V_i^*\}U_i^* \\
&\geq U_i\{|A_i|^{2p} - V_i|A_i|^{2p}V_i^*\}U_i^* \text{ (since } A_i \in \mathcal{H}U(p)) \\
&\geq U_i\{|A_i|^{2p} - V_i|\hat{A}_i|^{2p}V_i^*\}U_i^* \\
&\quad \text{(since } |A_i|^{2p} \leq |\hat{A}_i|^{2p} \text{ by Lemma 1)} \\
&= U_i\{|A_i|^{2p} - |\hat{A}_i^*|^{2p}\}U_i^* \\
&\geq 0 \text{ (since } |\hat{A}_i^*|^{2p} \leq |A_i|^{2p} \text{ - see Lemma 1).}
\end{aligned}$$

Clearly, $P_i \in \mathcal{H}U(p)$. Let P_i have the polar decomposition $P_i = u_i|P_i|$ and define (the pure $\mathcal{H}U(p+\frac{1}{2})$ operator) $\hat{P}_i = v_i|\hat{P}_i|$ by $\hat{P}_i = |P_i|^{\frac{1}{2}}u_i|P_i|^{\frac{1}{2}}$. Then $0 \leq |P_i|^{\frac{1}{2}} - u_iv_i|P_i|^{2p}v_i^*u_i^*$, u_i and v_i are unitaries; also

$$\begin{aligned}
|| \, |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*|| &= ||0 \oplus (|P_i|^{2p} - u_iv_i|P_i|^{2p}v_i^*u_i^*)|| \\
&= || \, |P_i|^{2p} - u_iv_i|P_i|^{2p}v_i^*u_i^*|| \\
&\leq \mu(\sigma(|P_i|^{2p})) \text{ (by [7, p. 143; Problem 5(b)])} \\
&\leq \mu(\sigma(|A_i|^{2p})).
\end{aligned}$$

Since $0 \leq |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*$, $|| \, |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*|| \leq || \, |A_i|^{2p}||m(\sigma(U_iV_i))$, (by [**7**, p. 143; Problem 5(a)]), the lemma is proved.

Proof of the Theorem. As seen in Lemmas 1 and 2, $0 \leq D_{pi} \leq \breve{D}_{pi}$ and $0 \leq D_p \leq \breve{D}_p$. Hence $||D_p|| \leq ||\breve{D}_p||$. Since the operator $U_iV_i|\tilde{A}_i|^p$ is hyponormal for all $1 \leq i \leq n$, $\breve{\mathbb{A}}$ is a doubly commuting n-tuple of hyponormal operators. Hence, by [**2**, Theorem 5], we have

$$||D_p|| \leq \frac{1}{\pi^n} \int_{\sigma(\breve{\mathbb{A}})} \cdots \int d\nu.$$

Combining this with inequality (5) we have inequality (3). We now prove inequality (4).

$$\text{Let } \prod_{i=1}^{n'} D_{pi} = D_{p1}D_{p2}\ldots D_{p(i-1)}D_{p(i+1)}\ldots D_{pn}, \quad \hat{D}_{pi} = U_iV_i|A_i|^{2p}V_i^*U_i^* - |A_i^*|^{2p}.$$

If either of the hypotheses (i) or (ii) of the statement of the theorem holds, then $|| \, |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*|| = 0$; (see the proof of Lemma 4). Suppose now that either (i) or (ii) holds. Then

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$$\begin{aligned}||D_p|| &= ||\left(\prod_{i=1}^{n'} D_{pi}\right)|A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^* + \hat{D}_{pi})|| \\ &\leq ||\left(\prod_{i=1}^{n'} D_{pi}\right)||\ ||\ |A_i|^{2p} - U_iV_i|A_i|^{2p}V_i^*U_i^*|| + ||\left(\prod_{i=1}^{n'} D_{pi}\right)\hat{D}_{pi}|| \\ &= ||\left(\prod_{i=1}^{n'} D_{pi}\right)\hat{D}_{pi}||,\end{aligned}$$

and so, by repeating the argument, it follows that

$$\begin{aligned}||D_p|| \leq \ldots \leq ||\prod_{i=1}^{n} \hat{D}_{pi}|| &\leq ||\prod_{i=1}^{n}\ U_iV_i(|\tilde{A}_i|^{2p} - |\tilde{A}_i^*|^{2p})V_i^*U_i^*|| \\ = ||\prod_{i=1}^{n} \tilde{D}_{pi}|| &= ||\left(\prod_{i=1}^{n'} \tilde{D}_{pi}\right)(|\tilde{A}_i|^{2p} - |\tilde{A}_i^*|^{2p})||.\end{aligned}$$

(See the proof of Lemma 1.)

Let $\prod_{i=1}^{n'} \tilde{D}_{pi} = \tilde{D}_{p1}\tilde{D}_{p2}\ldots\tilde{D}_{p(i-1)}\ldots\tilde{D}_{p(i+1)}\ldots\tilde{D}_{pn} = d$; then $d \geq 0$ and d commutes with $\tilde{A}_i$. Since $\tilde{A}_i$ is hyponormal, we have

$$\begin{aligned}||D_p|| &\leq ||(d^{\frac{1}{p}}|\tilde{A}_i|^2)^p - (d^{\frac{1}{p}}|\tilde{A}_i^*|^2)^p|| \\ &\leq ||\{d^{\frac{1}{p}}|\tilde{A}_i|^2 - d^{\frac{1}{p}}|\tilde{A}_i^*|^2\}^p||\ \text{(by [**1**, Theorem 1])} \\ &= ||d\,\tilde{D}_i^p||.\end{aligned}$$

Hence, by repeating the argument, we obtain

$$||D_p|| \leq ||d\,\tilde{D}_i^p|| \leq \ldots \leq ||\left(\prod_{i=1}^{n} \tilde{D}_i\right)^p|| \leq ||\prod_{i=1}^{n} \tilde{D}_i||^p,$$

(since $0 < p < \frac{1}{2}$). The n-tuple $\tilde{\mathbb{A}} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ is a doubly commuting n-tuple of hyponormal operators; applying [**2**, Theorem 5] we obtain

$$||D_p|| \leq ||\prod_{i=1}^{n} \tilde{D}_i||^p \leq \frac{1}{\pi^{np}}\left(\int_{\sigma(\tilde{\mathbb{A}})}\int d\nu\right)^p.$$

Recall that $\sigma(\tilde{\mathbb{A}}) = \sigma(\mathbb{A})$ by [**4**; Theorem 1]; hence

$$||D_p|| \leq \frac{1}{\pi^{np}}\left(\int_{\sigma(\tilde{\mathbb{A}})}\int d\nu\right)^p.$$

This completes the proof.

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