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A Quantitative Evaluation of Seismic Signals at Teleseismic Distances—I
Radiation from Point Sources

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Summary

The aim of the series of papers to be published under the above title is to follow up and improve Carpenter's (1966) attempt to construct realistic pulse shapes for teleseismic body waves from underground explosions. The present work deals with both explosions and shallow earthquakes and with the radiation of surface waves as well as body waves.

The extension of Carpenter's theory to take crustal layering more accurately into account involves the matrix theory first introduced by Thomson (1950) and developed by Haskell. We shall use the notation of Haskell's (1964) paper in the following analysis and the initial theoretical work is concerned with some of the groundwork and one or two results which have not been covered by Haskell in his series of papers (1953, 1962, 1964).

A point source can be represented either as a system of forces or as a discontinuity in the displacement or stress or their derivatives across an element of surface. We show here that a general source of either type is equivalent in the generation of elastic radiation to a discontinuity across a horizontal plane in the displacement and the stress acting on the plane. This means that any point source can be put into a form suitable for computations based on the Thomson–Haskell theory.

The theory is applied to the construction of theoretical models of earthquake and explosive sources. Some of the more realistic models so far proposed are given in the later section of the paper.

1. Introduction

In 1953 Haskell published a corrected version of Thomson's (1950) theory of elastic waves in a layered solid medium, applying it to the calculation of dispersion curves of surface waves. Since then, the theory has been developed principally by Haskell (1962, 1964) and Harkrider (1964) to deal with the surface wave motion caused by a source in a layered half-space and with the disturbance caused by plane waves incident from infinity.

The fundamental mathematical theory behind the method was presented by Gilbert & Backus (1966) in their paper on propagator matrices for two-dimensional problems (which include dispersion of surface waves and incidence of plane waves). The corresponding theory for three-dimensional problems (a point source in a layered half-space) is developed here, showing that the propagator matrices, or

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layer matrices, are the same for two and three dimensions. This leads to a close similarity of presentation for both types of problem.

The notation to be used is principally that of Haskell (1964), except that each of his (6×6) matrices are divided (in the obvious way) into a (4×4) and a (2×2) matrix corresponding to the first four rows and columns and the last two rows and columns respectively of Haskell's matrix. Each 6-vector is divided in the same manner into a 4-vector and a 2-vector.

In order to use the matrix theory to calculate the seismic radiation from a source of disturbance in a layered medium it is necessary to describe the source as discontinuities in the stress motion vectors across a horizontal plane. (The components of the stress motion vectors are time and space transforms of the displacements and the three components of stress acting on an element of surface.) There is a close connection between the type of source described by a discontinuity and the type described by a system of forces Burridge & Knopoff 1964). The reasoning leading to this conclusion can be extended to show that the general source of either type can be represented as discontinuities appropriate to the Thomson–Haskell theory.

2. Wave transmission in a medium whose properties vary with depth

Let $\mathbf{u}(r,\phi,z,t)$ be the elastic displacement in an isotropic medium whose Lamé parameters λ, μ and density ρ vary with the z-co-ordinate of cylindrical polar coordinates (r,ϕ,z). Following Haskell's (1964) notation we define operators

$$\left.\begin{aligned} L^{nc} &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty}\exp(-i\omega t)\,dt\int_{-\pi}^{\pi}\cos n\phi\,d\phi\int_{0}^{\infty}J_n(kr)\frac{r}{k}\,dk \\ L^{ns} &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty}\exp(-i\omega t)\,dt\int_{-\pi}^{\pi}\sin n\phi\,d\phi\int_{0}^{\infty}J_n(kr)\frac{r}{k}\,dk, \end{aligned}\right\} \tag{2.1}$$

and construct

$$\left.\begin{aligned} U_r^c(k,n,z,\omega) &= -L^{nc}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}\right\}, \qquad n=1,2,\ldots \\ U_r^c(k,\mathrm{O},z,\omega) &= -\tfrac{1}{2}L^{oc}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}\right\} \\ U_r^s(k,n,z,\omega) &= -L^{ns}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}\right\}, \qquad n=1,2,\ldots \\ U_r^s(k,\mathrm{O},z,\omega) &= 0. \end{aligned}\right\} \tag{2.2}$$

Similarly

$$\left.\begin{aligned} U_\phi^c(k,n,z,\omega) &= L^{ns}\left\{\frac{1}{r}\frac{\partial u_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)\right\} \\ U_\phi^s(k,n,z,\omega) &= -L^{nc}\left\{\frac{1}{r}\frac{\partial u_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)\right\} \\ U_z^c(k,n,z,\omega) &= L^{nc}\{k^2u_z\} \\ U_z^s(k,n,z,\omega) &= L^{ns}\{k^2u_z\}, \qquad n=1,2,\ldots, \end{aligned}\right\} \tag{2.3}$$

with corresponding equations for $n=0$.

We operate in a similar manner on the stresses τ_{rz}, $\tau_{\phi z}$, τ_{zz} corresponding to the displacements $\mathbf{u}$:

$$\left.\begin{aligned}\omega^2\, T_r^{\,c}(k,n,z,\omega) &= -L^{nc}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial\tau_{\phi z}}{\partial\phi}\right\}\\ \omega^2\, T_r^{\,s}(k,n,z,\omega) &= -L^{ns}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial\tau_{\phi z}}{\partial\phi}\right\},\qquad n=1,2,\ldots,\end{aligned}\right\}\tag{2.4}$$

and so on, defining $T_\phi^{\,c}$, $T_\phi^{\,s}$, $T_z^{\,c}$ and $T_z^{\,s}$ as well.

It can be shown that, as a result of the equations of motion and stress–strain relations of perfect elasticity, the stress–motion vectors

$$\mathbf{B}^c(k,n,z,\omega)=\begin{pmatrix}U_r^{\,c}\\ U_z^{\,c}\\ T_z^{\,c}\\ T_r^{\,c}\end{pmatrix}\quad\text{and}\quad \mathbf{b}^c(k,n,z,\omega)=\begin{pmatrix}U_\phi^{\,c}\\ T_\phi^{\,c}\end{pmatrix}\tag{2.5}$$

satisfy the equations

$$\frac{\partial}{\partial z}\mathbf{B}^c=\mathbf{N}\mathbf{B}\quad\text{and}\quad\frac{\partial}{\partial z}\mathbf{b}^c=\mathbf{n}\mathbf{b}\tag{2.6}$$

for all $n\geqslant 0$, where

$$\mathbf{N}=\begin{pmatrix}0 & -1 & 0 & \dfrac{\omega^2}{\mu}\\ \dfrac{k^2\lambda}{\lambda+2\mu} & 0 & \dfrac{\omega^2}{\lambda+2\mu} & 0\\ 0 & -\rho & 0 & k^2\\ \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\cdot\dfrac{k^2}{\omega^2}-\rho & 0 & \dfrac{-\lambda}{\lambda+2\mu} & 0\end{pmatrix}\tag{2.7}$$

and

$$\mathbf{n}=\begin{pmatrix}0 & \dfrac{\omega^2}{\mu}\\ \dfrac{\mu k^2}{\omega^2}-\rho & 0\end{pmatrix}.\tag{2.8}$$

The vectors $\mathbf{B}^s(k,n,z,\omega)$ and $\mathbf{b}^s(k,n,z,\omega)$ constructed in the same way, also satisfy equations 2.6.

Exactly the same equations may be obtained for two-dimensional displacements $\mathbf{u}(x,z,t)$ and stresses (referred to Cartesian co-ordinates (x,y,z)) where k is the transform variable connected with x (see Gilbert & Backus 1966).

Solutions of equations 2.6 can be written in terms of the square matrices $\mathbf{P}_B(z,z_0)$ and $\mathbf{p}_b(z,z_0)$ which have the properties

$$\mathbf{P}_B(z_0,z_0)=\mathbf{I}_4,\qquad \mathbf{p}_b(z_0,z_0)=\mathbf{I}_2,\tag{2.9}$$

(z_0 being any point) where $\mathbf{I}_n$ is the $(n\times n)$ identity matrix. $\mathbf{P}_B$ and $\mathbf{p}_b$ are the propagator matrices.

In a homogeneous medium

$$\mathbf{P}_B(z,z_0)=\exp\,[\mathbf{N}(z-z_0)],\qquad \mathbf{p}_b(z,z_0)=\exp\,[\mathbf{n}(z-z_0)],\tag{2.10}$$

where exp denotes the exponential series.

A simplified procedure for finding the propagators in this case is given by the use of a non-singular transformation which diagonalizes **N**. The exponential can then be found without difficulty. A slightly modified method is suggested by Haskell's (1964) work. We define $\mathbf{K}^c$ and $\mathbf{k}^c$ by

$$\mathbf{B}^c(k, n, z, \omega) = \mathbf{E}\mathbf{K}^c(k, n, z, \omega), \qquad \mathbf{b}^c(k, n, z, \omega) = \mathbf{e}\mathbf{k}^c(k, n, z, \omega), \tag{2.11}$$

where

$$\mathbf{E} = \begin{pmatrix} -1/\rho & 0 & -\nu_\beta/\rho & 0 \\ 0 & \nu_\alpha/\rho & 0 & k^2/\rho \\ 1-\gamma & 0 & -\gamma\nu_\beta & 0 \\ 0 & \gamma\nu_\alpha/k^2 & 0 & \gamma-1 \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} 1/\rho & 0 \\ 0 & -\gamma\nu_\beta/2k^2 \end{pmatrix} \tag{2.12}$$

and $\gamma = 2\beta^2 k^2/\omega^2$, $\nu_\alpha = (k^2 - \omega^2/\alpha^2)^{\frac{1}{2}}$, $\nu_\beta = (k^2-\omega^2/\beta^2)^{\frac{1}{2}}$ (α, β being the wave speeds).

$\mathbf{K}^c$ and $\mathbf{k}^c$ satisfy equations like (2.6) with coefficient matrices

$$\mathbf{E}^{-1}\mathbf{N}\mathbf{E} = \begin{pmatrix} 0 & -\nu_\alpha & 0 & 0 \\ -\nu_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu_\beta \\ 0 & 0 & -\nu_\beta & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}^{-1}\mathbf{n}\mathbf{e} = \begin{pmatrix} 0 & -\nu_\beta \\ -\nu_\beta & 0 \end{pmatrix} \tag{2.13}$$

respectively. The corresponding propagator matrices are (summing the series)

$$\mathbf{P}_k(z, z_0) = \begin{pmatrix} \cosh\nu_\alpha(z-z_0) & -\sinh\nu_\alpha(z-z_0) & 0 & \\ -\sinh\nu_\alpha(z-z_0) & \cosh\nu_\alpha(z-z_0) & 0 & \\ 0 & 0 & \cosh\nu_\beta(z-z_0) & -\sinh\nu_\beta(z-z_0) \\ 0 & 0 & -\sinh\nu_\beta(z-z_0) & \cosh\nu_\beta(z-z_0) \end{pmatrix} \tag{2.14}$$

$$\mathbf{p}_k(z, z_0) = \begin{pmatrix} \cosh\nu_\beta(z-z_0) & -\sinh\nu_\beta(z-z_0) \\ -\sinh\nu_\beta(z-z_0) & \cosh\nu_\beta(z-z_0) \end{pmatrix}. \tag{2.15}$$

The propagator matrices for $\mathbf{B}^c$ and $\mathbf{b}^c$ (and similarly $\mathbf{B}^s$ and $\mathbf{b}^s$) are given by

$$\left.\begin{aligned} \mathbf{P}_B(z, z_0) &= \mathbf{E}\mathbf{P}_k(z, z_0)\,\mathbf{E}^{-1} \\ \text{and} \qquad \mathbf{p}_b(z, z_0) &= \mathbf{e}\mathbf{p}_k(z, z_0)\,\mathbf{e}^{-1} \end{aligned}\right\} \tag{2.16}$$

respectively.

Continuing with Haskell's notation, we define the layer matrices for a homogeneous medium to be

$$\mathbf{A}(z-z_0) = \mathbf{P}_B(z, z_0), \qquad \mathbf{a}(z-z_0) = \mathbf{p}_b(z, z_0). \tag{2.17}$$

Haskell's source matrices, corresponding to discontinuities in stress and displacement across a plane of constant z, are given by discontinuities in $\mathbf{K}^c$, $\mathbf{k}^c$, etc. For a discontinuity across $z = z_1$,

$$\left.\begin{aligned} \mathbf{S}^{nc}(k, \omega) &= \mathbf{K}^c(k, n, z_1+\mathrm{O}, \omega) - \mathbf{K}^c(k, n, z_1-\mathrm{O}, \omega) \\ \mathbf{s}^{nc}(k, \omega) &= \mathbf{k}^c(k, n, z_1+\mathrm{O}, \omega) - \mathbf{k}^c(k, n, z_1-\mathrm{O}, \omega). \end{aligned}\right\} \tag{2.18}$$

$\mathbf{S}^{ns}$ and $\mathbf{s}^{ns}$ are similarly defined.

Once solutions for the stress–motion vectors are known, the displacements can be found from the inverse transformations

$$\left.\begin{aligned}
&u_r(r,\phi,z,t)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)\,d\omega \left\{\sum_{n=0}^{\infty}\int_0^{\infty}\left[(U_r{}^c\cos n\phi + U_r{}^s \sin n\phi)\frac{\partial J_n(kr)}{\partial r}\right.\right.\\
&\qquad\qquad\left.\left. -\frac{n}{r}(U_\phi{}^c\cos n\phi + U_\phi{}^s\sin n\phi)\,J_n(kr)\right]dk\right\}\\
&u_\phi(r,\phi,z,t)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)\,d\omega \left\{\sum_{n=0}^{\infty}\int_0^{\infty}\left[\frac{n}{r}(U_r{}^s\cos n\phi - U_r{}^c \sin n\phi)\,J_n(kr)\right.\right.\\
&\qquad\qquad\left.\left. -(U_\phi{}^s\cos n\phi - U_\phi{}^c\sin n\phi)\frac{\partial J_n(kr)}{\partial r}\right]dk\right\}\\
&u_z(r,\phi,z,t)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega t)\,d\omega \left\{\sum_{n=0}^{\infty}\int_0^{\infty}(U_z{}^c\cos n\phi + U_z{}^s \sin n\phi)\,J_n(kr)\,dk\right\}.
\end{aligned}\right\} \quad (2.19)$$

3. Wave transmission in a fluid overlying a solid medium

In an elastic fluid, the modulus of rigidity μ is zero and the second of equations (2.6) becomes null (i.e. $\mathbf{b}^c \equiv \mathbf{b}^s \equiv 0$). The first equation is modified into

$$\frac{\partial}{\partial z}\mathbf{b}_l{}^c = \mathbf{n}_l\,\mathbf{b}_l{}^{nc} \quad (3.1)$$

with the subsidiary conditions

$$U_r{}^c = -\frac{1}{\rho}T_z{}^c, \qquad T_r{}^c = 0, \quad (3.2)$$

where

$$\mathbf{b}_l{}^c(k,n,z,\omega) = \begin{pmatrix} U_z{}^c \\ T_z{}^c \end{pmatrix}, \qquad \mathbf{n}_l = \begin{pmatrix} 0 & \dfrac{\omega^2}{\lambda} - \dfrac{k^2}{\rho} \\ -\rho & 0 \end{pmatrix} \quad (3.3)$$

and λ is now the bulk modulus of the fluid. The components $U_r{}^s$, $U_z{}^s$, etc. obey similar equations.

From the propagator $\mathbf{p}_b(z,z_0)$ of $\mathbf{b}_l{}^c$ and $\mathbf{b}_l{}^s$ we construct a modified propagator matrix

$$\mathbf{P}_b{}^l(z,z_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{11}{}^l & p_{12}{}^l & 0 \\ 0 & p_{21}{}^l & p_{22}{}^l & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.4)$$

where $p_{ij}{}^l$ are the components of $\mathbf{p}_b{}^l(z,z_0)$. $\mathbf{P}_b{}^l$ is not a true propagator but has the following properties similar to those of a propagator matrix

$$\mathbf{P}_b{}^l(z,z_1)\,\mathbf{P}_b{}^l(z_1,z_0) = \mathbf{P}_b{}^l(z_1,z_0) \quad (3.5)$$

$$\mathbf{P}_b{}^l(z_0, z)\,\mathbf{P}_b{}^l(z, z_0) = \mathbf{P}_b{}^l(z_0, z_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.6}$$

$$\mathbf{B}^c(k, n, z, \omega) = \mathbf{P}_b{}^l(z, z_0) \begin{pmatrix} U_r{}^c(z) \\ U_z{}^c(z_0) \\ T_z{}^c(z_0) \\ T_r{}^c(z_0) \end{pmatrix}, \tag{3.7}$$

where $U_r{}^c(z) = U_r{}^c(k, n, z, \omega)$, etc.

If the top surface of the fluid ($z = 0$) is stress-free and there are no discontinuities in stress or displacement in the fluid, then, in the medium below a fluid–solid interface at $z = z_0$,

$$\mathbf{B}^c(k, n, z, \omega) = \mathbf{P}_B(z, z_0)\,\mathbf{P}_b{}^l(z_0, \mathrm{O}) \begin{pmatrix} U_r{}^c(z_0) \\ U_z{}^c(\mathrm{O}) \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

and

$$\mathbf{b}^c(k, n, z, \omega) = \mathbf{p}_b(z, z_0) \begin{pmatrix} U_\phi{}^c(z_0) \\ 0 \end{pmatrix}, \qquad z \geqslant z_0, \tag{3.9}$$

with similar equations for $\mathbf{B}^s$ and $\mathbf{b}^s$, where $\mathbf{P}_B$ and $\mathbf{p}_b$ are the propagators for the solid medium.

The modified propagator matrix may be used, according to its properties given above, in a similar way to the (4×4) propagator matrix for a solid medium. The unknowns appearing in equations (3.8) and (3.9) refer to motion at the free surface and at the fluid–solid interface.

To find the propagator $\mathbf{p}_b{}^l(z, z_0)$ for a homogeneous fluid we make the transformation

$$\mathbf{b}_l{}^c(k, n, z, \omega) = \mathbf{e}^l\,\mathbf{k}_l{}^c(k, n, z, \omega) \tag{3.10}$$

where

$$\mathbf{e}^l = \begin{pmatrix} \nu_\alpha/\rho & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha^2 = \lambda/\rho. \tag{3.11}$$

The final result is

$$\mathbf{p}_b{}^l(z, z_0) = \begin{pmatrix} \cosh \nu_\alpha(z-z_0) & -\dfrac{\nu_\alpha}{\rho}\sinh \nu_\alpha(z-z_0) \\ -\dfrac{\rho}{\nu_\alpha}\sinh \nu_\alpha(z-z_0) & \cosh \nu_\alpha(z-z_0) \end{pmatrix}. \tag{3.12}$$

A modified layer matrix may be defined as

$$\mathbf{A}^l(z-z_0) = \mathbf{P}_b{}^l(z, z_0). \tag{3.13}$$

This is similar to the modified matrix proposed by Haskell (1953), but the unknowns appearing in the equations (3.8 and 3.9) by this method are more directly useful.

At a discontinuity in the fluid we construct the source vector

$$\mathbf{s}_l{}^{nc}(k, \omega) = \mathbf{k}_l{}^c(k, n, z_0+\mathrm{O}, \omega) - \mathbf{k}_l{}^c(k, n, z_0-\mathrm{O}, \omega) \tag{3.14}$$

or

$$\mathbf{S}_l{}^{nc}(k, \omega) = \mathbf{K}_l{}^c(k, n, z_0+\mathrm{O}, \omega) - \mathbf{K}_l{}^c(k, n, z_0-\mathrm{O}, \omega)$$

where

$$\mathbf{B}^c(k, n, z, \omega) = \mathbf{E}^l\,\mathbf{K}_l{}^c(k, n, z, \omega) \tag{3.15}$$

and $\mathbf{E}^l$ is constructed from $\mathbf{e}^l$ in the same way as $\mathbf{P}_b{}^l$ from $\mathbf{p}_b{}^l$.

In a medium where fluid lies below the solid medium the procedure is more complicated (see Dorman 1962).

4. Discontinuities across an element of surface

Burridge & Knopoff (1964) showed that discontinuities across a surface in displacement and in the stress acting on the surface can be assigned arbitrary values. The discontinuities in the other components of the stress tensor will be specified by these.

Furthermore, it can be shown that discontinuities in the derivatives of the stress can also be found from the six discontinuities specified above. To prove this, we use the equation of motion

$$\frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{4.1}$$

(where u_i is the displacement, τ_{ij} the stress, and ρ the density).

Equation (4.1) holds on both sides of the surface of discontinuity. Let us take the x_3-axis in the direction of the normal to the surface at a given point. Then, denoting by square brackets the discontinuity across the surface at that point, we have

$$\left[\frac{\partial \tau_{i3}}{\partial x_3}\right] = \rho \frac{\partial^2 [u_i]}{\partial t^2} - \frac{\partial [\tau_{i2}]}{\partial x_2} - \frac{\partial [\tau_{i1}]}{\partial x_1}, \qquad i = 1, 2, 3. \tag{4.2}$$

The right-hand side is known and so the left-hand side can be calculated. We also have

$$\left[\frac{\partial \tau_{ij}}{\partial x_1}\right] = \frac{\partial}{\partial x_1}[\tau_{ij}], \qquad \left[\frac{\partial \tau_{ij}}{\partial x_2}\right] = \frac{\partial}{\partial x_2}[\tau_{ij}], \tag{4.3}$$

since each of the derivatives is along a direction tangential to the surface.

There are three further derivatives:

$$\left.\begin{aligned}
\left[\frac{\partial \tau_{12}}{\partial x_3}\right] &= \mu\left[\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3}\right] \\
&= \frac{\partial}{\partial x_1}[\tau_{23}] + \frac{\partial}{\partial x_2}[\tau_{13}] - 2\mu \frac{\partial^2 [u_3]}{\partial x_1 \partial x_2} \\
\left[\frac{\partial \tau_{11}}{\partial x_3}\right] &= \left(\frac{\lambda}{\lambda+2\mu}\right)\left\{\left[\frac{\partial \tau_{33}}{\partial x_3}\right] + 2\frac{\partial}{\partial x_2}[\tau_{23}] + 4\left(1+\frac{\mu}{\lambda}\right)\frac{\partial}{\partial x_1}[\tau_{13}]\right. \\
&\qquad \left. -2\mu\left(2\left\{1+\frac{\mu}{\lambda}\right\}\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)[u_3]\right\} \\
\left[\frac{\partial \tau_{22}}{\partial x_3}\right] &= \left(\frac{\lambda}{\lambda+2\mu}\right)\left\{\left[\frac{\partial \tau_{33}}{\partial x_3}\right] + 4\left(1+\frac{\mu}{\lambda}\right)\frac{\partial}{\partial x_2}[\tau_{23}] + 2\frac{\partial}{\partial x_1}[\tau_{13}]\right. \\
&\qquad \left. -2\mu\left(\frac{\partial^2}{\partial x_1^2} + 2\left\{1+\frac{\mu}{\lambda}\right\}\frac{\partial^2}{\partial x_2^2}\right)[u_3]\right\},
\end{aligned}\right\} \tag{4.4}$$

where λ and μ are the Lamé constants. The right-hand sides are known in terms of the six basic discontinuities in displacement and stress and so the left-hand sides can be calculated.

Higher derivatives of stress can be treated in the same way. Discontinuities in the derivatives of the displacements can be written in terms of the discontinuities in the stress. Hence, the most general discontinuity in stress and displacement, and their derivatives, across an arbitrary surface is specified by the discontinuities in displacement and in the stress acting on the surface.

We may represent such discontinuities across an element of surface with normal **n** by

$$\left.\begin{aligned} [u_i](\mathbf{x}, t) &= a_i\,\delta(x_1)\,\delta(x_2) \\ [\tau_{in}](\mathbf{x}, t) &= b_i\,\delta(x_1)\,\delta(x_2), \end{aligned}\right\} \quad (4.5)$$

where a_i, b_i are functions of time.

Burridge & Knopoff (1964) have given a formula by which the equivalent body force can be calculated (that is, the body force giving rise to the same elastic radiation). It is

$$e_i(\mathbf{x}, t) = -\int_{\Sigma} \{[u_p](\boldsymbol{\xi}, t)\,n_q(\boldsymbol{\xi})\,c_{pqij}\,\delta_j(\mathbf{x}, \boldsymbol{\xi}) + [\tau_{in}](\boldsymbol{\xi}, t)\,\delta(\mathbf{x}, \boldsymbol{\xi})\}\,d\Sigma, \quad (4.6)$$

where Σ is the surface of discontinuity, c_{ijpq} are the elastic coefficients, and

$$\delta(\mathbf{x}, \boldsymbol{\xi}) = \delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3)$$

$$\delta_j(\mathbf{x}, \boldsymbol{\xi}) = \frac{\partial}{\partial x_j}\delta(\mathbf{x}, \boldsymbol{\xi}).$$

Substituting equations (4.5) we find the equivalent body force to be

$$e_i(\mathbf{x}, t) = -\delta_j(\mathbf{x}, \mathrm{O})\,\{\lambda a_p\,n_p\,\delta_{ij} + \mu(a_i\,n_j + a_j\,n_i)\} - \delta(\mathbf{x}, \mathrm{O})\,b_i. \quad (4.7)$$

Suppose now that the plane $x_3 = 0$ is the horizontal. We choose discontinuities

$$\left.\begin{aligned} [u_i](\mathbf{x}, t) &= A_i\,\delta(x_1)\,\delta(x_2) \\ [\tau_{i3}](\mathbf{x}, t) &= \left\{B_i + \mu B_{ij}\frac{\partial}{\partial x_j}\right\}\delta(x_1)\,\delta(x_2) \end{aligned}\right\} \quad (4.8)$$

across the plane $x_3 = 0$. The equivalent body force is

$$e_i(\mathbf{x}, t) = -\delta_j(\mathbf{x}, \mathrm{O})\,\{\lambda A_3\,\delta_{ij} + \mu(A_i\,\delta_{j3} + A_j\,\delta_{i3} + B_{ij})\} - \delta(\mathbf{x}, \mathrm{O})\,B_i. \quad (4.9)$$

The two sets of discontinuities (2.5) and (2.8) are equivalent if the body forces are equal for all **x**.

This implies that

$$\left.\begin{aligned} & A_1 = a_1\,n_3 + a_3\,n_1, \quad A_2 = a_2\,n_3 + a_3\,n_2, \quad A_3 = (2\mu a_3\,n_3 + \lambda a_p\,n_p)/(\lambda+2\mu) \\ & B_i = b_i, \quad i = 1, 2, 3 \\ & B_{11} = \frac{2}{\lambda+2\mu}\{2(\lambda+\mu)\,a_1\,n_1 + \lambda a_2\,n_2\} \\ & B_{12} = B_{21} = a_1\,n_2 + a_2\,n_1 \\ & B_{22} = \frac{2}{\lambda+2\mu}\{2(\lambda+\mu)\,a_2\,n_2 + \lambda a_1\,n_1\} \end{aligned}\right\} \quad (4.10)$$

and all other components of B_{ij} are zero.

Thus, an arbitrary discontinuity as given by equations (4.5) can be replaced by a discontinuity across a horizontal plane.

5. Point forces and couples

We may describe the orientation of a couple by two perpendicular unit vectors **n** and **f**, where **n** is normal to, and **f** lies in, the null plane in such a way that the direction of the couple is given by $\mathbf{n} \times \mathbf{f}$. A double couple is composed of two couples, one with orientation vectors $(\mathbf{n}, \mathbf{f})$ and the other with vectors $(\mathbf{f}, \mathbf{n})$. It is uniquely defined by the specification of the two vectors without regard to order.

Burridge & Knopoff (1964) show that a unit discontinuity in tangential displacement in the direction of **f** across an element of surface with normal **n** is equivalent to a double couple with magnitude μ and orientation vectors **n** and **f**.

A unit discontinuity in normal displacement is equivalent to a dilatational force of magnitude λ and a dipole of magnitude 2μ in the direction of **n**.

Finally, a unit discontinuity in the stress acting on the fault plane is equivalent to a unit body force acting in the opposite direction.

The problem of a general force system acting at a point in a half-space has been dealt with by Chakrabarty (1967) who reformulated the Thomson–Haskell theory in terms of body forces rather than discontinuities across horizontal planes. However, a general point force system can be represented as a discontinuity across a plane as follows.

A simple body force acting at the point $\mathbf{x} = 0$ can be written as

$$\begin{aligned} e_i(\mathbf{x}, t) &= d_i \,\delta(\mathbf{x}, \mathrm{O}) \\ &= -\int_\Sigma \{-d_i \,\delta(\xi_1)\,\delta(\xi_2)\}\,\delta(\mathbf{x}, \boldsymbol{\xi})\,d\Sigma, \end{aligned} \tag{5.1}$$

where d_i is a function of time only and Σ is the plane $x_3 = 0$. By equation (4.6) it is clear that the body force is equivalent to a discontinuity in stress across Σ of

$$[\tau_{i3}](\mathbf{x}, t) = -d_i \,\delta(x_1)\,\delta(x_2). \tag{5.2}$$

A general point force system can be derived from the simple force by differentiation. A first derivative, for instance, gives a couple or dipole;

$$\begin{aligned} e_i(\mathbf{x}, t) &= d_i \frac{\partial}{\partial x_j}\delta(\mathbf{x}, o) \\ &= -\int_\Sigma \{-d_i \,\delta(\xi_1)\,\delta(\xi_2)\}\,\delta_j(\mathbf{x}, \boldsymbol{\xi})\,d\Sigma. \end{aligned} \tag{5.3}$$

If $j = 1, 2$, this can be rearranged to give

$$e_i(\mathbf{x}, t) = -\int_\Sigma \left\{-d_i \frac{\partial}{\partial \xi_j}\delta(\xi_1)\,\delta(\xi_2)\right\}\delta(\mathbf{x}, \boldsymbol{\xi})\,d\Sigma \tag{5.4}$$

and the couple is equivalent to a discontinuity in stress

$$[\tau_{i3}](\mathbf{x}, t) = -d_i \frac{\partial}{\partial x_j}\delta(x_1)\,\delta(x_2). \tag{5.5}$$

Any number of derivatives with respect to x_1 and x_2 can be dealt with in this way.

If $j = 3$, a rearrangement of the integral in equation 5.3 shows that the couple is equivalent to the more complex discontinuity

$$\left.\begin{aligned}
[u_1] &= -\frac{d_1}{\mu}\delta(x_1)\delta(x_2), & [\tau_{13}] &= \frac{\lambda}{(\lambda+2\mu)}d_3\frac{\partial}{\partial x_1}\delta(x_1)\delta(x_2)\\
[u_2] &= -\frac{d_2}{\mu}\delta(x_1)\delta(x_2), & [\tau_{23}] &= \frac{\lambda}{(\lambda+2\mu)}d_3\frac{\partial}{\partial x_2}\partial(x_2)\delta(x_1)\\
[u_3] &= -\frac{d_3}{(\lambda+2\mu)}\delta(x_1)\delta(x_2), & [\tau_{33}] &= \left(d_1\frac{\partial}{\partial x_1}+d_2\frac{\partial}{\partial x_2}\right)\delta(x_1)\delta(x_2)
\end{aligned}\right\} \tag{5.6}$$

across the plane $x_3 = 0$.

Higher derivatives of the body force with respect to x_1 and x_2 imply higher derivatives of the delta-functions in the expressions for the discontinuities. Higher derivatives with respect to x_3 are more difficult to deal with.

The equations of motion with body force **F** are

$$\frac{\partial}{\partial x_j}\left(c_{ijpq}\frac{\partial u_p}{\partial x_q}\right) - \rho\frac{\partial^2 u_i}{\partial t^2} = -F_i. \tag{5.7}$$

If we put

$$\frac{F_i}{K} = -\frac{\partial}{\partial x_j}\left(c_{ijpq}\frac{\partial e_p}{\partial x_q}\right) + \rho\frac{\partial^2 e_i}{\partial t^2}, \tag{5.8}$$

where K is an arbitrary constant and **e** is given by (5.1), then the solution to equation (4.7) is $\mathbf{u} = K\mathbf{e}$, with no radiation since **e** is a point force and is zero outside a restricted volume. Therefore, a body force $\partial^2 \mathbf{e}/\partial x_3{}^2$ is equivalent to a force $\mathbf{F}^*$ in elastic radiation, where

$$\left.\begin{aligned}
F_1{}^* &= \frac{\rho}{\mu}\frac{\partial^2 e_1}{\partial t^2} - \left(\frac{\lambda+\mu}{\mu}\right)\frac{\partial^2 e_j}{\partial x_1\,\partial x_j} - \left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}\right)e_1\\
F_2{}^* &= \frac{\rho}{\mu}\frac{\partial^2 e_2}{\partial t^2} - \left(\frac{\lambda+\mu}{\mu}\right)\frac{\partial^2 e_j}{\partial x_2\,\partial x_j} - \left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}\right)e_2\\
F_3{}^* &= \left(\frac{\rho}{\lambda+2\mu}\right)\frac{\partial^2 e_3}{\partial t^2} - \left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\frac{\partial}{\partial x_3}\left(\frac{\partial e_1}{\partial x_1}+\frac{\partial e_2}{\partial x_2}\right)\\
&\qquad - \left(\frac{\mu}{\lambda+2\mu}\right)\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}\right)e_3.
\end{aligned}\right\} \tag{5.9}$$

The right-hand sides of equations (5.9) can be represented in terms of equivalent discontinuities. Therefore, the left-hand sides have equivalent discontinuities.

Higher derivatives of **e** can be dealt with in the same way. Hence, a general body force system acting at a point is equivalent to a set of discontinuities in displacement and stress across a horizontal surface element.

Expressions for the source vectors (defined by equations (2.18)) for a unit force, a single or double couple, and for a dilatational source have been given by Haskell (1964) in terms of the orientation vectors **f** and **n** defined earlier in this section. The relation between the (x_1, x_2, x_3) axes and the (r, ϕ, z) co-ordinates is given by

$$x_1 = r\cos\phi, \qquad x_2 = r\sin\phi, \qquad x_3 = z. \tag{5.10}$$

Haskell's source vectors may be used for the equivalent discontinuities in displacement and stress.

6. Point sources in a fluid

In the study of the propagation of sound in a fluid, viscosity is neglected. This means that, in a disturbance starting from rest, the motion is irrotational. The particle displacement $\mathbf{u}$ is therefore the gradient of a scalar function $\varphi(\mathbf{x}, t)$;

$$\mathbf{u} = -\operatorname{grad}\varphi. \tag{6.1}$$

The elastic wave equation for a homogeneous fluid is

$$\lambda\frac{\partial^2 u_j}{\partial x_i \partial x_j} - \rho\frac{\partial^2 u_i}{\partial t^2} = -F_i, \tag{6.2}$$

where λ is the bulk modulus of the fluid and $\mathbf{F}$ is the body force.

Substituting equation (6.1), we get

$$\operatorname{grad}\left[\lambda\nabla^2\varphi - \rho\frac{\partial^2\varphi}{\partial t^2}\right] = \mathbf{F}, \tag{6.3}$$

which implies that the body force must be the gradient of a scalar. This is a direct consequence of the impossibility of imposing shear stress on an inviscid fluid.

We write, therefore,

$$\mathbf{F} = -\operatorname{grad}\Omega \tag{6.4}$$

so that the wave equation becomes

$$\lambda\nabla^2\varphi - \rho\frac{\partial^2\varphi}{\partial t^2} = -\Omega. \tag{6.5}$$

(Any function of time which may appear in the integration is absorbed into φ.)

It is clear from this analysis that a point source can be defined either by specifying a jump in the value of φ or its derivative of any order across an element of surface, or by specifying $\Omega(\mathbf{x}, t)$.

The elementary point force system is given by

$$\Omega = d\delta(\mathbf{x}, \mathrm{O}), \tag{6.6}$$

where d is a function of time only. This corresponds to a dilatational source

$$F_i = -d\delta_{,i}(\mathbf{x}, \mathrm{O}). \tag{6.7}$$

More complex force systems are given by derivatives of the dilatational force.

In order to find the equivalent body forces for a given discontinuity we need to write down Green's formula for the fluid. In an unbounded fluid with a radiation condition at infinity it is (Stakgold 1968)

$$\varphi(\mathbf{y}, s) = \int_{-\infty}^{\infty} dt \int_V G(\mathbf{y}, s;\ \mathbf{x}, t)\,\Omega(\mathbf{x}, t)\,dV + \int_{-\infty}^{\infty} dt \int_S \lambda n_i \left\{G(\mathbf{y}, s;\ \mathbf{x}, t)\frac{\partial\varphi(\mathbf{x}, t)}{\partial x_i} - \varphi(\mathbf{x}, t)\frac{\partial}{\partial x_i} G(\mathbf{y}, s;\ \mathbf{x}, t)\right\} dS, \tag{6.8}$$

where V is the volume of fluid bounded internally by a surface S with outward normal $\mathbf{n}$, and $G(\mathbf{x}, t;\ \mathbf{y}, s)$ is Green's function satisfying the equation

$$\lambda \frac{\partial^2 G}{\partial x_i^2} - \rho \frac{\partial^2 G}{\partial t^2} = -\delta(\mathbf{x}, \mathbf{y})\,\delta(t-s) \tag{6.9}$$

with $G = 0$ for $t < s$.

We now follow Burridge & Knopoff (1964) in taking S to be the two faces of a surface Σ of discontinuity and rewrite equation (6.9) as

$$\varphi(\mathbf{y}, s) = \int_{-\infty}^{\infty} dt \int_V G(\mathbf{y}, s;\ \mathbf{x}, t) \Big\{ \Omega(\mathbf{x}, t) - \int_\Sigma \lambda \nu_i ([\varphi](\boldsymbol{\xi}, t)\,\delta_i(\mathbf{x}, \boldsymbol{\xi}) + [\varphi,_i](\boldsymbol{\xi}, t)\,\delta(\mathbf{x}, \boldsymbol{\xi}))\, d\Sigma \Big\}\, dV, \tag{6.10}$$

where $[\varphi](\boldsymbol{\xi}, t)$ and $[\varphi,_i](\boldsymbol{\xi}, t)$ are the discontinuities in φ and $\partial\varphi/\partial x_i$ at a point $\boldsymbol{\xi}$ of Σ measured in the direction of the normal $\boldsymbol{\nu}$.

This means that the discontinuities are equivalent to the body force $\mathbf{e}(\mathbf{x}, t)$ with potential

$$\chi(\mathbf{x}, t) = -\int_\Sigma \lambda \nu_i \{[\varphi](\boldsymbol{\xi}, t)\,\delta_i(\mathbf{x}, \boldsymbol{\xi}) + [\varphi, i](\boldsymbol{\xi}, t)\,\delta(\mathbf{x}, \boldsymbol{\xi})\}\, d\Sigma. \tag{6.11}$$

A discontinuity in φ across an element of Σ,

$$[\varphi](\mathbf{x}, t) = b\delta(x_1)\,\delta(x_2) \tag{6.12}$$

corresponds to a discontinuity in pressure,

$$\begin{aligned} [p](\mathbf{x}, t) &= -\rho \frac{\partial^2}{\partial t^2}\, [\varphi](\mathbf{x}, t) \\ &= -\rho \frac{\partial^2 b}{\partial t^2}\, \delta(x_1)\,\delta(x_2). \end{aligned} \tag{6.13}$$

It is equivalent to the body force given by

$$e_i(\mathbf{x}, t) = -b\lambda \nu_k \frac{\partial}{\partial x_k} \delta_i(\mathbf{x}, \mathrm{O}). \tag{6.14}$$

This expression is in the form of the derivative along the normal direction to Σ of a dilatational source (equation (6.7)). We now proceed to find an equivalent form of this force.

If a body force $\mathbf{F}$ were given by

$$F_i = -\lambda \frac{\partial^2 e_j}{\partial x_i \partial x_j} + \rho \frac{\partial^2 e_i}{\partial t^2} \tag{6.15}$$

where $\mathbf{e}(\mathbf{x}, t)$ is a function which is bounded in space, the solution to equation (5.2) would be

$$\mathbf{u} = \mathbf{e} \tag{6.16}$$

and there would be no elastic radiation. Therefore, a body force of $\lambda \partial^2 e_j/\partial x_i \partial x_j$ is equivalent to one of $\rho \dfrac{\partial^2 e_i}{\partial t^2}$.

We now take

$$e_i = \nu_i b \delta(\mathbf{x}, \mathrm{O}) \tag{6.17}$$

and so the two body force systems

$$-\lambda b v_k \frac{\partial^2}{\partial x_k \partial x_i} \delta(\mathbf{x}, \mathrm{O}) \quad \text{and} \quad -\rho v_i \delta(\mathbf{x}, \mathrm{O}) \frac{\partial^2 b}{\partial t^2} \tag{6.18}$$

are equivalent. Hence, the equivalent body force to the discontinuity in φ, given in equation (6.12), is

$$e_i(\mathbf{x}, t) = -\rho v_i \frac{\partial^2 b}{\partial t^2} \delta(\mathbf{x}, \mathrm{O}). \tag{6.19}$$

Therefore, comparing this with equation (6.13), we see that a unit discontinuity in pressure is equivalent to a unit simple force acting in the direction of the normal to the surface of discontinuity.

This is the same result as achieved in the case of an elastic solid modified to take account of the fact that there are no shear stresses in the fluid and so discontinuities in the normal component of stress $(-p)$ only are allowed.

A discontinuity in displacement

$$\begin{aligned}[u_i](\mathbf{x}, t) &= -[\varphi, i](\mathbf{x}, t) \\ &= a_i \delta(x_1) \delta(x_2)\end{aligned} \tag{6.20}$$

is equivalent to the body force

$$e_i(\mathbf{x}, t) = -\lambda v_k a_k \delta_i(\mathbf{x}, \mathrm{O}). \tag{6.21}$$

This is a dilatational source; i.e. the same result as in the case of a solid with the shear modulus μ zero.

It is clear from equation (6.21) that the discontinuity in the normal component only of displacement $(v_k a_k)$ contributes to elastic radiation.

Discontinuities in the higher derivatives of φ across an element of surface are specified uniquely by $[\varphi]$ and $[\varphi, i]$. This can be shown in the same way as in the case of a solid by using the wave equation (6.5) with zero body force. In fact, the most general discontinuity is specified when values are given to $[\varphi]$ (or $[p]$) and $[\varphi,_v]$ $(= -[\mathbf{u}.\mathbf{v}])$ since the other components of $[\nabla\varphi]$ are given by tangential derivatives of $[\varphi]$.

The most general point body force is given by derivatives of the dilatational force (equation (6.7)). The dilatational source itself is equivalent to a point discontinuity in normal displacement across a surface. The derivative of this normal to the surface is equivalent to a discontinuity in pressure.

Other derivatives of the basic point source are given by derivatives tangential to the surface of discontinuity, by the use of the equivalence of the two-forces systems (6.18). Therefore, they are equivalent to tangential derivatives of the two basic discontinuities mentioned above.

Hence the most general discontinuity and the most general point force system can be specified by giving values to $[\varphi]$ and $[\varphi,_v]$.

We shall now show that a set of discontinuities across an element of surface of arbitrary orientation is equivalent to discontinuities across a horizontal plane $(x_3 = 0)$.

Discontinuities

$$\left.\begin{aligned}[\varphi](\mathbf{x}, t) &= b\delta(x_1)\delta(x_2) \\ [\varphi,_v](\mathbf{x}, t) &= v_j[\varphi,_j](\mathbf{x}, t) \\ &= a\delta(x_1)\delta(x_2)\end{aligned}\right\} \tag{6.22}$$

are equivalent to a body force with potential

$$\chi(\mathbf{x}, t) = -\lambda\{b v_i \delta_i(\mathbf{x}, \mathrm{O}) + a\delta(\mathbf{x}, \mathrm{O})\}. \tag{6.23}$$

This is equivalent to discontinuities

$$\left.\begin{aligned}[\varphi](\mathbf{x}, t) &= B\delta(x_1)\,\delta(x_2)\\ [\varphi,_3](\mathbf{x}, t) &= \left[A + A_j \frac{\partial}{\partial x_j}\right]\delta(x_1)\,\delta(x_2), \qquad j = 1, 2\end{aligned}\right\} \tag{6.24}$$

across the plane $x_3 = 0$ if the body force potential

$$-\lambda\{B\delta_3(\mathbf{x}, \mathrm{O}) + A_j\,\delta_j(\mathbf{x}, \mathrm{O}) + A\delta(\mathbf{x}, \mathrm{O})\} \tag{6.25}$$

is equal to that given by equation (6.23).

This implies that

$$\left.\begin{aligned}A &= a \qquad & B &= bv_3\\ A_1 &= bv_1 \qquad & A_2 &= bv_2.\end{aligned}\right\} \tag{6.26}$$

Therefore, by this method, we can represent the general point source, body force or discontinuity, by equivalent discontinuities in φ and $\varphi,_3$ across the horizontal plane $x_3 = 0$.

The discontinuities $[U_z^c]$ and $[T_z^c]$ in the components of the stress–motion vector are found by equations (2.3) and (2.4) in terms of a discontinuity in normal displacement

$$[u_z] = -[\varphi,_3] \tag{6.27}$$

and normal stress

$$\begin{aligned}[\tau_{zz}] &= -[p]\\ &= \rho \frac{\partial^2}{\partial t^2}[\varphi]\end{aligned} \tag{6.28}$$

across a horizontal plane.

The source vector for a dilatational source is the same as that given by Haskell (1964) for a solid. The only non-zero source vector for a unit simple force in the vertical direction $\big(f = (0, 0, 1)\big)$, however, is

$$\mathbf{S}_l^{0c} = \begin{pmatrix} 0 \\ 0 \\ -k/2\pi\omega^2 \\ 0 \end{pmatrix}, \tag{6.29}$$

where k and ω are the transform variables.

7. Source models for an earthquake

The spatial extent of an earthquake is not in general small compared with other typical lengths in the problem (Båth & Duda 1964). Therefore, we must deal with sources of finite size rather than point sources. This is most easily done by integrating the theoretical results from the Thomson–Haskell theory for a delta-function source.

Ben Menahem (1961) calculated the radiation from a simple force acting on a line of finite length which moved over a rectangle at constant speed. Knopoff & Gilbert (1960) extended the idea of a point source to a model where point discontinuities in displacement and stress moved along a line at constant speed. The single and double couple sources of Hirasawa & Stauder (1965) were assumed to move across a rectangle in a similar way to Ben Menahem's model.

With regard to the mechanism of the source, the dislocation or transverse slip model has lately aroused the most support. Ben Menahem has preferred the equivalent force system, the double couple, in his later work (e.g. Ben Menahem, Smith &

Teng 1965). It would seem that cracking or slipping along a surface is a probable explanation of the majority of shallow earthquakes, if not of deep earthquakes.

The hydrostatic pressure acting on the rock probably rules out any normal displacement of the faces of the fault except at the surface itself. Therefore, transverse slip or the double couple source seems to be the most likely source of earth tremors.

The most convincing model of an earthquake is that proposed by Savage (1966). Transverse slip is assumed to originate at a point and travel outwards in a plane at constant velocity. It is finally arrested at the edge of an ellipse so that the fault, after rupture, is elliptic in shape.

We shall generalize this idea slightly. Consider a fault plane Σ with normal $\mathbf{n}$. We assume that transverse slip occurs on this plane in a uniform direction $\mathbf{f}(\mathbf{n}.\mathbf{f} = 0)$ of magnitude $F(\xi_1, \xi_2, t)$, where (ξ_1, ξ_2) are rectangular co-ordinates in the fault plane.

$$[u_i] = f_i\, F(\xi_1, \xi_2, t). \tag{7.1}$$

The function F will be zero outside an area whose shape may be arbitrary chosen. The magnitude of slip as a function of position and time as well as the fault shape are subjects which are studied in the theory of cracks and which need a fairly deep understanding of the properties of rocks under pressure (see, for instance, Burridge 1969).

The model is equivalent to a double couple of magnitude F. Therefore, the Haskell (1964) source vector may be used in calculating the radiation and the result for a point source integrated over the co-ordinates (ξ_1, ξ_2) of points of Σ.

8. Source models for explosions

(a) *Underground explosions*

The behaviour of the medium in the neighbourhood of an underground explosion can be regarded as perfectly elastic outside a certain sphere surrounding the source. If the explosion were contained in an infinite homogeneous medium, the displacements in the elastic region would be spherically symmetric and would have a radial component u_R only.

In order to satisfy the equations of motion, u_R takes the form

$$u_R = \frac{\partial}{\partial R}\left[\frac{\psi(t-R/\alpha)}{R}\right], \tag{8.1}$$

where R is the radial distance from the centre of the sphere. Experimental measurements of the magnitude and shape of the potential function ψ for nuclear explosions of various yields and in various media are available in the literature (e.g. Werth & Herbst 1963).

The displacement may be thought of as being due to a dilatational point force $\mathbf{e}$ acting at the centre of the sphere,

$$e_i(\mathbf{x}, t) = P(t)\,\delta_i(\mathbf{x}, O) \tag{8.2}$$

where $R = (x_1{}^2+x_2{}^2+x_3{}^2)^{\frac{1}{2}}$ and P is some function of time.

Alternatively, $\mathbf{e}$ may be thought of as having a radial component only

$$e_R(\mathbf{x}, t) = -P(t)\frac{\partial}{\partial R}\left\{\frac{\delta(R)}{4\pi R^2}\right\}. \tag{8.3}$$

The equations of motion give

$$\nabla^2(\psi/R) - \frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}(\psi/R) = \frac{P(t)}{\alpha^2\rho}\cdot\frac{\delta(R)}{4\pi R^2} \tag{8.4}$$

whose solution in an unbounded medium is (Love 1903)

$$\frac{\psi(t-R/\alpha)}{R} = \frac{-1}{16\pi^2\alpha^2\rho}\iiint \frac{\delta(R')\,P\left(t-\dfrac{|\mathbf{x}-\mathbf{x}'|}{\alpha}\right)}{|\mathbf{x}-\mathbf{x}'|(R')^2}(R')^2\,dR'\,d\Omega. \tag{8.5}$$

The integration covers all points $\mathbf{x}'$ in space ($R' = |\mathbf{x}'|$) and $d\Omega$ is an element of solid angle.

Therefore,

$$\frac{\psi(t-R/\alpha)}{R} = \frac{-P(t-R/\alpha)}{4\pi R\alpha^2\rho} \tag{8.6}$$

or

$$P(t) = -4\pi\alpha^2\rho\psi(t). \tag{8.7}$$

This means that if the explosion is approximately spherically symmetric into the elastic region we may use the Thomson–Haskell theory for a point dilatational source of magnitude $-4\pi\alpha^2\rho\psi(t)$ to calculate the radiation. The method will not apply if the region in which the laws of infinitesimal perfect elasticity do not hold extends into inhomogeneities of structure or to the surface itself.

(*b*) *Underwater explosions*

If the non-linear region of disturbance due to the explosion does not break surface, nor extend into the lower solid region, the analysis used above is applicable. We define a potential function (equation (8.1))

$$\mathbf{u} = \mathrm{grad}\,\{\psi(t-R/\alpha)/R\} \tag{8.8}$$

which is related to the magnitude P of the equivalent dilatational source by equation (8.7).

Experimental data is given in terms of the pressure,

$$p(R,t) = -\frac{\rho}{R}\frac{\partial^2\psi(t-R/\alpha)}{\partial t^2}. \tag{8.9}$$

If we use data in the form

$$\mathscr{P}(t-R/\alpha) = Rp(R,t) \tag{8.10}$$

then we have the relation

$$\mathscr{P}(t) = -\rho\frac{\partial^2\psi(t)}{\partial t^2}. \tag{8.11}$$

The magnitude of the dilatational source which represents the explosion is given in terms of the data by

$$P(t) = 4\pi\alpha^2\int_0^t\int_0^{\tau'}\mathscr{P}(\tau)\,d\tau\,d\tau'. \tag{8.12}$$

(*c*) *Atmospheric explosions*

The effect of an atmospheric explosion on a layered structure below is to create a moving pressure pulse which spreads out symmetrically from a point directly below the centre of the explosion (see Glasstone 1964).

Wave propagation in the solid medium will probably be linear so long as the fireball does not touch the surface.

The pressure pulse may be represented in the Thomson–Haskell theory as a moving discontinuity in normal stress acting at zero depth;

$$[\tau_{zz}] = -p(r, t), \tag{8.13}$$

where p is the pressure and r is the radial distance on the surface away from the centre of symmetry.

This is equivalent to a moving normal force (see Section 5) acting in the downward direction $(\mathbf{f} = (0, 0, 1))$ of magnitude $p(r, t)$.

We may use the Thomson–Haskell theory to find the response to such a disturbance by integrating over the response to a series of point sources.

9. Conclusions

Simple models of earthquake and explosive sources can be represented in terms of simple force systems or discontinuities acting at a point or over an extended area.

Magnitudes of explosive sources can be calculated from experimental data, but the details of the earthquake model need further hypotheses concerning the properties of the surrounding rock.

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