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ERRATUM

# Erratum to: A $\mathbb{Q}$-factorial complete toric variety is a quotient of a poly weighted space

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After the publication of [2], we realized that Proposition 3.1, in that paper, contains an error, whose consequences are rather pervasive along the whole section 3 and for some aspects of Examples 5.1 and 5.2. Here we give a complete account of needed corrections.

First of all, [2, Prop. 3.1] has to be replaced by the following:

**Proposition 3.1** *Let $X(\Sigma)$ be a $\mathbb{Q}$-factorial complete toric variety and $Y(\widehat{\Sigma})$ be its universal 1-covering. Let $\{D_\rho\}_{\rho\in\Sigma(1)}$ and $\{\widehat{D}_\rho\}_{\rho\in\widehat{\Sigma}(1)}$ be the standard bases of $\mathcal{W}_T(X)$ and $\mathcal{W}_T(Y)$, respectively, given by the torus orbit closures of the rays. Then*

$$D=\sum_{\rho\in\Sigma(1)} a_\rho D_\rho \in \mathcal{C}_T(X) \quad\Longrightarrow\quad \widehat{D}=\sum_{\rho\in\widehat{\Sigma}(1)} a_\rho \widehat{D}_\rho \in \mathcal{C}_T(Y).$$

*Therefore under the identification $\mathbb{Z}^{|\Sigma(1)|}\cong\mathcal{W}_T(X)\overset{\alpha}{\cong}\mathcal{W}_T(Y)\cong\mathbb{Z}^{|\widehat{\Sigma}(1)|}$ realized by the isomorphism $D_\rho\overset{\alpha}{\mapsto}\widehat{D}_\rho$,*

$$\mathcal{C}_T(X)\cong\alpha(\mathcal{C}_T(X))\leq\mathcal{C}_T(Y)\leq\mathcal{W}_T(Y)$$

*is a chain of subgroup inclusions. Moreover the induced morphism $\overline{\alpha}\colon \mathrm{Cl}(X)\to\mathrm{Cl}(Y)$ is injective when restricted to $Pic(X)$, realizing the following further chain of subgroup inclusions*

$$\mathrm{Pic}(X)\cong\overline{\alpha}(\mathrm{Pic}(X))\leq\mathrm{Pic}(Y)\leq\mathrm{Cl}(Y).$$

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*Proof* Let us fix a basis $\mathcal{B}$ of the $\mathbb{Z}$-module $M \cong \mathbb{Z}^n$ and let $V$ and $\widehat{V}$ be fan matrices representing the standard morphisms

$$div_X \colon M \cong \mathbb{Z}^n \xrightarrow{V^T} \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X), \quad div_Y \colon M \cong \mathbb{Z}^r \xrightarrow{\widehat{V}^T} \mathbb{Z}^{|\widehat{\Sigma}(1)|} \cong \mathcal{W}_T(Y)$$

Let $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ be such that $V = \beta\widehat{V}$ and so realizing an injective endomorphism of the $\mathbb{Z}$-module $M$. The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

$$\begin{aligned} \mathcal{I}^\Sigma &= \{I \subseteq \{1,\dots,n+r\} \colon \left\langle V^I \right\rangle \in \Sigma(n)\} \\ &= \{I \subseteq \{1,\dots,n+r\} \colon \left\langle \widehat{V}^I \right\rangle \in \widehat{\Sigma}(n)\} = \mathcal{I}^{\widehat{\Sigma}} \end{aligned} \tag{1}$$

by the construction of $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, given the choice of $\Sigma \in \mathcal{SF}(V)$. Notice that $\mathcal{I}^\Sigma$ describes the complements of those sets described by $\mathcal{I}_\Sigma$, as defined in [2, Rem. 2.4]. In particular, the Weil divisor $\sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X)$ is Cartier if and only if

$$\forall\, I \in \mathcal{I}^\Sigma \quad \exists\, \mathbf{m}_I \in M \colon \forall\, j \notin I \;\; \mathbf{v}_j^T \mathbf{m}_I = a_j, \tag{2}$$

where $\mathbf{v}_j$ is the $j$th column of $V$. Then $\alpha(\sum_{j=1}^{n+r} a_j D_j) = \sum_{j=1}^{n+r} a_j \widehat{D}_j$ is a Cartier divisor since

$$\forall\, I \in \mathcal{I}^\Sigma \quad \forall\, j \notin I \quad \widehat{\mathbf{v}}_j^T (\beta^T \mathbf{m}_I) = a_j$$

where $\widehat{\mathbf{v}}_j$ is the $j$th column of $\widehat{V}$.
The injectivity of $\overline{\alpha}$ follows from the well-known freeness of $\mathrm{Pic}(X)$. □

As a consequence, parts 1, 4, 5 of [2, Thm. 3.2] still hold, while parts 2, 3, 6, 7 have to be replaced by the following:

**Theorem 3.2** *Let $X = X(\Sigma)$ be a $n$-dimensional $\mathbb{Q}$-factorial complete toric variety of rank $r$ and $Y = Y(\widehat{\Sigma})$ be its universal* 1*-covering. Let $V$ be a reduced fan matrix of $X$, $Q = \mathcal{G}(V)$ a weight matrix of $X$ and $\widehat{V} = \mathcal{G}(Q)$ be a CF-matrix giving a fan matrix of $Y$.*

2. *Define $\mathcal{I}^\Sigma$ as in (1). For any $I \in \mathcal{I}^\Sigma$ let $E_I$ be the $r \times (n+r)$ matrix admitting as rows the standard basis vectors $e_i = (0,\dots,0,\underset{i}{1},0,\dots,0)$, for $i \in I$, representing the $i$th basis divisor $D_i \in \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Set $\widetilde{V}_I := \left(V^T \mid E_I^T\right) \in \mathbf{M}_{n+r}(\mathbb{Z})$. Then Cartier divisors give rise to the following maximal rank subgroup of $\mathcal{W}_T(X)$*

$$\mathcal{C}_T(X) \cong \bigcap_{I \in \mathcal{I}^\Sigma} \mathcal{L}_c\left(\widetilde{V}_I\right) \leq \mathbb{Z}^{|\Sigma(1)|} \cong \mathcal{W}_T(X)$$

   *and a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X)$ can be explicitly computed by applying the procedure described in* [1, § 1.2.3].
3. *Let $C_X \in \mathrm{GL}_{n+r}(\mathbb{Q}) \cap \mathbf{M}_{n+r}(\mathbb{Z})$ be a matrix whose rows give a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$, as obtained in the previous part* 2*. Identify $\mathrm{Cl}(X)$ with $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$ by item* (*c*) *of part* 4 *in* [2, Thm. 3.2] *and represent the morphism $d_X$ by $Q \oplus \Gamma$, according to parts* 1 *and* 5*. Let $A \in \mathrm{GL}_{n+r}(\mathbb{Z})$ be a matrix such that $A \cdot C_X \cdot Q^T$ is in* HNF*. Let $\mathbf{c}_1,\dots,\mathbf{c}_r$ be the first $r$ rows of the matrix $A \cdot C_X$ and for $i = 1,\dots r$ put $\mathbf{b}_i = Q \cdot \mathbf{c}_i^T + \Gamma \cdot \mathbf{c}_i^T$. Then $\mathbf{b}_1,\dots \mathbf{b}_r$ is a basis of the free group $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$.*
6. *Given the choice of $\widehat{V}$ and $V$ as in the previous parts* 4 *and* 5 *of* [2, Thm. 3.2], *consider*

$$U := \begin{pmatrix} {}^rU_Q \\ \widehat{V} \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Z})$$

$$\begin{aligned} W &\in \mathrm{GL}_{n+r}(\mathbb{Z})\colon W\cdot ({}^{n+r-s}U)^T = \mathrm{HNF}\left(({}^{n+r-s}U)^T\right)\\ G &:= {}_s\widehat{V}\cdot ({}_sW)^T \in \mathbf{M}_s(\mathbb{Z})\\ U_G &\in \mathrm{GL}_s(\mathbb{Z})\colon U_G\cdot G^T = \mathrm{HNF}(G^T). \end{aligned}$$

*Then a "torsion matrix" representing the "torsion part" of the morphism $d_X$, that is, $\tau_X\colon \mathcal{W}_T(X)\to \mathrm{Tors}(\mathrm{Cl}(X))$, is given by*

$$\varGamma = U_G\cdot {}_sW \mod \boldsymbol{\tau} \tag{3}$$

*where this notation means that the $(k,j)$-entry of $\varGamma$ is given by the class in $\mathbb{Z}/\tau_k\mathbb{Z}$ represented by the corresponding $(k,j)$-entry of ${}^sU_G\cdot {}_sW$, for every $1\le k\le s$, $1\le j\le n+r$.*

7. *Setting $\delta_\Sigma := \mathrm{lcm}\left(\det(Q_I)\colon I\in\mathcal{I}^\Sigma\right)$ then*

$$\delta_\Sigma\mathcal{W}_T(X)\subseteq\mathcal{C}_T(X)\quad and\quad \delta_\Sigma\mathcal{W}_T(Y)\subseteq\mathcal{C}_T(Y)$$

*and there are the following divisibility relations*

$$\delta_\Sigma \mid [\mathrm{Cl}(Y)\colon \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y)\colon\mathcal{C}_T(Y)] \mid [\mathrm{Cl}(X)\colon \mathrm{Pic}(X)] = [\mathcal{W}_T(X)\colon\mathcal{C}_T(X)].$$

*Proof* (2): Recalling relation (2) in the proof of Proposition 3.1, set

$$\forall\, I\in\mathcal{I}^\Sigma\quad \mathcal{P}^I = \left\{L=\sum_{j=1}^{n+r} a_jD_j\in\mathcal{W}_T(X)\mid \exists\,\mathbf{m}\in M\colon\forall\, j\notin I\ \mathbf{m}\cdot\mathbf{v}_j = a_j\right\}.$$

Then $\mathcal{P}^I$ contains $\mathrm{Im}(div_X\colon M\to\mathcal{W}_T(X)) = \mathcal{L}_c\left(V^T\right)$ and a $\mathbb{Z}$-basis of $\mathcal{P}^I$ is given by

$$\{D_j,\ j\in I\}\cup\left\{\sum_{k=1}^{n+r} v_{ik}D_k,\ i=1,\dots,n\right\},$$

where $\{v_{ik}\}$ is the $i$th entry of $\mathbf{v}_k$, so giving the rows of the matrix $\widetilde{V}_I$ defined in the statement.
(3): By definition

$$\mathrm{Pic}(X) = \mathrm{Im}\left(\mathcal{C}_T(X)\hookrightarrow\mathcal{W}_T(X)\stackrel{d_X}{\to}\mathrm{Cl}(X)\right)$$

so that $\mathrm{Pic}(X)$ is generated by the image under $Q\oplus\varGamma$ of the transposed of the rows of $C_X$. Since $\mathrm{rk}(C_X)=n+r$ and $\mathrm{rk}(Q)=r$, the matrix $C_X\cdot Q^T$ has rank $r$ and therefore its HNF has the last $n-r$ rows equal to zero. Therefore the rows of the matrix $A\cdot C_X$ provide a basis of $\mathcal{C}_T(X)$ in $\mathcal{W}_T(X)$ such that its last $n$ rows are a basis of $\mathcal{L}_r(\widehat{V})\cap\mathcal{C}_T(X)=\mathcal{L}_r(V)$. Since $\mathrm{Pic}(X)$ is free of rank $r$, it is freely generated by the images under $d_X$ of the first $r$ rows.
(6): A representative matrix of the torsion part $\tau_X\colon\mathcal{W}_T(X)\to\mathrm{Cl}(X)$ of the morphism $d_X$ is any matrix satisfying the following properties:

(i) $\varGamma=(\gamma_{kj})$ with $\gamma_{kj}\in\mathbb{Z}/\tau_k\mathbb{Z}$,
(ii) $\varGamma\cdot({}^rU_Q)^T = \mathbf{0}_{s,r} \mod \boldsymbol{\tau}$, meaning that $\varGamma$ kills the generators of the free part $F\le \mathrm{Cl}(X)$ defined in display (4) of part 1 of [2, Thm. 3.2],
(iii) $\varGamma\cdot V^T = \mathbf{0}_{s,n} \mod \boldsymbol{\tau}$, where $V$ is a fan matrix satisfying condition 4.(b) in [2, Thm. 3.2]: this is due to the fact that the rows of $V$ span $\ker(d_X)$,
(iv) $\varGamma\cdot({}_s\widehat{V})^T = \mathbf{I}_s \mod \boldsymbol{\tau}$, since the rows of ${}_s\widehat{V}$ give the generators of $\mathrm{Tors}(\mathrm{Cl}(X))$, as in display (6) of part 5 of [2, Thm. 3.2].

Therefore it suffices to show that the matrix $U_G \cdot {}_sW$ in (3) satisfies the previous conditions $(ii)$, $(iii)$ and $(iv)$ without any reduction mod $\boldsymbol{\tau}$, that is,

$$U_G \cdot {}_sW \cdot ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s}, \quad U_G \cdot {}_sW \cdot ({}_s\widehat{V})^T = \mathbf{I}_s.$$

The first equation follows by the definition of $W$, in fact

$$W \cdot ({}^{n+r-s}U)^T = \mathrm{HNF}\left(({}^{n+r-s}U)^T\right) = \begin{pmatrix} \mathbf{I}_{n+r-s} \\ \mathbf{0}_{s,n+r-s} \end{pmatrix} \Rightarrow {}_sW \cdot ({}^{n+r-s}U)^T = \mathbf{0}_{s,n+r-s}$$

The second equation follows by the definition of $U_G$, in fact

$$U_G \cdot {}_sW \cdot ({}_s\widehat{V})^T = U_G \cdot G^T = \mathrm{HNF}(G^T) = \mathbf{I}_s.$$

(7): Part (4) of [1, Thm. 2.9] gives that $\delta_\Sigma \mid [\mathrm{Cl}(Y)\colon \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y)\colon \mathcal{C}_T(Y)]$. On the other hand, Proposition 3.1 gives that $[\mathcal{W}_T(Y)\colon \mathcal{C}_T(Y)] \mid [\mathcal{W}_T(X)\colon \mathcal{C}_T(X)] = [\mathrm{Cl}(X)\colon \mathrm{Pic}(X)]$. □

Considerations i, ii, iii, iv, v of [2, Rem. 3.3] still holds, while vi, vii and the remaining part of Remark 3.3 have to be replaced by the following

*Remark 3.3* vi. apply procedure [1, § 1.2.3], based on the HNF algorithm, to get a $(n+r) \times (n+r)$ matrix $C_X$ whose rows give a basis of $\mathcal{C}_T(X) \leq \mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$;

vii. apply procedure described in part 6 of Theorem 3.2 to get a system of generators of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$. Precisely, let $A \in \mathrm{GL}_{n+r}(\mathbb{Z})$ be a switching matrix such that $\mathrm{HNF}(C_X \cdot Q^T) = A \cdot C_X \cdot Q^T$, and put

$$B_X = {}^r(A \cdot C_X \cdot Q^T), \quad \Theta_X = {}^r(A \cdot C_X \cdot \Gamma^T) \tag{4}$$

Then the rows of the matrices $B_X$ and $\Theta_X$ represent, respectively, the free part and the torsion part of a basis of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$, where the latter is identified to $\mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k\mathbb{Z}$.

Moreover:

– recall that, for the universal 1-covering $Y$ of $X$, once fixed the basis $\{\widehat{D}_j\}_{j=1}^{n+r}$ of $\mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_Y(\widehat{L}_i)\}_{i=1}^r$ of $\mathrm{Cl}(Y) \cong \mathbb{Z}^r$, (see (11) in [1, Thm. 2.9]), one gets the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\left(\mathbf{0}_{n,r} \mid \mathbf{I}_n\right)"] \arrow[d, equal] & \mathcal{C}_T(Y) \cong \mathrm{Pic}(Y) \oplus M \arrow[r, "\left(\mathbf{I}_r \mid \mathbf{0}_{r,n}\right)"] \arrow[d, "C_Y^T"] & \mathrm{Pic}(Y) \arrow[r] \arrow[d, "B_Y^T"] & 0 \\
0 \arrow[r] & M \arrow[r, "div_Y", "\widehat{V}^T"'] \arrow[d] & \mathcal{W}_T(Y) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j \arrow[r, "d_Y", "Q"'] \arrow[d] & \mathrm{Cl}(Y) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{T}_Y \arrow[r, "\cong"] \arrow[d] & \mathcal{T}_Y \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $B_Y$ is the $r \times r$ matrix constructed in [1, Thm. 2.9(3)] and

$$C_Y = \begin{pmatrix} B_Y & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B_Y \cdot {}^r U_Q \\ \widehat{V} \end{pmatrix},$$

– once fixed the basis $\{D_j\}_{j=1}^{n+r}$ for $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_X(L_i)\}_{i=1}^{r}$ of the free part $F \cong \mathbb{Z}^r$ of $\mathrm{Cl}(X)$, constructed in part 1 of [2, Thm. 3.2], one gets the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)"] \arrow[d, equal] & \mathcal{C}_T(X) \cong \mathrm{Pic}(X) \oplus M \arrow[r, "\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)"] \arrow[d, "C_X^T"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "B_X^T \oplus \Theta_X^T"] & 0 \\
0 \arrow[r] & M \arrow[r, "div_X", "V^T"'] \arrow[d] & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j \arrow[r, "d_X = f_X \oplus \tau_X", "Q \oplus \Gamma"'] \arrow[d] & \mathrm{Cl}(X) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{T}_X \arrow[r, "\cong"] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Moreover:

– recall the following commutative diagram of short exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \ker(\overline{\alpha}) = \mathrm{Tors}(\mathrm{Cl}(X)) \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "div_X", "V^T"'] \arrow[d, "\beta^T"'] & \mathcal{W}_T(X) = \mathbb{Z}^{|\Sigma(1)|} \arrow[r, "d_X"] \arrow[d, "\mathbf{I}_{n+r}"', "\alpha"] & \mathrm{Cl}(X) \arrow[r] \arrow[d, "\overline{\alpha}"] & 0 \\
0 \arrow[r] & M \arrow[r, "div_Y", "\widehat{V}^T"'] \arrow[d] & \mathcal{W}_T(Y) = \mathbb{Z}^{|\widehat{\Sigma}(1)|} \arrow[r, "d_Y"] \arrow[d] & \mathrm{Cl}(Y) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{coker}(\beta^T) \cong \mathrm{Tors}(\mathrm{Cl}(X)) \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd} \tag{5}$$

then, putting all together, one gets the following 3-dimensional commutative diagram

$$
\begin{tikzcd}[column sep=small, row sep=small]
M \arrow[rrr, hook, "div_X", "\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)"'] \arrow[dr, hook, "\beta^T"'] \arrow[dddd, equal] & & & \mathcal{C}_T(X) \arrow[rrr, two heads, "d_{X\,|}", "\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)"'] \arrow[dr, hook, "\alpha_|", "(C_X\cdot C_Y^{-1})^T"'] \arrow[dddd, hook, "C_X^T"'] & & & \mathrm{Pic}(X) \arrow[dr, hook, "\overline{\alpha}_|", "(B_X\cdot B_Y^{-1})^T"'] \arrow[dddd, hook, "B_X^T\oplus\Theta_X^T"] & & \\
& M \arrow[rrr, hook, "div_Y", "\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)"'] \arrow[dr] \arrow[dddd, equal] & & & \mathcal{C}_T(Y) \arrow[rrr, two heads, "d_{Y\,|}", "\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)"'] \arrow[dr] \arrow[dddd, hook, "C_Y^T"'] & & & \mathrm{Pic}(Y) \arrow[dr] \arrow[dddd, hook, "B_Y^T"'] & \\
& & \mathrm{coker}(\beta^T) \arrow[rrr, hook] \arrow[dddd, equal] & & & \mathrm{coker}(\alpha_|) \arrow[rrr, two heads] & & & \mathrm{coker}(\overline{\alpha}_|) \\
& & & & & \ker(\overline{\alpha}) \arrow[dr, hook] \arrow[dddd, hook] & & & \\
M \arrow[rrr, hook, "div_X", "V^T"'] \arrow[dr, hook, "\beta^T"'] & & & \mathcal{W}_T(X) \arrow[rrr, two heads, "d_X=f_X\oplus\tau_X", "Q\oplus\Gamma"'] \arrow[dr, "\alpha", "\mathbf{I}_{n+r}"'] \arrow[dddd, two heads] & & & \mathrm{Cl}(X) \arrow[dr, "\overline{\alpha}", "\mathbf{I}_r\oplus\mathbf{0}_r"'] \arrow[dddd, two heads] & & \\
& M \arrow[rrr, hook, "div_Y", "\hat{V}^T"'] \arrow[dr] & & & \mathcal{W}_T(Y) \arrow[rrr, two heads, "d_Y", "Q"'] \arrow[dddd, two heads] & & & \mathrm{Cl}(Y) \arrow[dddd, two heads] & \\
& & \mathrm{coker}(\beta^T) & & & & & & \\
& & \mathcal{K} \arrow[rrr, hook, "\cong"'] \arrow[dr, hook] & & & \mathcal{K} \arrow[dr, hook] & & & \\
& & & \mathcal{T}_X \arrow[rrr, "\cong"'] \arrow[dr] & & & \mathcal{T}_X \arrow[dr] & & \\
& & & & \mathcal{T}_Y \arrow[rrr, "\cong"'] & & & \mathcal{T}_Y &
\end{tikzcd}
\tag{6}
$$

The Snake lemma implies

$$
\begin{aligned}
\mathrm{coker}(\beta^T) &\cong \ker(\overline{\alpha}) \cong \mathrm{Tors}(\mathrm{Cl}(X)) \\
\mathcal{K} &\cong \mathrm{coker}(\alpha_|) \cong \mathcal{C}_T(Y)/\mathcal{C}_T(X)
\end{aligned}
$$

so giving the following short exact sequences on torsion subgroups

$$
\begin{tikzcd}
& & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Tors}(\mathrm{Cl}(X)) \arrow[r] & \mathcal{C}_T(Y)/\mathcal{C}_T(X) \arrow[r] \arrow[d] & \mathrm{Pic}(Y)/\,\mathrm{Pic}(X) \arrow[r] & 0 \\
& & \mathrm{Cl}(X)/\,\mathrm{Pic}(X) \arrow[d] & & \\
& & \mathrm{Cl}(Y)/\,\mathrm{Pic}(Y) \arrow[d] & & \\
& & 0 & &
\end{tikzcd}
\tag{7}
$$

For what concerns the examples given in section 5, considerations related with parts v, vi and vii of Remark 3.3 have to be replaced as follows

*Example 5.1* v. A matrix $W \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}\left(({}^3U)^T\right) = W \cdot ({}^3U)^T$ is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

giving

$$G := {}_1\widehat{V} \cdot ({}_1W)^T = \begin{pmatrix} 1 \end{pmatrix}$$

Therefore

$$\Gamma = {}_1W \mod 5 = \begin{pmatrix} [0]_5 & [4]_5 & [2]_5 & [1]_5 \end{pmatrix}.$$

Consequently display (16) in [2], giving the action of $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_5$ on $Y = \mathbb{P}^3$, should be replaced by the following (equivalent) one:

$$\begin{array}{ccc} \mu_5 \times \mathbb{P}^3 & \longrightarrow & \mathbb{P}^3 \\ (\varepsilon, [x_1 \colon \ldots \colon x_4]) & \mapsto & \left[x_1 \colon \varepsilon^4 x_2 \colon \varepsilon^2 x_3 \colon \varepsilon x_4\right]. \end{array} \tag{8}$$

vi. Applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a $4 \times 4$ matrix $C_X$ whose rows give a basis of $\mathcal{C}_T(X)$ inside $\mathcal{W}_T(X) \cong \mathbb{Z}^{|\Sigma(1)|}$. Namely

$$C_X = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{pmatrix}$$

meaning that

$$\mathcal{C}_T(X) = \mathcal{L}\left(5D_1, 5D_2, -3D_1 - 3D_2 + D_3, -2D_1 - 4D_2 + D_4\right).$$

On the other hand, by part (3) of [1, Thm. 2.9], a basis of $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$ is given by the rows of

$$C_Y = \mathbf{I}_4 \cdot U_Q = U_Q \in \mathrm{GL}_n(\mathbb{Z})$$

giving $\mathcal{C}_T(Y) = \mathcal{W}_T(Y)$, as expected for $Y = \mathbb{P}^3$.
vii. A basis of $\mathrm{Pic}(X)$ inside $\mathrm{Cl}(X)$ is then obtained by applying part 6 of Theorem 3.2. With the notation of Remark 3.3 vii, a switching matrix $A$ such that $A \cdot C_X \cdot Q^T$ is in HNF is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} B_X &= {}^1(A \cdot C_X \cdot Q^T) = \begin{pmatrix} 5 \end{pmatrix} \\ \Theta_X &= {}^1(A \cdot C_X \cdot \Gamma^T) = \begin{pmatrix} 0 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} &\mathrm{Pic}(X) \cong \mathbb{Z}[5d_X(D_1)] \leq \mathbb{Z}[d_X(D_1)] \oplus \mathbb{Z}/5\mathbb{Z}[d_X(D_3 - D_4)] \cong \mathrm{Cl}(X) \\ &\quad \Rightarrow \ \mathrm{Cl}(X)/\mathrm{Pic}(X) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}. \end{aligned}$$

*Example 5.2* v. A matrix $U$ as defined in part 6 of Theorem 3.2 is given by

$$U = \begin{pmatrix} {}^{2}U_Q \\ \widehat{V}' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -6 & 3 & 1 & 0 & 0 & 0 \\ 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -184 & 105 & 53 & -2 & -23 & -1 \\ 191 & -109 & -55 & 2 & 24 & 1 \end{pmatrix}$$

A matrix $W \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}(({}^{4}U)^T) = W \cdot (({}^{4}U)^T)$ is given by

$$W = \begin{pmatrix} -57 & -115 & 3 & -549 & 17 & 0 \\ 4 & 8 & 1 & 3 & 7 & 0 \\ -125 & -250 & 0 & -1090 & 14 & 0 \\ -170 & -340 & 0 & -1482 & 19 & 0 \\ -188 & -376 & 0 & -1639 & 21 & 0 \\ -126 & -252 & 0 & -1092 & 13 & 1 \end{pmatrix}$$

then

$$G = {}_2\widehat{V}' \cdot ({}_2W)^T = \begin{pmatrix} -2093 & -1392 \\ 2302 & 1531 \end{pmatrix}$$

A matrix $U_G \in \mathrm{GL}_2(\mathbb{Z})$ such that $\mathrm{HNF}(G^T) = U_G \cdot G^T$ is given by

$$U_G = \begin{pmatrix} 1531 & -2302 \\ 1392 & -2093 \end{pmatrix}$$

hence giving

$$\begin{aligned} \Gamma &= U_G \cdot {}_2W \mod \boldsymbol{\tau} \\ &= \begin{pmatrix} 2224 & 4448 & 0 & 4475 & 2225 & -2302 \\ 2022 & 4044 & 0 & 4068 & 2023 & -2093 \end{pmatrix} \mod \begin{pmatrix} 3 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} [1]_3 & [2]_3 & [0]_3 & [2]_3 & [2]_3 & [2]_3 \\ [12]_{15} & [9]_{15} & [0]_{15} & [3]_{15} & [13]_{15} & [7]_{15} \end{pmatrix} \end{aligned}$$

Consequently display (20) in [2] should be replaced by the following (equivalent) one

$$\begin{aligned} & g\left(((t_1, t_2), \varepsilon, \eta), (x_1, \dots : x_6)\right) \\ & := \left(t_1^2 t_2 \varepsilon \eta^{12}\, x_1, t_1^4 t_2 \varepsilon^2 \eta^9\, x_2, t_1 t_2^3\, x_3, t_1^5 t_2^2 \varepsilon^2 \eta^3\, x_4, t_1^4 t_2^3 \varepsilon^2 \eta^{13}\, x_5, t_1^3 t_2^7 \varepsilon^2 \eta^7\, x_6\right) \end{aligned} \tag{9}$$

vi. Depending on the choice of the fan $\Sigma_i \in \mathcal{SF}(V)$, by applying procedure [1, § 1.2.3] as described in part 2 of Theorem 3.2, one gets a $6 \times 6$ matrix $C_{X,i}$ whose rows give a basis of $\mathcal{C}_T(X_i)$ inside $\mathcal{W}_T(X_i) \cong \mathbb{Z}^{|\Sigma_i(1)|}$. Namely

$$C_{X,1} = \begin{pmatrix} 265926375 & 0 & 0 & 0 & 0 & 0 \\ -148978500 & 825 & 0 & 0 & 0 & 0 \\ -58474020 & -375 & 15 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ -58473933 & -417 & -3 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,2} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -34716000 & 15 & 0 & 0 & 0 & 0 \\ -594165 & 0 & 30 & 0 & 0 & 0 \\ -34715963 & -3 & -7 & 1 & 0 & 0 \\ 17655087 & -12 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{X,3} = \begin{pmatrix} 43543500 & 0 & 0 & 0 & 0 & 0 \\ -37009500 & 825 & 0 & 0 & 0 & 0 \\ -6534165 & -750 & 30 & 0 & 0 & 0 \\ 37 & -18 & -7 & 1 & 0 & 0 \\ 87 & -42 & -18 & 0 & 3 & 0 \\ 19 & -8 & -5 & 0 & -1 & 1 \end{pmatrix}$$

vii. A basis of $\mathrm{Pic}(X_i)$ inside $\mathrm{Cl}(X_i)$ is then obtained by applying part 6 of Theorem 3.2. For $i = 1, 2, 3$, matrices $A_i$ switching $C_{X_i} \cdot Q^T$ in Hermite normal form are, respectively,

$$A_1 = \begin{pmatrix} -351039 & -449987 & -449987 & 0 & 0 & 0 \\ -502913 & -644670 & -644670 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -93838 & -117699 & 0 & 0 & 0 & 0 \\ -1157199 & -1451450 & 0 & 0 & 0 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -10317 & -12139 & 0 & 0 & 0 & 0 \\ -22429 & -26390 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$B_{X_1} = {}^2(A_1 \cdot C_{X_1} \cdot Q^T) = \begin{pmatrix} 825 & 185620050 \\ 0 & 265926375 \end{pmatrix}$$

$$B_{X_2} = {}^2(A_2 \cdot C_{X_2} \cdot Q^T) = \begin{pmatrix} 60 & 1765515 \\ 0 & 21771750 \end{pmatrix}$$

$$B_{X_3} = {}^2(A_3 \cdot C_{X_3} \cdot Q^T) = \begin{pmatrix} 3300 & 10016325 \\ 0 & 21771750 \end{pmatrix}$$

$$\Theta_{X_i} = {}^2(A_i \cdot C_{X_i} \cdot \Gamma^T) = \begin{pmatrix} [0]_3 & [0]_{15} \\ [0]_3 & [0]_{15} \end{pmatrix}, \quad \text{for } i = 1, 2, 3.$$

## References

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