A quantum geometric model of similarity

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Abstract
No other study has had as great an impact on the development of the similarity literature as that of Tversky (1977), which provided compelling demonstrations against all the fundamental assumptions of the popular, and extensively employed, geometric similarity models. Notably, similarity judgments were shown to violate symmetry and the triangle inequality, and also be subject to context effects, so that the same pair of items would be rated differently, depending on the presence of other items. Quantum theory provides a generalized geometric approach to similarity and can address several of Tversky's (1997) main findings. Similarity is modeled as quantum probability, so that asymmetries emerge as order effects, and the triangle equality violations and the diagnosticity effect can be related to the context-dependent properties of quantum probability. We so demonstrate the promise of the quantum approach for similarity and discuss the implications for representation theory in general.

Keywords: similarity, metric axioms, symmetry, triangle inequality, diagnosticity, quantum probability

I. Introduction

The notion of similarity is, in equal measure, a famous hero and a notorious villain in psychology. Across most areas of psychology, similarity plays a fundamental role (e.g., Goldstone, 1994; Pothos, 2005; Sloman & Rips, 1998), but equally its various formalizations have been the source of much criticism and debate (e.g., Goodman, 1972). A popular approach to similarity is a geometric one, according to which stimuli/ exemplars/ concepts are represented as points in a multidimensional psychological space, with similarity being a function of distance in that space. This geometric approach is exemplified in Shepard's (1987) famous law of generalization, according to which similarity is an exponentially decaying function of distance, and is heavily used in influential cognitive models of categorization, such as exemplar and prototype theory. It is fair to say that cognitive psychology cannot resist using a geometric approach to similarity.

This reliance on the geometric approach to similarity is surprising because it has been subject to intense, and, in some cases, highly compelling criticisms. The most complete and impactful expression of this criticism is that of Tversky (1977). Tversky's work has had a profound influence on the development of the similarity literature (over 2,200 citations), partly because his objections to geometric similarity models concern the most basic properties of such models – the metric axioms, that is, the fundamental properties that any similarity measure based on distance must obey. Thus, if the metric axioms are shown to be inconsistent with psychological similarity, then any distance model of similarity is essentially incorrect. Tversky's (1977) demonstration is a rare one, in that he has been able to convincingly argue against an entire modeling framework, rather than particular models. This is because his arguments were not dependent on e.g. particular parametric configurations, rather they concerned the fundamental properties of *any* model of similarity based on distance in psychological space (though see Nosofsky, 1991, for a parametric way to produce an asymmetric distance-based similarity measure). It is not surprising that Tversky's (1977) demonstrations have come to be accepted as the golden standard of key results any successful similarity model should cover (Ashby & Perrin, 1988; Bowdle & Gentner, 1997; Goldstone & Son, 2005; Krumhansl, 1978).

In brief, Tversky (1977) showed that similarity judgments violate minimality (identical objects are not always judged to be maximally similar), symmetry (the similarity of A to B can be different from that of B to A), and the triangle inequality (the distance between two points is always shorter directly, than via a third point). Moreover, he showed that the similarity between the same two objects can be affected by which other objects are present (called the diagnosticity effect). In the typical tradition of his work, part of the reason why his findings have had the influence they did is because they go against basic logic. For example, concerning his most famous result, violations of symmetry, if similarity is determined by distance, then how could it be the case that the similarity/ distance between two objects depends on the order in which they are considered? Yet, when he asked participants to choose between the statements 'China is similar to Korea' vs. 'Korea is similar to China' (actually North Korea and Red China, but for simplicity we will just talk about Korea and China'), 66 out of 69 participants selected the latter statement as more agreeable, implying that the similarity of Korea to China (denoted as S*im*(Korea, China)) is higher than that of China to Korea (denoted as *Sim* (China, Korea)). Thus, this result provided a compelling (and *retrospectively* intuitive) violation of symmetry in similarity. Tversky employed several other pairs of countries, stimuli from other domains, and alternative procedures (see also Bowdle & Gentner, 1997, Catrambone, Beike, & Niedenthal, 1996, Holyoak & Gordon, 1983, Op de Beeck, Wagemans, & Vogels, 2003, Ortony et al., 1985, and Rosch, 1975). Note that some researchers have questioned the reality of asymmetries in similarity. For example, Gleitman et al. (1996) suggested that in directional similarity statements, we cannot assume that, e.g., Korea gives rise to the same representation in the target position, as it does in the referent position. But Gleitman et al.'s (1996) analysis cannot explain why it is more intuitive to place, e.g., Korea in the referent, as opposed to the target, position, the absence of asymmetries in some cases (Aguilar & Medin, 1999), and the demonstration of similarity asymmetries with non-linguistic measures (Hodgetts & Hahn, 2012).

We will present what can be labeled a *quantum similarity* model. Quantum probability (QP) theory is a theory for how to assign probabilities to events (for more refined characterizations see e.g. Aerts & Gabora, 2005; Atmanspacher, Romer, & Wallach, 2006; Busemeyer & Bruza, 2012; Khrennikov, 2010). QP theory is a geometric theory of probability. It is analogous to classical probability theory, though QP theory and classical theory are founded from different sets of axioms (the Kolmogorov and Dirac/ von Neumann axioms respectively) and so are subject to alternative constraints. QP theory is based on linear algebra, augmented with a range of assumptions and theorems (such as the Kochen-Specker theorem and Gleason's theorem; Busemeyer & Bruza, 2012; Hughes, 1989; Isham, 1989; Khrennikov, 2010). Note that a quantum approach to cognitive modeling does not introduce assumptions regarding neural implementation and we are agnostic on this issue. Specifically, operations which are quantum-like can emerge at the computational level from a classical brain (Atmanspacher & beim Graben, 2007) and do not assume quantum neural computations (this latter thesis is very controversial; Hameroff, 2007; Litt et al., 2006).

A unique feature of the quantum similarity model is that, whereas previous models would equate objects with individual points or distributions of points, in the quantum model, objects are entire subspaces of potentially very high dimensionality. This is an important generalization of geometric models of similarity, as it leads to a naturally asymmetric similarity measure.

The quantum similarity model follows the recent interest in the application of quantum probability (QP) theory to cognitive modeling. Applications of QP theory have been presented in decision making (Blutner et al., in press; Busemeyer, Wang, & Townsend, 2006; Busemeyer et al., 2011; Bordley, 1998; Lambert-Mogiliansky, Zamir, & Zwirn, 2009; Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011; Wang & Busemeyer, in press; Yukalov & Sornette, 2010), conceptual combination (Aerts, 2009; Aerts & Gabora, 2005; Blutner, 2008; Bruza et al., under review), memory (Bruza, 2010; Bruza et al., 2009), and perception (Atmanspacher, Filk, & Romer, 2004). Psychological models based on quantum probability seem to work well (for overviews see Busemeyer & Bruza, 2009; Bruza et al., 2009; Khrennikov, 2004; Pothos & Busemeyer, in press) and add to the increasing realization that the application of QP need not be restricted to physics. For example, QP has also been applied to areas as diverse as economics (Baaquie, 2004) and information theory (Nielsen & Chuang, 2010).

We first present QP theory and motivate our similarity model. Subsequently, we consider three main results from Tversky (1977). Violations of symmetry, violations of the triangle inequality, and the diagnosticity effect. Violations of symmetry provide and most compelling and intuitive evidence against (simple) geometric representational models. Moreover, the diagnosticity effect is obviously impossible to reconcile with similarity models based on distance alone, as it shows that similarity judgments between the same two elements might be affected by the presence of other elements. Note, we do not consider violations of minimality, i.e., the finding that naïve observers do not always assign the maximum similarity rating for pairs of identical stimuli. Violations of minimality can simply be explained by noise in the system, so that the same stimulus presented twice would lead to slightly different internal representations. Violations of minimality have been typically demonstrated in confusability experiments, whereby participants have to decide whether two consecutively presented stimuli are identical or not. But, lack of identity judgments for identical stimuli can be explained if the time course of sampling stimulus information is stochastic and, moreover, it is straightforward to couple minimality violations and stimulus complexity (Lamberts, 2000; Nosofsky & Palmeri, 1997).

II. QP theory and geometric similarity

Representation in QP theory is based on a multidimensional space, in which different subspaces correspond to different entities. The current state of the system is described by a vector in this space (the knowledge state vector). Projecting the state onto different subspaces and computing the squared length of the projected vector tells us about the consistency between the state vector and these other entities in the quantum space. We present the quantum model in three steps. First, we outline the relevant elements of quantum theory. Second, we discuss the assumptions for how operations in quantum theory can be employed to provide a model of psychological similarity. Third, we briefly consider some prior general motivating considerations and criticisms. Finally, each empirical situation we consider involves some specific information. In each corresponding section, we discuss how this information can be incorporated in the quantum model.

Main elements of quantum theory

We require a psychologically realistic quantum space, which represents all the knowledge of a person. Therefore, this knowledge space can have a very high dimensionality (potentially infinite [Footnote 1]). The state vector, one dimensional subspace, this means that we can have a greater range of thoughts for China, than for Korea, which is equivalent to assuming that we have greater knowledge for China than for Korea.

The representation of China as a subspace is consistent with the idea that properties are not uniquely chained to particular concepts. For example, suppose that my current thought is

Furthermore, if we assume that the China concept is represented by a subspace spanned by vectors process of thinking about Korea (subject, mentioned first) and then China (object, mentioned second), which corresponds to projections, though note that eventually this process must break down (there must be a limit to how many proximal items can impact on a decision).

As before, the link with probability justifies the choice of judgments. This equivalence predicts that similarity findings (such as the diagnosticity effect) may well have analogues in decision-making situations (cf. Roe et al., 2001).

Specific aspects of the present proposal can be related to prior proposals. The squared distance between two unit length vectors *X*, *Y* in a psychological space is given by

The prototypical experimental finding we are interested in is that

Tversky (1977) considered how similarity judgments can lead to violations of the triangle inequality, another one of the metric axioms. The triangle inequality can be expressed as *Distance* (A, B) < *Distance* (A, C) + *Distance* (C, B). If we equate distance with dissimilarity, and assume similarity is the negative of dissimilarity, then the triangle inequality states that *Dissimilarity* (A, B) would always be less than *Dissimilarity* (A, C) plus *Dissimilarity* (C, B) or that *Similarity* (A, B) would always be greater than *Similarity* (A, C) plus *Similarity* (C, B). Tversky (1977) reported an example where the latter relation is violated. Consider A=Russia and B=Jamaica, so that *Similarity* (A, B) = *Similarity* (Russia, Jamaica) is low. Consider also C=Cuba. But, *Similarity* (A, C) = *Similarity* (Russia, Cuba) is high (because of political affiliation) and *Similarity* (C, B) = *Similarity* (Cuba, Jamaica) is also high (because of geographical proximity). Thus, Tversky's example shows that *Similarity* (Russia, Jamaica) < *Similarity* (Russia, Cuba) + *Similarity* (Cuba, Jamaica), which suggests a violation of the triangle inequality (Figure 3). Note that this demonstration does not depend on an assumption that one country is more salient than the others.

If one employs an exponentially decaying function to link distance and similarity (e.g., Nosofsky, 1984; Shepard, 1987), then similarities can violate the triangle inequality, even if the underlying distances obey the triangle inequality. For example, consider *Distance* (A,B)=5 units, *Distance* (A,C)=4 units, and *Distance* (C,B)=4 units; these distances obey the triangle inequality. For the similarities to follow Tversky's results we need that *Similarity*(A,B) < *Similarity*(A,C) + *Similarity*(C,B), and this relation is obtained from

Caribbean one, but far from the Communist one (Figure 4). Thus, different properties for the three countries are implied by proximity to the subspaces corresponding to these properties. Also, the two countries which are near the Communist subspace would both have large projections onto the Communist subspace and, therefore, they would be similar to each other by virtue of the fact that they both share the property of Communism.

Note that, as we are dealing with subspaces of the same dimensionality, any similarity computations here are insensitive to order (one order would not systematically produce a higher similarity than the opposite order, with an unbiased initial state, when averaging across a random sample of projector pairs). Note also that, as before, we require that the state vector (the activated thoughts just prior to the similarity comparisons) is set up so that guide the similarity between two concepts. Thus, in comparing Russia and Cuba, the Communism feature is invoked. However, this mechanism is underspecified: why is only the Communism feature invoked? Why not also features relating to trade ties or similar political leaders? In fact, there is an infinite number of possible features one could invoke. We are just re-expressing Goodman's (1972) concerns regarding the arbitrariness of similarity, though we do not think the problem is with similarity, rather it has to do with guessing features. The quantum model avoids this problem: proximity in psychological space *implies* the existence of common features, but identifying these features is not required for the similarity computation.

-------------------FIGURES 4, 5 ABOUT HERE--------------------

V. Diagnosticity effect

This finding concerns the context dependence of similarity relations. Tversky (1977) employed a forced choice similarity task, whereby participants were asked to decide which country was most similar to Austria, amongst a set of candidate choices. Performance in such a task clearly depends on the pairwise similarity between the target country, Austria, and each of the candidate countries. Equally, each pairwise similarity may be affected by the alternative choices. Indeed, when the candidate choices were Sweden, Hungary, and Poland, participants tended to select Sweden as most similar to Austria (49% of participants favored this choice). When the candidate choices were Sweden, Norway, and Hungary, participants, tended to select Hungary as most similar to Austria (60% of participants favored this choice; analogous demonstrations were provided with schematic stimuli). The exact task Tversky (1977) employed involved presenting two groups of participants with 20 sets of four countries. Participants were required to choose, for each set of four countries, the country in the set most similar to a target country. Regarding the critical comparisons between matched quadruples of countries (e.g., Austria, Sweden, Hungary, Poland vs. Austria, Sweden, Norway, Hungary), the design was a between-participants one. We consider how, in the set of countries Hungary, Poland, Sweden, and Austria, Sweden ends up being most similar to Austria, since the situation involving Hungary, Sweden, Norway, and Austria is entirely analogous.

Tversky's (1977) explanation for the diagnosticity effect was partly the idea that a diagnostic feature of Eastern vs. Western Europe emerges, and it is this feature that makes Sweden more similar to Austria, than Hungary and Poland. Because this approach is popular in the literature, we note that it can be expressed in quantum terms. Suppose there is an Austria ray in a 3D space, which is equidistant to Hungary/Poland rays and a Sweden ray. A suitable 2D space corresponding to the Eastern vs. Western feature could be identified, such that, when expressing the Sweden, Austria, Hungary, Poland rays with the basis of that subspace, the Sweden ray becomes most similar to the Austria one. But, per our discussion for the triangle inequality, the idea of emergent features is underspecified. For example, what determines the particular diagnostic features that emerge, why would there be only one pair of (Eastern, Western Europe) diagnostic features, and how does the similarity between the countries in the comparison moderate the emergence of diagnostic features? Therefore this explanation is questionable, and the findings demand an alternative approach.

Tversky (1977) selected his materials so that two options are grouped together (Hungary, Poland) and these are approximately equally similar to the target (Austria) as the third option (Sweden). Representing these concepts in a quantum knowledge space and computing similarity in a way that takes into account the relevant context leads to a diagnosticity effect.

We seek the country which maximizes the similarity with the target, that is, the one for which

It is straightforward to provide a computational illustration of the effect. Figure 5 shows a plausible geometrical arrangement for Austria, Sweden, Poland, and Hungary. As we did for the Russia-Cuba-Jamaica example, rather than directly compute an initial state vector which leads to the same projection for all four countries (that is, an initial state vector which is not biased towards any of the countries), it is simpler to assume that make the remaining alternative more similar to the target country. But, as discussed, an explanation relying on diagnostic features can have problems.

In the quantum model, context corresponds to successive projections between the context elements. When the context elements are grouped together (as for Hungary, Poland), projecting across them leads to little loss of amplitude of the state vector, so that the similarity judgment ends up being higher. When there is no grouping across any of the possible contexts, then the effect of context is simply to uniformly scale the similarity judgments. So, context can make the same similarity comparison appear higher or lower, depending exactly on the grouping of the context elements. Finally, a grouping of alternatives means that there are subspaces along which these alternatives have a high projection, that is, properties or features that are common to some alternatives, but not others. So, overall, the intuition for how the quantum model produces the diagnosticity effect is not much different from that of Tversky's (1977). But, in Tversky's (1977) model it has to be assumed that diagnostic features are invoked, as a result of the grouping, while in the quantum model, the diagnosticity effect emerges directly from the presence of a grouping.

-------------------FIGURES 6,7 ABOUT HERE--------------------

VI. Conclusions and future directions

The objective of this paper was to generalize the notion of geometric representations. In the quantum proposal, the representation of an object need not be restricted to a single vector in a multidimensional psychological space, rather it can be a subspace of arbitrary dimensionality. The QP framework was developed to do exactly this, that is, associate knowledge with subspaces. The idea of representations as subspaces allows us to capture the intuition that a concept is the span of all the thoughts produced by combinations of the basic features that form the basis for the concept. Such relevant thoughts can include a central tendency, individual relevant instances, and properties. Moreover, the insight that concepts are about relevant thoughts, as well as instances, prototypes, etc., has been often expressed in discussions on representation and similarity (cf. Fodor, 1983; Murphy & Medin, 1985), but particular schemes for formalization have been lacking. Finally, the proposal for similarity is that this involves a process of thinking of the first and then the second of the compared entities. We were so able to cover some key empirical results: the basic violation of symmetry and the triangle inequality (Tversky, 1977) and the diagnosticity effect (Tversky, 1977).

One challenge for future work is to expand the range of empirical issues considered and motivate novel empirical demonstrations. For example, violations of symmetry can also arise from differences in perceptual salience or even frequency (Polk et al., 2002) and similarity judgments sometimes reflect correspondence between the parts of the compared stimuli (Larkey & Love, 2003; Markman & Gentner, 1993). More generally, a theoretical link between similarity and subjective probability has been the basis of influential ideas (Medin, Goldstone, & Markman, 1995; Shafir, Smith, & Osherson, 1990; Sloman, 1993; Tversky & Kahneman, 1983). Quantum theory provides a way to formalize this link. For example, in directional probability judgment tasks, one might seek to induce asymmetries in the consideration of predicates, analogous to the ones observed in similarity judgments.

The present emphasis was on the mathematical specification of the quantum model. One challenge for future work is detailed comparisons with alternative similarity models. We make some preliminary comments, in relation to symmetry. Tversky's (1977) own contrast model is that

documents and other documents etc., usually in terms of the cosines of corresponding angles. A key difference between the LSA and the quantum similarity model is that in the former all documents, regardless of extent or complexity, would still be represented as single vectors. In the context of LSA applications, does it matter if more extensive documents are represented in a way equivalent to that of shorter ones? A related issue is that the similarity metric in LSA is symmetric, so the method would fare poorly with Tversky's (1977) key results (but see Griffiths, Steyvers, & Tenenbaum, 2007, for a generalization of these ideas, in a way allowing violations of the metric axioms). Again, one can ask whether empirical results in LSA application reveal any asymmetries or not. Moreover, LSA provides a data-driven mechanism for creating representations. If this work can be adapted to the specification of subspaces, instead of individual vectors, then this would enable a major development in the quantum similarity model.

It is natural to compare applications of quantum theory to psychology with ones of classical probability (CP) theory, since both are general, formal frameworks for assigning probabilities to events. Classical models have had an enormous influence in psychological theory (Chater, Tenenbaum, Yuille, 2006; Griffiths et al., 2010; Oaksford & Chater, 2007; Tenenbaum et al, 2011) and many researchers recognize the appeal of the kind of psychological explanation provided by CP models. Researchers interested in the application of QP theory in cognition aim to develop models with the same general characteristics as CP models. Specifically, quantum models aspire to cognitive explanations emphasizing the nature of representations, the operations on these representations, and the identification of the computational biases which guide cognitive process (Griffiths et al., 2010). Because of the sequential nature of projection in quantum theory, it will perhaps be easier to extend quantum models to include process assumptions, than it is generally the case for CP models (Jones & Love, 2011).

Quantum and classical probability theories arise from sharply different axiomatic foundations and so quantum theory has many unique characteristics, which have no analogue in classical theory. Notably, in quantum theory, computation can be order and context dependent and states are often superposition states, relative to the outcomes of a question. A superposition state vector cannot be said to possess a specific value for any of these possible outcomes and possible outcomes may interfere with each other, as the state vector develops in time. These features of quantum theory have enabled probabilistic models for situations which have been puzzling from a classical perspective (Aerts, 2009; Atmanspacher et al., 2004; Blutner, 2008; Bruza, 2010; Busemeyer & Bruza, 2012; Khrennikov, 2004; Yukalov & Sornette, 2010). A general difference between quantum and classical theories is that the latter require that there is always a complete joint probability distribution for all the questions relevant for a system (this is the principle of unicity; Griffiths, 2003). We suggest that such a requirement is psychologically unrealistic and, rather, the perspective dependent nature of calculation in quantum theory provides a more plausible framework for cognitive modeling (e.g., Pothos & Busemeyer, in press).

In this vein, regarding the quantum similarity model, its foremost characteristic is its sensitivity to the order and context of evaluating projections (and so similarities). There is a growing realization that analogous order effects are common in psychology, which recommends further study of QP cognitive models. Indeed, researchers interested in cognitive modeling have been increasingly employing computational machinery from this important class of models (Aerts & Gabora, 2005; Atmanspacher, Filk, & Romer, 2004; Bruza, Busemeyer, & Gabora, 2009; Bruza, Busemeyer, & Gabora, 2009; Busemeyer, Wang, & Townsend, 2006; Busemeyer et al., 2011; Franco, 2009; Khrennikov, 2004; Pothos & Busemeyer, 2009; Van Rijsbergen, 2004). The present work adds to this effort. As discussed, important challenges remain. We hope that the current analyses will motivate the application of the model in more specific problems and its further elaboration.

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Footnotes

Footnote 1. Infinite dimensional spaces are routinely employed in QP theory. Such spaces, when subject to certain completeness properties (e.g., that the infinite sum

Appendix 1: Computing the initial state vector (

ψ), so that it does not bias the similarity comparison towards China or Korea, when assessing the similarity between the two. This requires that

<u>Appendix 2</u>: An analytic examination of the quantum model for the Korea-China example, in the simple case in which Korea is a one-dimensional subspace and China a two-dimensional one.

Let us assume that the Korea subspace is spanned by

Appendix 3: Analytic computations for the series of projections relevant to the demonstration for Tversky's (1977) diagnosticity effect.

We first consider the problem of identifying the country most similar to Austria, from the set of Sweden, Poland, Hungary (this was Tversky's original experiment for the diagnosticity effect).
Let us consider, for example,

Note, finally, that the model assumes that the Austria ray is approximately halfway the Poland/Hungary rays and the Sweden one. That is, we also have

Figure captions

Figure 1. An illustration of the idea of projection.

Figure 2. The figure illustrates how a successive projection from a ray and then to a plane will preserve more amplitude than a successive projection to a plane and then to a ray. The first projection is assumed to retain the same amount of amplitude, regardless of whether it is to a ray or to a plane.

Figure 3. Tversky's (1977) example of when human similarity judgments can violate the triangle inequality, assuming that dissimilarity is some linear function of distance in a psychological space. The diagram implies that Similarity(Russia, Jamaica)> Similarity (Russia, Cuba) + Similarity (Cuba, Jamaica), but in practice the opposite is true.

Figure 4. A geometric, two-dimensional representation of the information in Tversky's Russia-Cuba-Jamaica example. The triangle inequality requires that *Dissimilarity* (Russia, Cuba) + *Dissimilarity* (Cuba, Jamaica) > *Dissimilarity* (Russia, Jamaica). The black perforated lines illustrate how the corresponding projections allow for *Similarity* (Russia, Cuba) (green) + *Similarity* (Cuba, Jamaica) (yellow) > *Similarity* (Russia, Jamaica) (blue), thus violating the triangle inequality.

Figure 5. Illustrating a model for Tversky's (1977) diagnosticity effect. The top panel shows the series of projections in

Figure 1.

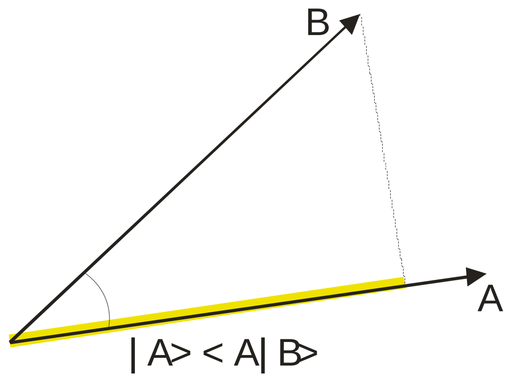


Figure 2.

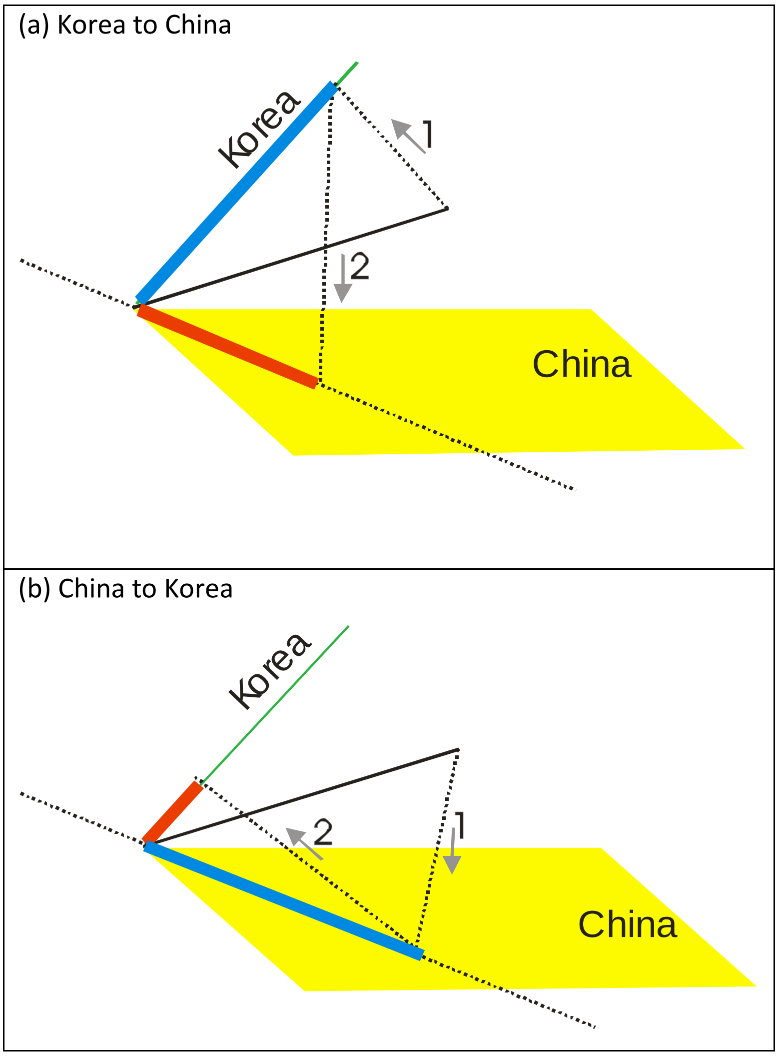

(a) Korea to China
Korea
1
2
China
(b) China to Korea
Korea
2
1
China

Figure 3.

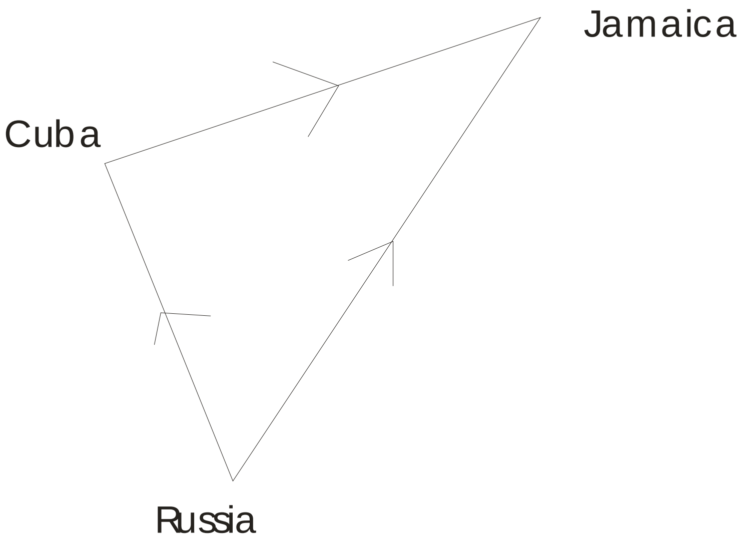


Figure 4.

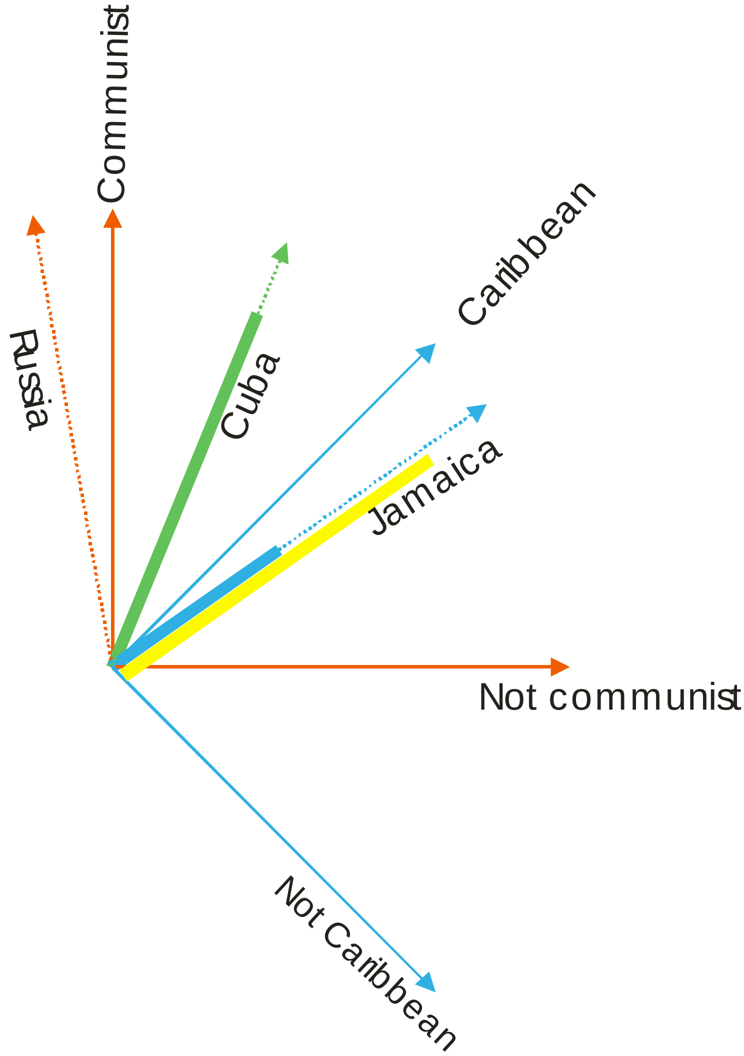

Communist
Russia
Cuba
Caribbean
Jamaica
Not communist
Not Caribbean

Figure 5.

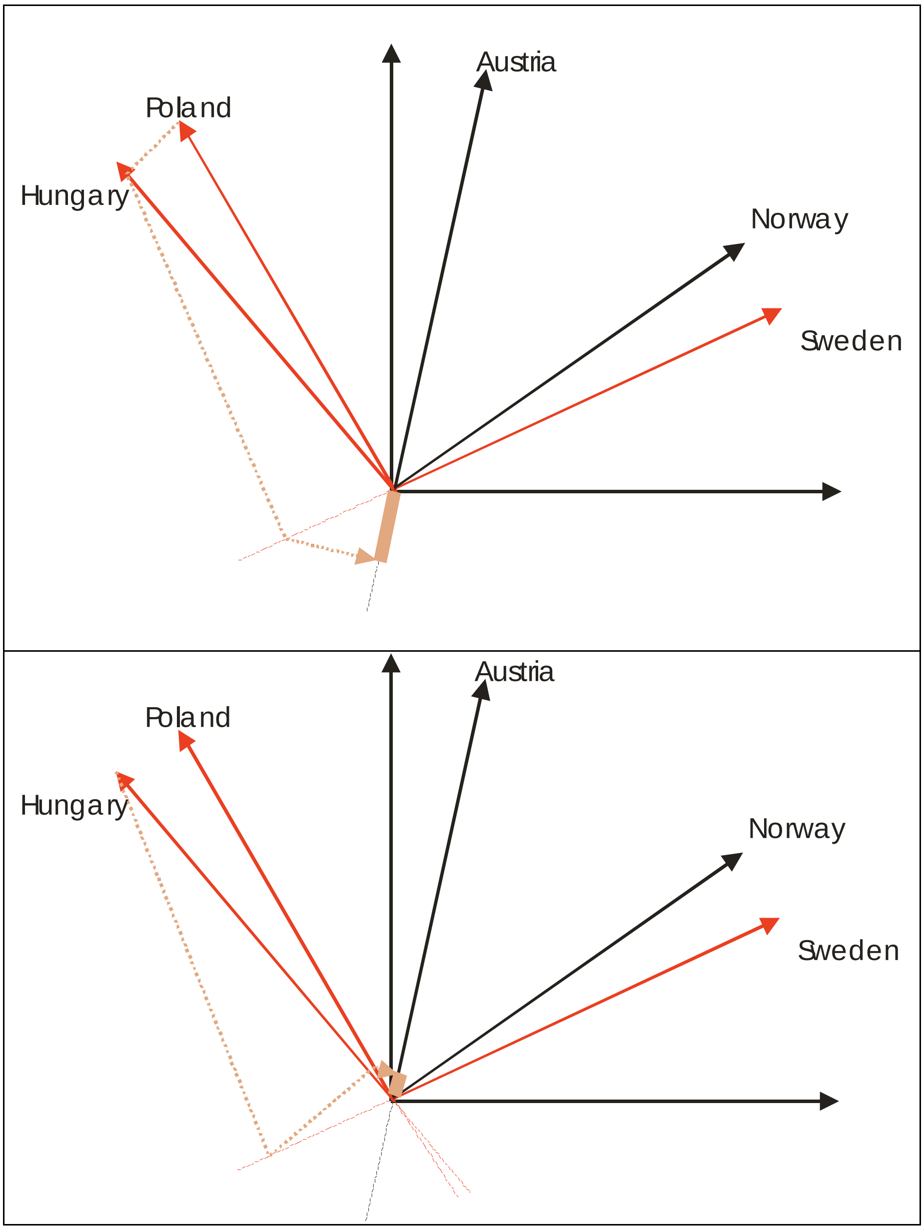
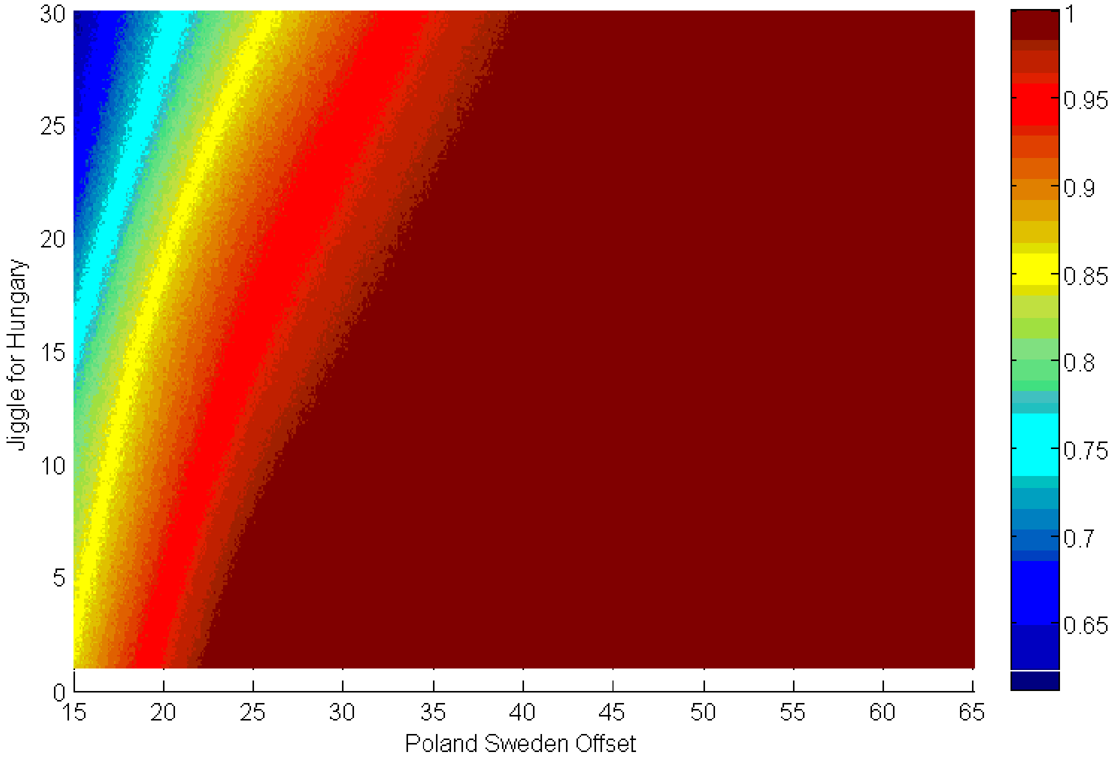

Austria
Poland
Hungary
Norway
Sweden
Austria
Poland
Hungary
Norway
Sweden

Figure 6.


Jiggle for Hungary
Poland Sweden Offset
0
5
10
15
20
25
30
15
20
25
30
35
40
45
50
55
60
65
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65