Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2013, Article ID 637375, 20 pages
http://dx.doi.org/10.1155/2013/637375

Research Article

A Quantum Mermin-Wagner Theorem for a Generalized Hubbard Model

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Received 20 March 2013; Accepted 14 May 2013

Academic Editor: Christian Maes

This paper is the second in a series of papers considering symmetry properties of bosonic quantum systems over 2D graphs, with continuous spins, in the spirit of the Mermin-Wagner theorem. In the model considered here the phase space of a single spin is $\mathscr{H}_1 = \mathrm{L}_2(M)$, where M is a d-dimensional unit torus $M = \mathbb{R}^d/\mathbb{Z}^d$ with a flat metric. The phase space of k spins is $\mathscr{H}_k = \mathrm{L}_2^{\rm sym}(M^k)$, the subspace of $\mathrm{L}_2(M^k)$ formed by functions symmetric under the permutations of the arguments. The Fock space $\boldsymbol{\mathscr{H}} = \oplus_{k=0,1,\dots}\mathscr{H}_k$ yields the phase space of a system of a varying (but finite) number of particles. We associate a space $\boldsymbol{\mathscr{H}} \simeq \boldsymbol{\mathscr{H}}(i)$ with each vertex $i \in \Gamma$ of a graph $(\Gamma, \mathscr{E})$ satisfying a special bidimensionality property. (Physically, vertex i represents a heavy "atom" or "ion" that does not move but attracts a number of "light" particles.) The kinetic energy part of the Hamiltonian includes (i) $-\Delta/2$, the minus a half of the Laplace operator on M, responsible for the motion of a particle while "trapped" by a given atom, and (ii) an integral term describing possible "jumps" where a particle may join another atom. The potential part is an operator of multiplication by a function (the potential energy of a classical configuration) which is a sum of (a) one-body potentials $U^{(1)}(x)$, $x \in M$, describing a field generated by a heavy atom, (b) two-body potentials $U^{(2)}(x, y)$, $x, y \in M$, showing the interaction between pairs of particles belonging to the same atom, and (c) two-body potentials $V(x, y)$, $x, y \in M$, scaled along the graph distance $\mathtt{d}(i, j)$ between vertices $i, j \in \Gamma$, which gives the interaction between particles belonging to different atoms. The system under consideration can be considered as a generalized (bosonic) Hubbard model. We assume that a connected Lie group $\mathtt{G}$ acts on M, represented by a Euclidean space or torus of dimension $d' \le d$, preserving the metric and the volume in M. Furthermore, we suppose that the potentials $U^{(1)}, U^{(2)}$, and V are $\mathtt{G}$-invariant. The result of the paper is that any (appropriately defined) Gibbs states generated by the above Hamiltonian is $\mathtt{G}$-invariant, provided that the thermodynamic variables (the fugacity z and the inverse temperature β) satisfy a certain restriction. The definition of a Gibbs state (and its analysis) is based on the Feynman-Kac representation for the density matrices.

1. Introduction

1.1. Basic Facts on Bi-Dimensional Graphs. As in [1], we suppose that the graph $(\Gamma, \mathscr{E})$ has been given, with the set of vertices Γ and the set of edges $\mathscr{E}$. The graph has the property that whenever edge $(j', j'') \in \mathscr{E}$, the reversed edge (j'', j') belongs to $\mathscr{E}$ as well. Furthermore, graph $(\Gamma, \mathscr{E})$ is without multiple edges and has a bounded degree; that is, the number of edges (j, j') with a fixed initial or terminal vertex is uniformly bounded:

$$\sup\left[\max\left(\sharp\left\{j' \in \Gamma : \left(j, j'\right) \in \mathscr{E}\right\},\right.\right. \\ \left.\left.\sharp\left\{j' \in \Gamma : \left(j', j\right) \in \mathscr{E}\right\}\right) : j \in \Gamma\right] < \infty. \tag{1}$$

The bi-dimensionality property is expressed in the bound

$$0 < \sup\left[\frac{1}{n}\sharp\Sigma(j,n) : j \in \Gamma,\ n = 1,2,\ldots\right] < \infty, \quad (2)$$

where $\Sigma(j,n)$ stands for the set of vertices in Γ at the graph distance n from $j \in \Gamma$ (a sphere of radius n around j):

$$\Sigma(j,n) = \left\{j' \in \Gamma : \mathtt{d}\left(j,j'\right) = n\right\}. \quad (3)$$

(The graph distance $\mathtt{d}(j,j') = \mathtt{d}_{\Gamma,\mathscr{E}}(j,j')$ between $j, j' \in \Gamma$ is determined as the minimal length of a path on $(\Gamma,\mathscr{E})$ joining j and j'.) This implies that for any $o \in \Gamma$ the cardinality $\sharp\Lambda(o,n)$ of the ball

$$\Lambda(o,n) = \left\{j' \in \Gamma : \mathtt{d}\left(o,j'\right) \le n\right\} \quad (4)$$

grows at most quadratically with n.

A justification for putting a quantum system on a graph can be that graph-like structures become increasingly popular in rigorous Statistical Mechanics, for example, in quantum gravity. Namely, see [2–4]. On the other hand, a number of properties of Gibbs ensembles do not depend upon "regularity" of an underlying spatial geometry.

1.2. A Bosonic Model in the Fock Space. With each vertex $i \in \Gamma$ we associate a copy of a compact manifold M which we take in this paper to be a unit d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ with a flat metric ρ and the volume ν. We also associate with $i \in \Gamma$ a bosonic Fock-Hilbert space $\mathscr{H}(i) \simeq \mathscr{H}$. Here $\mathscr{H} = \oplus_{k=0,1,\ldots}\mathscr{H}_k$ where $\mathscr{H}_k = \mathrm{L}_2^{\mathrm{sym}}(M^k)$ is the subspace in $\mathrm{L}_2(M^k)$ formed by functions symmetric under a permutation of the variables. Given a finite set $\Lambda \subset \Gamma$, we set $\mathscr{H}(\Lambda) = \otimes_{i\in\Lambda}\mathscr{H}(i)$. An element $\boldsymbol{\phi} \in \mathscr{H}(\Lambda)$ is a complex function:

$$\mathbf{x}_\Lambda^* \in M^{*\Lambda} \longmapsto \boldsymbol{\phi}\left(\mathbf{x}_\Lambda^*\right). \quad (5)$$

Here $\mathbf{x}_\Lambda^*$ is a collection $\{\mathbf{x}^*(j),\ j \in \Lambda\}$ of finite point sets $\mathbf{x}^*(j) \subset M$ associated with sites $j \in \Lambda$. Following [1], we call $\mathbf{x}^*(j)$ a particle configuration at site j (which can be empty) and $\mathbf{x}_\Lambda^*$ a particle configuration in, or over, Λ. The space $M^{*\Lambda}$ of particle configurations in Λ can be represented as the Cartesian product $(M^*)^{\times\Lambda}$ where M^* is the disjoint union $\bigcup_{k=0,1,\ldots} M^{(k)}$ and $M^{(k)}$ is the collection of (unordered) k-point subsets of M. (One can consider $M^{(k)}$ as the factor of the "off-diagonal" set $M_{\neq}^k$ in the Cartesian power M^k under the equivalence relation induced by the permutation group of order k.) The norm and the scalar product in $\mathscr{H}_\Lambda$ are given by

$$\|\boldsymbol{\phi}\| = \left(\int_{M^{*\Lambda}} \left|\boldsymbol{\phi}(\mathbf{x}_\Lambda^*)\right|^2 \mathrm{d}\mathbf{x}_\Lambda^*\right)^{1/2},$$
$$\left\langle \boldsymbol{\phi}_1,\boldsymbol{\phi}_2\right\rangle = \int_{M^{*\Lambda}} \boldsymbol{\phi}_1\left(\mathbf{x}_\Lambda^*\right)\overline{\boldsymbol{\phi}_2\left(\mathbf{x}_\Lambda^*\right)}\mathrm{d}\mathbf{x}_\Lambda^*, \quad (6)$$

where measure $\mathrm{d}\mathbf{x}_\Lambda^*$ is the product $\times_{j\in\Lambda}\mathrm{d}\mathbf{x}^*(j)$ and $\mathrm{d}\mathbf{x}^*(j)$ is the Poissonian sum measure on M^*:

$$\mathrm{d}\mathbf{x}^*(j) = \sum_{k=0,1,\ldots} \mathbf{1}\left(\sharp\mathbf{x}^*(j) = k\right)\frac{1}{k!}\prod_{x\in\mathbf{x}^*(j)} \mathrm{d}\nu(x)\, e^{-\nu(M)}. \quad (7)$$

Here $\nu(M)$ is the volume of torus M.

As in [1], we assume that an action

$$(\mathtt{g},x) \in \mathtt{G}\times M \longmapsto \mathtt{g}x \in M \quad (8)$$

is given, of a group $\mathtt{G}$ that is a Euclidean space or a torus of dimension $d' \le d$. The action is written as

$$\mathtt{g}x = x + \theta A \bmod 1. \quad (9)$$

Here vector $\theta = (\theta_1,\ldots,\theta_{d'})$ with components $\theta_l \in [0,1)$ and θA is the d-dimensional vector $\underline{\theta} = ((\theta A)_1,\ldots,(\theta A)_d)$ representing the element $\mathtt{g}$, where A is a $(d'\times d)$ matrix of column rank d' with rational entries. The action of $\mathtt{G}$ is lifted to unitary operators $\mathbf{U}_\Lambda(\mathtt{g})$ in $\mathscr{H}_\Lambda$:

$$\mathbf{U}_\Lambda(\mathtt{g})\,\boldsymbol{\phi}\left(\mathbf{x}_\Lambda^*\right) = \boldsymbol{\phi}\left(\mathtt{g}^{-1}\mathbf{x}_\Lambda^*\right), \quad (10)$$

where $\mathtt{g}^{-1}\mathbf{x}_\Lambda^* = \{\mathtt{g}^{-1}\mathbf{x}^*(j),\ j\in\Lambda\}$ and $\mathtt{g}^{-1}\mathbf{x}^*(j) = \{\mathtt{g}^{-1}x,\ x \in \mathbf{x}^*(j)\}$.

The generally accepted view is that the Hubbard model is a highly oversimplified model for strongly interacting electrons in a solid. The Hubbard model is a kind of minimum model which takes into account quantum mechanical motion of electrons in a solid, and nonlinear repulsive interaction between electrons. There is little doubt that the model is too simple to describe actual solids faithfully [5]. In our context the Hubbard Hamiltonian $\mathbf{H}_\Lambda$ of the system in Λ acts as follows:

$$\begin{aligned}
\left(\mathbf{H}_\Lambda\boldsymbol{\phi}\right)\left(\mathbf{x}_\Lambda^*\right) = \Bigg[& -\frac{1}{2}\sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)}\Delta_j^{(x)} + \sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)} U^{(1)}(x) \\
& + \frac{1}{2}\sum_{j\in\Lambda}\sum_{x,x'\in\mathbf{x}^*(j)} \mathbf{1}\left(x\neq x'\right)U^{(2)}\left(x,x'\right) \\
& + \frac{1}{2}\sum_{j,j'\in\Lambda}\mathbf{1}\left(j\neq j'\right)J\left(\mathtt{d}\left(j,j'\right)\right) \\
& \times \sum_{x\in\mathbf{x}^*(j),x'\in\mathbf{x}^*(j')} V\left(x,x'\right)\Bigg]\boldsymbol{\phi}\left(\mathbf{x}_\Lambda^*\right) \\
& + \sum_{j,j'\in\Lambda}\lambda_{j,j'}\mathbf{1}\left(\sharp\mathbf{x}^*(j)\ge 1, \sharp\mathbf{x}^*\left(j'\right) < \kappa\right) \\
& \times \sum_{x\in\mathbf{x}^*(j)}\int_M \nu(\mathrm{d}y) \\
& \times\left[\boldsymbol{\phi}\left(\mathbf{x}_\Lambda^{*(j,x)\to(j',y)}\right) - \boldsymbol{\phi}\left(\mathbf{x}_\Lambda^*\right)\right].
\end{aligned} \quad (11)$$

Here $\Delta_j^{(x)}$ means the Laplacian in variable $x \in \mathbf{x}^*(j)$. Next, $\sharp\mathbf{x}^*$ stands for the cardinality of the particle configuration $\mathbf{x}^*$ (i.e., $\sharp\mathbf{x}^* = k$ when $\mathbf{x}^* \in M^{(k)}$), and the parameter κ is introduced in (17). (Symbol $\sharp$ will be used for denoting the cardinality of a general (finite) set; for example, $\sharp\Lambda$ means the number of vertices in Λ.) Further, $\mathbf{x}_\Lambda^{*(j,x)\to(j',y)}$ denotes the particle configuration with the point $x \in \mathbf{x}^*(j)$ removed and point y added to $\mathbf{x}^*(j')$.

As in [1], we also consider a Hamiltonian $\mathbf{H}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ in an external field generated by a configuration $\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda} = \{\overline{\mathbf{x}}^*(j'), j' \in \overline{\Gamma}\setminus\Lambda\} \in M^{*\overline{\Gamma}\setminus\Lambda}$ where $\overline{\Gamma} \subseteq \Gamma$ is a (finite or infinite) collection of vertices. More precisely, we only consider $\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}$ with $\sharp\overline{\mathbf{x}}^*(j') \le \kappa$ (see (17) below) and set

$$\begin{aligned}
&\left(\mathbf{H}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}\phi\right)(\mathbf{x}_\Lambda^*) \\
&\quad = \Bigg[-\frac{1}{2}\sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)}\Delta_j^{(x)} + \sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)}U^{(1)}(x) \\
&\qquad + \frac{1}{2}\sum_{j\in\Lambda}\sum_{x,x'\in\mathbf{x}^*(j)}\mathbf{1}(x\neq x')\,U^{(2)}(x,x') \\
&\qquad + \frac{1}{2}\sum_{j,j'\in\Lambda}\mathbf{1}(j\neq j')\,J(\mathrm{d}(j,j')) \\
&\qquad \times \sum_{x\in\mathbf{x}^*(j),x'\in\mathbf{x}^*(j')}V(x,x')\Bigg]\phi(\mathbf{x}_\Lambda^*) \\
&\quad + \sum_{j\in\Lambda,j'\in\overline{\Gamma}\setminus\Lambda}J(\mathrm{d}(j,j')) \\
&\quad \times \sum_{x\in\mathbf{x}^*(j),\overline{x}'\in\overline{\mathbf{x}}^*(j')}V(x,\overline{x}')\,\phi(\mathbf{x}_\Lambda^*) \\
&\quad + \sum_{j,j'\in\Lambda}\lambda_{j,j'}\mathbf{1}\left(\sharp\mathbf{x}^*(j)\ge 1,\ \sharp\mathbf{x}^*(j')<\kappa\right) \\
&\quad \times \sum_{x\in\mathbf{x}^*(j)}\int_M \nu(\mathrm{d}y)\left[\phi\left(\mathbf{x}_\Lambda^{*(j,x)\to(j',y)}\right) - \phi(\mathbf{x}_\Lambda^*)\right].
\end{aligned} \tag{12}$$

The novel elements in (11) and (12) compared with [1] are the presence of on-site potentials $U^{(1)}$ and $U^{(2)}$ and the summand involving transition rates $\lambda_{j,j'} \ge 0$ for jumps of a particle from site j to j'.

We will suppose that $\lambda_{j,j'}$ vanishes if the graph distance $\mathrm{d}(j,j') > 1$. We will also assume uniform boundedness:

$$\sup\left[\lambda_{j,j'}(x,M),\ j,j'\in\Gamma,\ x\in M\right] < \infty; \tag{13}$$

in view of (1) it implies that the total exit rate $\sum_{j':\mathrm{d}(j,j')=1}\lambda_{j,j'}(x,M)$ from site j is uniformly bounded. These conditions are not sharp and can be liberalized.

The model under consideration can be considered as a generalization of the Hubbard model [6] (in its bosonic version). Its mathematical justification includes the following. (a) An opportunity to introduce a Fock space formalism incorporates a number of new features. For instance, a fermonic version of the model (not considered here) emerges naturally when the bosonic Fock space $\mathscr{H}(i)$ is replaced by a fermonic one. Another opening provided by this model is a possibility to consider random potentials $U^{(1)}$, $U^{(2)}$ and V which would yield a sound generalization of the Mott-Anderson model. (b) Introducing jumps makes a step towards a treatment of a model of a quantum (Bose-) gas where particles "live" in a single Fock space. For example, a system of interacting quantum particles is originally confined to a "box" in a Euclidean space, with or without "internal" degrees of freedom. In the thermodynamical limit the box expands to the whole Euclidean space. In a two-dimensional model of a quantum gas one expects a phenomenon of invariance under space-translations; one hopes to be able to address this issue in future publications. (c) A model with jumps can be analysed by means of the theory of Markov processes which provides a developed methodology.

Physically speaking, the model with jumps covers a situation where "light" quantum particles are subject to a "random" force and change their "location." This class of models is interesting from the point of view of transport phenomena that they may display. (An analogy with the famous Anderson model, in its multiparticle version, inevitably comes to mind; cf., e.g., [7].) Methodologically, such systems occupy an "intermediate" place between models where quantum particles are "fixed" forever to their designated locations (as in [1]) and models where quantum particles move in the same space (a Bose-gas, considered in [8, 9]). In particular, this work provides a bridge between [1, 8, 9]; reading this paper ahead of [8, 9] might help an interested reader to get through [8, 9] at a much quicker pace.

We would like to note an interesting problem of analysis of the small-mass limit (cf. [10]) from the point of Mermin-Wagner phenomena.

1.3. Assumptions on the Potentials. The between-sites potential V is assumed to be of class C^2. Consequently, V and its first and second derivatives satisfy uniform bounds. Namely, $\forall x', x'' \in M$

$$-V(x',x''),\ \left|\nabla_{\mathrm{x}}V(x',x'')\right|,\ \left|\nabla_{\mathrm{x},\mathrm{x}'}V(x',x'')\right| \le \overline{V}. \tag{14}$$

Here x and x' run through the pairs of variables x, x'. A similar property is assumed for the on-site potential $U^{(1)}$ (here we need only a C^1 smoothness):

$$-U^{(1)}(x),\ \left|\nabla_{\mathrm{x}}U^{(1)}(x)\right| \le \overline{U}^{(1)}, \quad x \in M. \tag{15}$$

Note that for V and $U^{(1)}$ the bounds are imposed on their negative parts only.

As to $U^{(2)}$, we suppose that (a)

$$U^{(2)}(x,x') = +\infty \quad \text{when } |x-x'| \le \rho, \tag{16}$$

and (b) $\exists$ a C^1-function $(x,x') \mapsto \widetilde{U}^{(2)}(x,x') \in \mathbb{R}$ such that $U^{(2)}(x,x') = \widetilde{U}^{(2)}(x,x')$ whenever $\rho(x,x') > \rho$. Here $\rho(x,x')$ stands for the (flat) Riemannian distance between points $x, x' \in M$. As a result of (16), there exists a "hard core" of diameter ρ, and a given atom cannot "hold" more than

$$\kappa = \left[\frac{\nu(M)}{\nu(B(\rho))} \right] \quad (17)$$

particles where $\nu(B(\rho))$ is the volume of a d-dimensional ball of diameter ρ. We will also use the bound

$$-\widetilde{U}^{(2)}(x,x'), \left|\nabla_x \widetilde{U}^{(2)}(x,x')\right| \le \overline{U}^{(2)}, \quad x,x' \in M. \quad (18)$$

Formally, (16) means that the operators in (11) and (12) are considered for functions $\phi(\mathbf{x}^*_\Lambda)$ vanishing when in the particle configuration $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(j),\ j \in \Lambda\}$, the cardinality $\sharp\mathbf{x}^*(j) > \kappa$ for some $j \in \Lambda$.

The function $J : r \in (0,\infty) \mapsto J(r) \ge 0$ is assumed monotonically nonincreasing with r and obeying the relation $\overline{J}(l) \to 0$ as $l \to \infty$, where

$$\overline{J}(l) = \sup \left[\sum_{j'' \in \Gamma} J\left(\mathtt{d}(j',j'')\right) \mathbf{1}\left(\mathtt{d}(j',j'') \ge l\right) : j' \in \Gamma \right] < \infty. \quad (19)$$

Additionally, let $J(r)$ be such that

$$J^* = \sup \left[\sum_{j' \in \Gamma} J\left(\mathtt{d}(j,j')\right) \mathtt{d}(j,j')^2 : j \in \Gamma \right] < \infty. \quad (20)$$

Next, we assume that the functions $U^{(1)}$, $U^{(2)}$, and V are $\mathtt{g}$-invariant: $\forall x, x' \in M$ and $\mathtt{g} \in \mathtt{G}$,

$$\begin{aligned} U^{(1)}(x) &= U^{(1)}(\mathtt{g}x), \\ U^{(2)}(x,x') &= U^{(2)}(\mathtt{g}x,\mathtt{g}x'), \\ V(x,x') &= V(\mathtt{g}x,\mathtt{g}x'). \end{aligned} \quad (21)$$

In the following we will need to bound the fugacity (or activity, cf. (25)) z in terms of the other parameters of the model

$$ze^{\Theta} < 1, \quad \text{where } \Theta = \kappa\beta\left(\overline{U}^{(1)} + \kappa\overline{U}^{(2)} + \kappa\overline{J}(1)\overline{V}\right). \quad (22)$$

1.4. The Gibbs State in a Finite Volume. Define the particle number operator $\mathbf{N}_\Lambda$, with the action

$$\mathbf{N}_\Lambda \phi(\mathbf{x}^*_\Lambda) = \sharp\mathbf{x}^*_\Lambda \phi(\mathbf{x}^*_\Lambda), \quad \mathbf{x}^*_\Lambda \in M^{*\Lambda}. \quad (23)$$

Here, for a given $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(j),\ j \in \Lambda\}$, $\sharp\mathbf{x}^*_\Lambda$ stands for the total number of particles in configuration $\mathbf{x}^*_\Lambda$:

$$\sharp\mathbf{x}^*_\Lambda = \sum_{j \in \Lambda} \sharp\mathbf{x}^*(j). \quad (24)$$

The standard canonical variable associated with $\mathbf{N}_\Lambda$ is activity $z \in (0,\infty)$.

The Hamiltonians (11) and (12) are self-adjoint (on the natural domains) in $\mathscr{H}(\Lambda)$. Moreover, they are positive definite and have a discrete spectrum, cf. [14]. Furthermore, $\forall z, \beta > 0$, $\mathbf{H}_\Lambda$ and $\mathbf{H}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ give rise to positive-definite trace-class operators $\mathbf{G}_\Lambda = \mathbf{G}_{z,\beta,\Lambda}$ and $\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}} = \mathbf{G}_{z,\beta,\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$:

$$\begin{aligned} \mathbf{G}_\Lambda &= z^{\mathbf{N}_\Lambda} \exp\left[-\beta\mathbf{H}_\Lambda\right], \\ \mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}} &= z^{\mathbf{N}_\Lambda} \exp\left[-\beta\mathbf{H}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\right]. \end{aligned} \quad (25)$$

We would like to stress that the full range of variables $z, \beta > 0$ is allowed here because of the hard-core condition (16): it does not allow more than $\kappa\sharp\Lambda$ particles in Λ where $\sharp\Lambda$ stands for the number of vertices in Λ. However, while passing to the thermodynamic limit, we will need to control z and β.

Definition 1. We will call $\mathbf{G}_\Lambda$ and $\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ the Gibbs operators in volume Λ, for given values of z and β (and—in the case of $\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$—with the boundary condition $\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}$).

The Gibbs operators in turn give rise to the Gibbs states $\boldsymbol{\varphi}_\Lambda = \boldsymbol{\varphi}_{\beta,z,\Lambda}$ and $\boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}_{\Gamma\setminus\Lambda}} = \boldsymbol{\varphi}_{\beta,z,\Lambda|\overline{\mathbf{x}}_{\Gamma\setminus\Lambda}}$, at temperature β^{-1} and activity z in volume Λ. These are linear positive normalized functionals on the C^*-algebra $\mathfrak{B}_\Lambda$ of bounded operators in space $\mathscr{H}_\Lambda$:

$$\begin{aligned} \boldsymbol{\varphi}_\Lambda(\mathbf{A}) &= \mathrm{tr}_{\mathscr{H}_\Lambda}(\mathbf{R}_\Lambda\mathbf{A}), \\ \boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}_{\Gamma\setminus\Lambda}}(\mathbf{A}) &= \mathrm{tr}_{\mathscr{H}_\Lambda}\left(\mathbf{R}_{\Lambda|\overline{\mathbf{x}}_{\Gamma\setminus\Lambda}}\mathbf{A}\right), \quad \mathbf{A} \in \mathfrak{B}_\Lambda, \end{aligned} \quad (26)$$

where

$$\mathbf{R}_\Lambda = \frac{\mathbf{G}_\Lambda}{\Xi(\Lambda)}, \qquad \text{with } \Xi(\Lambda) = \Xi_{z,\beta}(\Lambda) = \mathrm{tr}_{\mathscr{H}_\Lambda}\mathbf{G}_\Lambda, \quad (27)$$

$$\begin{aligned} \mathbf{R}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}} &= \frac{\mathbf{G}_{\Lambda|\overline{\mathbf{x}}_{\Gamma\setminus\Lambda}}}{\Xi\left(\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)}, \\ &\text{with } \Xi\left(\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) = \Xi_{z,\beta}\left(\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) \\ &= \mathrm{tr}_{\mathscr{H}_\Lambda}\left(z^{\mathbf{N}_\Lambda} \exp\left[-\beta\mathbf{H}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\right]\right). \end{aligned} \quad (28)$$

The hard-core assumption (16) yields that the quantities $\Xi(\Lambda)$ and $\Xi_{z,\beta}(\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ are finite; formally, these facts will be verified by virtue of the Feynman-Kac representation.

Definition 2. Whenever $\Lambda^0 \subset \Lambda$, the C^*-algebra $\mathfrak{B}_{\Lambda^0}$ is identified with the C^* subalgebra in $\mathfrak{B}_\Lambda$ formed by the operators of the form $\mathbf{A}_0 \otimes \mathbf{I}_{\Lambda\setminus\Lambda^0}$. Consequently, the restriction $\boldsymbol{\varphi}^{\Lambda^0}_\Lambda$ of state $\boldsymbol{\varphi}_\Lambda$ to C^*-algebra $\mathfrak{B}_{\Lambda^0}$ is given by

$$\boldsymbol{\varphi}^{\Lambda^0}_\Lambda(\mathbf{A}_0) = \mathrm{tr}_{\mathscr{H}_{\Lambda^0}}\left(\mathbf{R}^{\Lambda^0}_\Lambda\mathbf{A}_0\right), \quad \mathbf{A}_0 \in \mathfrak{B}_{\Lambda^0}, \quad (29)$$

where

$$\mathbf{R}^{\Lambda^0}_\Lambda = \mathrm{tr}_{\mathscr{H}_{\Lambda\setminus\Lambda^0}}\mathbf{R}_\Lambda. \quad (30)$$

Operators $\mathbf{R}_\Lambda^{\Lambda^0}$ (we again call them RDMs) are positive definite and have $\mathrm{tr}_{\mathscr{H}_{\Lambda^0}}\mathbf{R}_\Lambda^{\Lambda^0}=1$. They also satisfy the compatibility property: $\forall\Lambda^0\subset\Lambda^1\subset\Lambda$,

$$\mathbf{R}_\Lambda^{\Lambda^0}=\mathrm{tr}_{\mathscr{H}_{\Lambda^1\setminus\Lambda^0}}\mathbf{R}_\Lambda^{\Lambda^1}. \tag{31}$$

In a similar fashion one defines functionals $\boldsymbol{\varphi}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ and operators $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$, with the same properties.

1.5. Limiting Gibbs States. The concept of a limiting Gibbs state is related to notion of a quasilocal C^*-algebra; see [14]. For the class of systems under consideration, the construction of the quasilocal C^*-algebra $\mathfrak{B}_\Gamma$ is done along the same lines as in [1]: $\mathfrak{B}_\Gamma$ is the norm completion of the C^* algebra $(\mathfrak{B}^0_\Gamma)=\lim\,\mathrm{ind}_{n\to\infty}\mathfrak{B}_{\Lambda_n}$. Any family of positive-definite operators $\mathbf{R}^{\Lambda^0}$ in spaces $\mathscr{H}_{\Lambda^0}$ of trace one, where Λ^0 runs through finite subsets of Γ, with the compatibility property

$$\mathbf{R}^{\Lambda^1}=\mathrm{tr}_{\mathscr{H}_{\Lambda^0\setminus\Lambda^1}}\mathbf{R}^{\Lambda^0},\quad \Lambda^1\subset\Lambda^0, \tag{32}$$

determines a state of $\mathfrak{B}_\Gamma$, see [12, 13].

Finally, we introduce unitary operators $\mathbf{U}_{\Lambda^0}(\mathtt{g})$, $\mathtt{g}\in\mathtt{G}$, in $\mathscr{H}_{\Lambda^0}$:

$$\mathbf{U}_\Lambda\phi\left(\mathbf{x}^*_{\Lambda^0}\right)=\phi\left(\mathtt{g}^{-1}\mathbf{x}^*_{\Lambda^0}\right), \tag{33}$$

where

$$\begin{gathered}\mathtt{g}^{-1}\mathbf{x}^*_{\Lambda^0}=\left\{\mathtt{g}^{-1}\mathbf{x}^*(j),\,j\in\Lambda^0\right\},\\ \mathtt{g}^{-1}\mathbf{x}^*(j)=\left\{\mathtt{g}^{-1}x:x\in\mathbf{x}^*(j)\right\}.\end{gathered} \tag{34}$$

Theorem 3. *Assuming the conditions listed above, for all $z,\beta\in(0,+\infty)$ satisfying* (22) *and a finite $\Lambda^0\subset\Gamma$, operators $\mathbf{R}_\Lambda^{\Lambda^0}$ form a compact sequence in the trace-norm topology in $\mathscr{H}_{\Lambda^0}$ as $\Lambda\nearrow\Gamma$. Furthermore, given any family of (finite or infinite) sets $\overline{\Gamma}=\overline{\Gamma}(\Lambda)\subseteq\Gamma$ and configurations $\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}=\{\overline{\mathbf{x}}^*(i),\ i\in\overline{\Gamma}\setminus\Lambda\}$ with $\sharp\overline{\mathbf{x}}^*(i)<\kappa$, operators $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}_{\overline{\Gamma}\setminus\Lambda}}$ also form a compact sequence in the trace-norm topology. Any limit point, $\mathbf{R}^{\Lambda^0}$, for $\{\mathbf{R}_\Lambda^{\Lambda^0}\}$ or $\{\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}\}$ as $\Lambda\nearrow\Gamma$, is a positive-definite operator in $\mathscr{H}(\Lambda^0)$ of trace one. Moreover, if $\Lambda^1\subset\Lambda^0$ and $\mathbf{R}^{\Lambda^0}$ and $\mathbf{R}^{\Lambda^1}$ are the limits for $\mathbf{R}_\Lambda^{\Lambda^0}$ and $\mathbf{R}_\Lambda^{\Lambda^1}$ or for $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ and $\mathbf{R}^{\Lambda^1}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ along the same subsequence $\Lambda_s\nearrow\Gamma$, then the property* (32) *holds true.*

Consequently, the Gibbs states $\boldsymbol{\varphi}_\Lambda$ and $\boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ form compact sequences as $\Lambda\nearrow\Gamma$.

Remark 4. In fact, the assertion of Theorem 3 holds without assuming the bidimensionality condition on graph $(\Gamma,\mathscr{E})$, only under an assumption that the degree of the vertices in Γ is uniformy bounded.

Definition 5. Any limit point $\boldsymbol{\varphi}$ for states $\boldsymbol{\varphi}_\Lambda$ and $\boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ is called a limiting Gibbs state (for given $z,\beta\in(0,+\infty)$).

Theorem 6. *Under the condition* (22)*, any limiting point, $\mathbf{R}^{\Lambda^0}$, for $\{\mathbf{R}_\Lambda^{\Lambda^0}\}$ or $\{\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}\}$, as $\Lambda\nearrow\Gamma$, is a positive-definite operator of trace one commuting with operators $\mathbf{U}_{\Lambda^0}(\mathtt{g})$: $\forall\mathtt{g}\in\mathtt{G}$,*

$$\mathbf{U}_{\Lambda^0}(\mathtt{g})^{-1}\mathbf{R}^{\Lambda^0}\mathbf{U}_{\Lambda^0}(\mathtt{g})=\mathbf{R}^{\Lambda^0}. \tag{35}$$

Accordingly, any limiting Gibbs state $\boldsymbol{\varphi}$ of $\mathfrak{B}$ determined by a family of limiting operators $\mathbf{R}^{\Lambda^0}$ obeying (35) *satisfies the corresponding invariance property: $\forall$ finite $\Lambda^0\subset\Gamma$, any $\mathbf{A}\in\mathfrak{B}_{\Lambda^0}$, and $\mathtt{g}\in\mathtt{G}$,*

$$\boldsymbol{\varphi}(\mathbf{A})=\boldsymbol{\varphi}\left(\mathbf{U}_{\Lambda^0}(\mathtt{g})^{-1}\mathbf{A}\mathbf{U}_{\Lambda^0}(\mathtt{g})\right). \tag{36}$$

Remarks. (1) Condition (22) does not imply the uniqueness of an infinite-volume Gibbs state (i.e., absence of phase transitions).

(2) Properties (35) and (36) are trivially fulfilled for the limiting points $\mathbf{R}^{\Lambda^0}$ and $\boldsymbol{\varphi}$ of families $\{\mathbf{R}_\Lambda^{\Lambda^0}\}$ and $\{\boldsymbol{\varphi}_\Lambda\}$. However, they require a proof for the limit points of the families $\{\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\}$ and $\{\boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\}$.

The set of limiting Gibbs states (which is nonempty due to Theorem 3) is denoted by $\mathfrak{G}^0$. In Section 3 we describe a class $\mathfrak{G}\supset\mathfrak{G}^0$ of states of C^*-algebra $\mathfrak{B}$ satisfying the FK-DLR equation, similar to that in [1].

2. Feynman-Kac Representations for the RDM Kernels in a Finite Volume

2.1. The Representation for the Kernels of the Gibbs Operators. A starting point for the forthcoming analysis is the Feynman-Kac (FK) representation for the kernels $\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)=\mathbf{K}_{\beta,z,\Lambda}(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)$ and $\mathbf{F}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)=\mathbf{F}_{\beta,z,\Lambda}(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)$ of operators $\mathbf{G}_\Lambda$ and $\mathbf{R}_\Lambda$.

Definition 7. Given $(x,i),(y,j)\in M\times\Gamma$, $\overline{W}^\beta_{(x,i),(y,j)}$ denotes the space of path, or trajectories, $\overline{\omega}=\overline{\omega}_{(x,i),(y,j)}$ in $M\times\Gamma$, of time-length β, with the end-points (x,i) and (y,j). Formally, $\overline{\omega}\in\overline{W}^\beta_{(x,i),(y,j)}$ is defined as follows:

$$\begin{gathered}\overline{\omega}:\tau\in[0,\beta]\longmapsto\overline{\omega}(\tau)=(u(\overline{\omega},\tau),l(\overline{\omega},\tau))\in M\times\Gamma,\\ \overline{\omega}\text{ is cádlág;}\quad\overline{\omega}(0)=(x,i),\quad\overline{\omega}(\beta-)=(y,j),\\ \overline{\omega}\text{ has finitely many jumps on }[0,\beta];\\ \text{if a jump occurs at time }\tau,\text{ then }\mathtt{d}[l(\overline{\omega},\tau-),l(\overline{\omega},\tau)]=1.\end{gathered} \tag{37}$$

The notation $\overline{\omega}(\tau)$ and its alternative, $(u(\overline{\omega},\tau),l(\overline{\omega},\tau))$, for the position and the index of trajectory $\overline{\omega}$ at time τ will be employed as equal in rights. We use the term the temporal section (or simply the section) of path $\overline{\omega}$ at time τ.

Definition 8. Let $\mathbf{x}^*_\Lambda=\{\mathbf{x}^*(i),\ i\in\Lambda\}\in M^{*\Lambda}$, and $\mathbf{y}^*_\Lambda=\{\mathbf{y}^*(j),\ j\in\Lambda\}\in M^{*\Lambda}$ be particle configurations over Λ, with $\sharp\mathbf{x}^*_\Lambda=\sharp\mathbf{y}^*_\Lambda$. A matching (or pairing) γ between $\mathbf{x}^*_\Lambda$ and $\mathbf{y}^*_\Lambda$ is defined as a collection of pairs $[(x,i),(y,j)]_\gamma$, with $i,j\in\Lambda$, $x\in\mathbf{x}^*(i)$, and $y\in\mathbf{y}^*(j)$, with the properties that (i) $\forall i\in\Lambda$ and $x\in\mathbf{x}^*(i)$: there exist unique $j\in\Lambda$ and $y\in\mathbf{y}^*(j)$ such that (x,i) and (y,j) form a pair, and (ii) $\forall j\in\Lambda$ and $y\in\mathbf{y}^*(j)$: there exist unique $i\in\Lambda$ and $x\in\mathbf{x}^*(i)$ such that (x,i) and (y,j) form a pair. (Owing to the condition $\sharp\mathbf{x}^*_\Lambda=\sharp\mathbf{y}^*_\Lambda$, these properties are equivalent.) It is convenient to write $[(x,i),(y,j)]_\gamma=[(x,i),\gamma(x,i)]$.

Next, consider the Cartesian product

$$\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}=\mathop{\times}_{i\in\Lambda}\mathop{\times}_{x\in\mathbf{x}^*(i)}\overline{W}^\beta_{(x,i),\gamma(x,i)},\tag{38}$$

and the disjoint union

$$\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}=\bigcup_\gamma\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}.\tag{39}$$

Accordingly, an element $\overline{\boldsymbol{\omega}}_\Lambda\in\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$ in (38) represents a collection of paths $\overline{\omega}_{x,i}$, $x\in\mathbf{x}^*(i)$, $i\in\Lambda$, of time-length β, starting at (x,i) and ending up at $\gamma(x,i)$. We say that $\overline{\boldsymbol{\omega}}_\Lambda$ is a path configuration in (or over) Λ.

The presence of matchings in the above construction is a feature of the bosonic nature of the systems under consideration.

We will work with standard sigma algebras (generated by cylinder sets) in $\overline{W}^\beta_{(x,i),(y,j)}$, $\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$, and $\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}$.

Definition 9. In what follows, $\xi(\tau)$, $\tau\geq 0$, stands for the Markov process on $M\times\Gamma$, with cádlág trajectories, determined by the generator $\mathscr{L}$ acting on a function $(x,i)\in M\times\Lambda\mapsto\psi(x,i)$ by

$$\begin{aligned}\mathscr{L}\psi(x,i)=&-\frac{1}{2}\Delta\psi(x,i)\\&+\sum_{j:\mathrm{d}(i,j)=1}\lambda_{i,j}\int_M\nu(\mathrm{d}y)\left[\psi(y,j)-\psi(x,i)\right].\end{aligned}\tag{40}$$

In the probabilistic literature, such processes are referred to as Lévy processes; see, for example, [14].

Pictorially, a trajectory of process ξ moves along M according to the Brownian motion with the generator $-\Delta/2$ and changes the index $i\in\Gamma$ from time to time in accordance with jumps occurring in a Poisson process of rate $\sum_{j:\mathrm{d}(i,j)=1}\lambda_{i.j}$. In other words, while following a Brownian motion rule on M, having index $i\in\Gamma$ and being at point $x\in M$, the moving particle experiences an urge to jump from i to a neighboring vertex j and to a point y at rate $\lambda_{i,j}\nu(\mathrm{d}y)$. After a jump, the particle continues the Brownian motion on M from y and keeps its new index j until the next jump, and so on.

For a given pairs $(x,i),(y,j)\in M\times\Gamma$, we denote by $\mathbb{P}^\beta_{(x,i),(y,j)}$ the nonnormalised measure on $\overline{W}^\beta_{(x,i),(y,j)}$ induced by ξ. That is, under measure $\mathbb{P}^\beta_{(x,i),(y,j)}$ the trajectory at time $\tau=0$ starts from the point x and has the initial index i while at time $\tau=\beta$ it is at the point y and has the index j. The value $\widehat{p}_{(x,i),(y,j)}=\mathbb{P}^\beta_{(x,i),(y,j)}(\overline{W}^\beta_{(x,i),(y,j)})$ is given by

$$\begin{aligned}\widehat{p}_{(x,i),(y,j)}=&\;\mathbf{1}(i=j)\,p^\beta_M(x,y)\exp\left[-\beta\sum_{\widetilde{j}:\mathrm{d}(i,\widetilde{j})=1}\lambda_{i,\widetilde{j}}\right]\\&+\sum_{k\geq 1}\sum_{l_0=i,l_1,\dots,l_k,l_{k+1}=j}\prod_{0\leq s\leq k}\mathbf{1}(\mathrm{d}(l_s,l_{s+1})=1)\\&\times\lambda_{l_s,l_{s+1}}\int_0^\beta\mathrm{d}\tau_s\exp\left[-(\tau_{s+1}-\tau_s)\sum_{\widetilde{j}:\mathrm{d}(l_s,\widetilde{j})=1}\lambda_{l_s,\widetilde{j}}\right]\\&\times\mathbf{1}(0=\tau_0<\tau_1<\cdots<\tau_k<\tau_{k+1}=\beta),\end{aligned}\tag{41}$$

where $p^\beta_M(x,y)$ denotes the transition probability density for the Brownian motion to pass from x to y on M in time β:

$$\begin{aligned}p^\beta_M(x,y)=&\;\frac{1}{(2\pi\beta)^{d/2}}\\&\times\sum_{\underline{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d}\exp\left(\frac{-|x-y+\underline{n}|^2}{2\beta}\right).\end{aligned}\tag{42}$$

In view of (13), the quantity $\widehat{p}_{(x,i),(y,j)}$ and its derivatives are uniformly bounded:

$$\begin{aligned}\widehat{p}_{(x,i),(y,j)},\left|\nabla_x\widehat{p}_{(x,i),(y,j)}\right|,\left|\nabla_y\widehat{p}_{(x,i),(y,j)}\right|\leq\widehat{p}_M,\\x,y\in M,\;i,j\in\Gamma,\end{aligned}\tag{43}$$

where $\widehat{p}_M=\widehat{p}_M(\beta)\in(0,+\infty)$ is a constant.

We suggest a term "non-normalised Brownian bridge with jumps" for the measure but expect that a better term will be proposed in future.

Definition 10. Suppose that $\mathbf{x}^*_\Lambda=\{\mathbf{x}^*(i),\;i\in\Lambda\}\in M^{*\Lambda}$ and $\mathbf{y}^*_\Lambda=\{\mathbf{y}^*(j),\;j\in\Lambda\}\in M^{*\Lambda}$ are particle configurations over Λ, with $\sharp\mathbf{x}^*_\Lambda=\sharp\mathbf{y}^*_\Lambda$. Let γ be a pairing between $\mathbf{x}^*_\Lambda$ and $\mathbf{y}^*_\Lambda$. Then $\mathbb{P}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$ denotes the product measure on $\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$:

$$\overline{\mathbb{P}}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}=\mathop{\times}_{i\in\Lambda}\mathop{\times}_{x\in\mathbf{x}^*(i)}\overline{\mathbb{P}}^\beta_{(x,i),\gamma(x,i)}.\tag{44}$$

Furthermore, $\overline{\mathbb{P}}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}$ stands for the sum measure on $\overline{W}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}$:

$$\overline{\mathbb{P}}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}=\sum_\gamma\overline{\mathbb{P}}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}.\tag{45}$$

According to Definition 10, under the measure $\overline{\mathbb{P}}^\beta_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$, the trajectories $\overline{\omega}_{x,i}\in\overline{W}^\beta_{(x,i),\gamma(x,i)}$ constituting $\overline{\boldsymbol{\omega}}_\Lambda$ are independent components. (Here the term independence is used in the measure-theoretical sense.)

As in [1], we will work with functionals on $\overline{W}^{\beta}_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$ representing integrals along trajectories. The first such functional, $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$, is given by

$$\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) = \sum_{i\in\Lambda}\sum_{x\in\mathbf{x}^*(i)}\mathbf{h}^{x,i}(\overline{\omega}_{x,i}) + \frac{1}{2}\sum_{(i,i')\in\Lambda\times\Lambda}\sum_{x\in\mathbf{x}^*(i),x'\in\mathbf{x}^*(i')}\mathbf{h}^{(x,i),(x',i')} \times (\overline{\omega}_{x,i},\overline{\omega}_{x',i'}). \tag{46}$$

Here, introducing the notation $u_{x,i}(\tau) = u(\overline{\omega}_{x,i},\tau)$ and $u_{x',i'}(\tau) = u(\overline{\omega}_{x',i'},\tau)$ for the positions in M of paths $\overline{\omega}_{x,i} \in \overline{W}^{\beta}_{(x,i),\gamma(x,i)}$ and $\overline{\omega}_{x',i'} \in \overline{W}^{\beta}_{(x',i'),\gamma(x',i')}$ at time τ, we define

$$\mathbf{h}^{x,i}(\overline{\omega}_{x,i}) = \int_0^\beta \mathrm{d}\tau U^{(1)}(u_{i,x}(\tau)). \tag{47}$$

Next, with $l_{x,i}(\tau)$ and $l_{x',i'}(\tau)$ standing for the indices of $\overline{\omega}_{x,i}$ and $\overline{\omega}_{x',i'}$ at time τ,

$$\begin{aligned}\mathbf{h}^{(x,i),(x',i')}(\overline{\omega}_{x,i},\overline{\omega}_{x',i'}) &= \int_0^\beta \mathrm{d}\tau\Bigg[\sum_{j'\in\Gamma}U^{(2)}(u_{i,x}(\tau),u_{i',x'}(\tau)) \\ &\quad\times \mathbf{1}(l_{x,i}(\tau) = j' = l_{x',i'}(\tau)) \\ &\quad+\frac{1}{2}\sum_{(j',j'')\in\Gamma\times\Gamma}J(\mathtt{d}(j',j'')) \\ &\quad\times V(u_{i,x}(\tau),u_{i',x'}(\tau)) \\ &\quad\times\mathbf{1}(l_{x,i}(\tau) = j' \neq j'' = l_{x',i'}(\tau))\Bigg].\end{aligned} \tag{48}$$

Next, consider the functional $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$: for $\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda} = \{\overline{\mathbf{x}}^*(j),\ j\in\overline{\Gamma}\setminus\Lambda\}$. As before, we assume that $\sharp\overline{\mathbf{x}}^*(j)\le\kappa$. Define

$$\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}) = \mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) + \mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}). \tag{49}$$

Here $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$ is as in (46) and

$$\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}) = \sum_{(i,i')\in\Lambda\times(\overline{\Gamma}\setminus\Lambda)}\sum_{x\in\mathbf{x}^*(i),\overline{x}'\in\overline{\mathbf{x}}^*(i')}\mathbf{h}^{(x,i),(\overline{x}',i')} \times(\overline{\omega}_{x,i},(\overline{x}',i')), \tag{50}$$

where, in turn,

$$\begin{aligned}\mathbf{h}^{(x,i),(\overline{x}',i')}(\overline{\omega}_{x,i},(\overline{x}',i')) &= \int_0^\beta \mathrm{d}\tau\Big[U^{(2)}(u_{i,x}(\tau),\overline{x}')\mathbf{1}(l_{x,i}(\tau) = i') \\ &\quad+\sum_{j\in\Gamma:j\neq i'}J(\mathtt{d}(j,i')) \\ &\quad\times V(u_{i,x}(\tau),\overline{x}')\mathbf{1}(l_{x,i}(\tau) = j)\Big].\end{aligned} \tag{51}$$

The functionals $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$ and $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ are interpreted as energies of path configurations. Compare (2.1.4) and (2.3.8) in [1].

Finally, we introduce the indicator functional $\alpha_\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$:

$$\alpha_\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) = \begin{cases}1, & \text{if index } l_{x,i}(\tau)\in\Lambda, \\ & \forall\tau\in[0,\beta],\ i\in\Lambda,\ x\in\mathbf{x}^*(i), \\ 0, & \text{otherwise.}\end{cases} \tag{52}$$

It can be derived from known results [11, 15–17] (for a direct argument, see [18]) that the following assertion holds true.

Lemma 11. *For all $z,\beta>0$ and a finite Λ, the Gibbs operators $\mathbf{G}_\Lambda$ and $\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ act as integral operators in $\mathscr{H}(\Lambda)$:*

$$\begin{aligned}(\mathbf{G}_\Lambda\phi)(\mathbf{x}^*_\Lambda) &= \int_{M^{*\Lambda}}\prod_{j\in\Lambda}\prod_{y\in\mathbf{y}^*(j)}\nu(\mathrm{d}y)\,\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)\,\phi(\mathbf{y}^*_\Lambda), \\ (\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}\phi)(\mathbf{x}^*_\Lambda) &= \int_{M^{*\Lambda}}\prod_{j\in\Lambda}\prod_{y\in\mathbf{y}^*(j)}\nu(\mathrm{d}y)\,\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})\,\phi(\mathbf{y}^*_\Lambda).\end{aligned} \tag{53}$$

Moreover, the integral kernels $\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)$ and $\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ vanish if $\sharp\mathbf{x}^*_\Lambda \neq \sharp\mathbf{y}^*_\Lambda$. On the other hand, when $\sharp\mathbf{x}^*_\Lambda = \sharp\mathbf{y}^*_\Lambda$, the kernels $\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda)$ and $\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ admit the following representations:

$$\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda) = z^{\sharp\mathbf{x}^*_\Lambda}\int_{\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}}\overline{\mathbb{P}}^{\beta}_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}(\mathrm{d}\overline{\boldsymbol{\omega}}_\Lambda)\,\alpha_\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) \times\exp[-\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)], \tag{54}$$

$$\mathbf{K}_\Lambda(\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}) = z^{\sharp\mathbf{x}^*_\Lambda}\int_{\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}}\overline{\mathbb{P}}^{\beta}_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}(\mathrm{d}\overline{\boldsymbol{\omega}}_\Lambda)\,\alpha_\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) \times\exp[-\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})]. \tag{55}$$

The ingredients of these representations are determined in (46)–(51).

Remark 12. As before, we stress that, owing to (16) and (17), a nonzero contribution to the integral in the RHS of (54) can only come from a path configuration $\overline{\boldsymbol{\omega}}_\Lambda = \{\overline{\omega}_{x,i}\}$ such that $\forall \tau \in [0,\beta]$ and $\forall j \in \Gamma$, the number of paths $\overline{\omega}_{x,i}$ with index $l_{x,i}(\tau) = j$ is less than or equal to κ. Likewise, the integral in the RHS of (55) receives a non-zero contribution only from configurations $\overline{\boldsymbol{\omega}}_\Lambda = \{\overline{\omega}^*_{x,i}\}$ such that, $\forall$ site $j \in \Gamma$, the number of paths $\overline{\omega}_{x,i}$ with index $l_{x,i}(\tau) = j$ plus the cardinality $\sharp\overline{\mathbf{x}}^*(j)$ does not exceed κ.

2.2. The Representation for the Partition Function. The FK representations of the partition functions $\Xi(\Lambda) = \Xi_{\beta,z}(\Lambda)$ in (27) and $\Xi(\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (1.4.6) reflect a specific character of the traces $\operatorname{tr}\mathbf{G}_\Lambda$ and $\operatorname{tr}\mathbf{G}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ in $\mathscr{H}(\Lambda)$. The source of a complication here is the jump terms in the Hamiltonians H_Λ and $H_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ in (11) and (12), respectively. In particular, we will have to pass from trajectories of fixed time-length β to loops of a variable time length. To this end, a given matching γ is decomposed into a product of cycles, and the trajectories associated with a given cycle are merged into closed paths (loops) of a time-length multiple of β. (A similar construction has been performed in [18].)

To simplify the notation, we omit, wherever possible, the index β.

Definition 13. For given $(x,i),(y,j) \in M\times\Gamma$, the symbol $\overline{W}^*_{(x,i),(y,j)}$ denotes the disjoint union:

$$\overline{W}^*_{(x,i),(y,j)} = \bigcup_{k=0,1,\ldots} \overline{W}^{k\beta}_{(x,i),(y,j)}. \tag{56}$$

In other words, $\overline{W}^*_{(x,i),(y,j)}$ is the space of paths $\overline{\Omega}^* = \overline{\Omega}^*_{(x,i),(y,j)}$ in $M\times\Gamma$, of a variable time-length $k\beta$, where $k = k(\overline{\Omega}^*)$ takes values $1,2,\ldots$ and called the length multiplicity, with the endpoints (x,i) and (y,j). The formal definition follows the same line as in (37), and we again use the notation $\overline{\Omega}^*(\tau)$ and the notation $(u(\overline{\Omega}^*,\tau), l(\overline{\Omega}^*,\tau))$ for the pair of the position and the index of path $\overline{\Omega}^*$ at time τ. Next, we call the particle configuration $\{\overline{\Omega}^*(\tau+\beta m),\ 0 \le m < k(\overline{\Omega}^*)\}$ the temporal section (or simply the section) of $\overline{\Omega}^*$ at time $\tau \in [0,\beta]$. We also call $\overline{\Omega}^*_{(x,i),(y,j)} \in \overline{W}^*_{(x,i),(y,j)}$ a path (from (x,i) to (y,j)).

A particular role will be played by closed paths (loops), with coinciding endpoints (where $(x,i) = (y,j)$). Accordingly, we denote by $W^*_{x,i}$ the set $\overline{W}^*_{(x,i),(x,i)}$. An element of $W^*_{x,i}$ is denoted by $\Omega^*_{x,i}$ or, in short, by Ω^* and called a loop at vertex i. (The upper index $*$ indicates that the length multiplicity is unrestricted.) The length multiplicity of a loop $\Omega^*_{x,i} \in W^*_{x,i}$ is denoted by $k(\boldsymbol{\Omega}^*_{x,i})$ or $k_{x,i}$. It is instructive to note that, as topological object, a given loop $\boldsymbol{\Omega}^*$ admits a multiple choice of the initial pair (x,i): it can be represented by any pair $(u(\Omega^*,\tau), l(\Omega^*,\tau))$ at a time $\tau = l\beta$ where $l = 1,\ldots,k(\boldsymbol{\Omega}^*)$. As above, we use the term the temporal section at time $\tau \in [0,\beta]$ for the particle configuration $\{\Omega^*_{x,i}(\tau+\beta m),\ 0 \le m < k_{x,i}\}$ and employ the alternative notation $(u(\tau+\beta m;\Omega^*), l(\tau+\beta m;\Omega^*))$ addressing the position and the index of Ω^* at time $\tau+\beta m \in [0,\beta k(\Omega^*)]$.

Definition 14. Suppose $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(i),\ i\in\Lambda\} \in M^{*\Lambda}$ and $\mathbf{y}^*_\Lambda = \{\mathbf{y}^*(j),\ j\in\Lambda\} \in M^{*\Lambda}$ are particle configurations over Λ, with $\sharp\mathbf{x}^*_\Lambda = \sharp\mathbf{y}^*_\Lambda$. Let γ be a matching between $\mathbf{x}^*_\Lambda$ and $\mathbf{y}^*_\Lambda$. We consider the Cartesian product:

$$\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma} = \mathop{\times}_{i\in\Lambda}\ \mathop{\times}_{x\in\mathbf{x}^*(i)} \overline{W}^*_{(x,i),\gamma(x,i)}, \tag{57}$$

and the disjoint union:

$$\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda} = \bigcup_\gamma \overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}. \tag{58}$$

Accordingly, an element $\overline{\boldsymbol{\Omega}}^*_\Lambda \in \overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$ in (58) represents a collection of paths $\overline{\boldsymbol{\Omega}}^*_{x,i}$, $x\in\mathbf{x}^*(i)$, $i\in\Lambda$, of time-length $k\beta$, starting at (x,i) and ending up at $(y,j) = \gamma(x,i)$. We say that $\overline{\boldsymbol{\Omega}}^*_\Lambda \in \overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}$ is a path configuration in (or over) Λ.

Again, loops play a special role and deserve a particular notation. Namely, $W^*_{\mathbf{x}^*_\Lambda}$ denotes the Cartesian product:

$$W^*_{\mathbf{x}^*_\Lambda} = \mathop{\times}_{i\in\Lambda}\ \mathop{\times}_{x\in\mathbf{x}^*(i)} W^*_{x,i}, \tag{59}$$

and W^*_Λ stands for the disjoint union (or equivalently, the Cartesian power):

$$W^*_\Lambda = \bigcup_{\mathbf{x}^*_\Lambda\in M^{*\Lambda}} W^*_{\mathbf{x}^*_\Lambda} = \mathop{\times}_{i\in\Lambda} W^*_{\{i\}},$$
$$\text{where } W^*_{\{i\}} = \bigcup_{\mathbf{x}^*\in M^*}\left(\mathop{\times}_{x\in\mathbf{x}^*} W^*_{x,i}\right). \tag{60}$$

Denote by $\boldsymbol{\Omega}^* = \{\boldsymbol{\Omega}^*(i),\ i\in\Lambda\} \in W^*_\Lambda$ a collection of loop configurations at vertices $i\in\Lambda$ starting and ending up at particle configurations $\mathbf{x}^*(i) \in M^*$ (note that some of the $\boldsymbol{\Omega}^*(i)$'s may be empty). The temporal section (or, in short, the section), $\boldsymbol{\Omega}^*(\tau)$, of $\boldsymbol{\Omega}^*$ at time τ is defined as the particle configuration formed by the points $\Omega^*_{x,i}(\tau+\beta m)$ where $i\in\Lambda$, $x\in\mathbf{x}^*(i)$, and $0\le m<k_{x,i}$.

As before, consider the standard sigma algebras of subsets in the spaces $\overline{W}^*_{(x,i),(y,j)}$, $W_{x,i}$, $\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$, $\overline{W}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda}$, $W^*_{\mathbf{x}^*_\Lambda}$, and W^*_Λ introduced in Definitions 13 and 14. In particular, the sigma algebra of subsets in W^*_Λ will be denoted by $\mathfrak{W}_\Lambda$; we comment on some of its specific properties in Section 3.1. (An infinite-volume version W^*_Γ of W^*_Λ is treated in Section 3.2 and after.)

Definition 15. Given points $(x,i),(y,j)\in M\times\Gamma$, we denote by $\overline{\mathbb{P}}^*_{(x,i),(y,j)}$ the sum measure on $\overline{W}^*_{(x,i),(y,j)}$:

$$\overline{\mathbb{P}}^*_{(x,i),(y,j)} = \sum_{k=0,1,\ldots} \overline{\mathbb{P}}^{k\beta}_{(x,i),(y,j)}. \tag{61}$$

Further, $\mathbb{P}^*_{x,i}$ denotes the similar measure on $W^*_{x,i}$:

$$\mathbb{P}^*_{x,i} = \sum_{k=0,1,\ldots} \mathbb{P}^{k\beta}_{x,i}. \tag{62}$$

Definition 16. Let $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(i),\ i \in \Lambda\} \in M^{*\Lambda}$ and $\mathbf{y}^*_\Lambda = \{\mathbf{y}^*(j),\ j \in \Lambda\} \in M^{*\Lambda}$ be particle configurations over Λ, with $\sharp\mathbf{x}^*_\Lambda = \sharp\mathbf{y}^*_\Lambda$. Let γ be a matching between $\mathbf{x}^*_\Lambda$ and $\mathbf{y}^*_\Lambda$, and we define the product measure $\overline{\mathbb{P}}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}$:

$$\overline{\mathbb{P}}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma} = \mathop{\times}_{i\in\Lambda} \mathop{\times}_{x\in\mathbf{x}^*(i)} \overline{\mathbb{P}}^*_{(x,i),\gamma(x,i)} \tag{63}$$

and the sum measure

$$\overline{\mathbb{P}}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda} = \sum_{\gamma} \overline{\mathbb{P}}^*_{\mathbf{x}^*_\Lambda,\mathbf{y}^*_\Lambda,\gamma}. \tag{64}$$

Next, symbol $\mathbb{P}^*_{\mathbf{x}^*_\Lambda}$ stands for the product measure on $W^*_{\mathbf{x}^*_\Lambda}$:

$$\mathbb{P}^*_{\mathbf{x}^*_\Lambda} = \mathop{\times}_{i\in\Lambda} \mathop{\times}_{x\in\mathbf{x}^*(i)} \mathbb{P}^*_{x,i}. \tag{65}$$

Finally, $\mathrm{d}\mathbf{\Omega}^*_\Lambda$ yields the measure on W^*_Λ:

$$\mathrm{d}\mathbf{\Omega}^*_\Lambda = \mathrm{d}\mathbf{x}^*_\Lambda \times \mathbb{P}^*_{\mathbf{x}^*_\Lambda}\left(\mathrm{d}\mathbf{\Omega}^*_\Lambda\right). \tag{66}$$

Here, for $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(i),\ i \in \Lambda\}$, we set: $\mathrm{d}\mathbf{x}^*_\Lambda = \prod_{i\in\Lambda}\prod_{x\in\mathbf{x}^*(i)}\nu(\mathrm{d}x)$. For sites i with $\mathbf{x}^*(i) = \emptyset$, the corresponding factors are trivial measures sitting on the empty configurations.

We again need to introduce energy-type functionals represented by integrals along loops. More precisely, we define the functionals $\mathbf{h}^\Lambda(\mathbf{\Omega}^*_\Lambda)$ and $\mathbf{h}^\Lambda(\mathbf{\Omega}^*_\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ which are modifications of the above functionals $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$ and $\mathbf{h}^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$; confer (46) and (50). Say, for a loop configuration $\mathbf{\Omega}^*_\Lambda = \{\Omega^*_{x,i}\}$ over Λ with an initial and final particle configuration $\mathbf{x}^*_\Lambda = \{\mathbf{x}^*(i),\ i \in \Lambda\}$,

$$\begin{aligned}\mathbf{h}^\Lambda\left(\mathbf{\Omega}^*_\Lambda\right) &= \sum_{i\in\Lambda}\sum_{x\in\mathbf{x}^*(i)} \mathbf{h}^{x,i}\left(\Omega^*_{x,i}\right) \\ &\quad + \frac{1}{2}\sum_{(i,i')\in\Lambda\times\Lambda}\ \sum_{x\in\mathbf{x}^*(i),x'\in\mathbf{x}^*(i')} \\ &\quad \times \mathbf{1}\left((x,i) \neq \left(x',i'\right)\right)\mathbf{h}^{(x,i),(x',i')}\left(\Omega^*_{x,i},\Omega^*_{x',i'}\right).\end{aligned} \tag{67}$$

To determine the functionals $\mathbf{h}^{x,i}(\Omega^*_{x,i})$ and $\mathbf{h}^{(x,i),(x',i')}(\Omega^*_{x,i}, \Omega^*_{x',i'})$, we set, for given $m = 0,1,\ldots,k_{x,i}-1$ and $m' = 0,1,\ldots,k_{x',i'}-1$:

$$\begin{aligned} u_{i,x}(\tau+\beta m) &= u\left(\tau+\beta m;\Omega^*_{x,i}\right), \\ l_{i,x}(\tau+\beta m) &= l\left(\tau+\beta m;\Omega^*_{x,i}\right), \\ u_{i',x'}\left(\tau+\beta m'\right) &= u\left(\tau+\beta m';\Omega^*_{x',i'}\right), \\ l_{i',x'}\left(\tau+\beta m'\right) &= l\left(\tau+\beta m';\Omega^*_{x',i'}\right). \end{aligned} \tag{68}$$

A (slightly) shortened notation $l_{i,x}(\tau+\beta m)$ is used for the index $l_{x,i}(\tau+\beta m;\Omega^*_{x,i})$ and $u_{i,x}(\tau+\beta m)$ for the position $u(\tau+\beta m;\Omega^*_{x,i})$ for $\Omega^*_{x,i}(\tau) \in M\times\Gamma$, of the section $\Omega^*_{x,i}(\tau)$ of the loop $\Omega^*_{x,i}$ at time τ, and similarly with $l_{i',x'}(\tau+\beta m')$ and $u_{i',x'}(\tau+\beta m')$. (Note that the pairs (x,i) and (x',i') may coincide.) Then

$$\begin{aligned}&\mathbf{h}^{(x,i)}\left(\Omega^*_{x,i}\right) \\ &= \int_0^\beta \mathrm{d}\tau \Bigg[\sum_{0\le m<k_{x,i}} U^{(1)}\left(u_{x,i}(\tau+\beta m)\right) \\ &\quad + \sum_{0\le m<m'<k_{x,i}}\sum_{j\in\Gamma}\mathbf{1}\left(l_{x,i}(\tau+\beta m) = j = l_{x,i}\left(\tau+\beta m'\right)\right) \\ &\quad \times U^{(2)}\left(u_{x,i}(\tau+\beta m), u_{x,i}\left(\tau+\beta m'\right)\right)\Bigg], \\ &\mathbf{h}^{(x,i),(x',i')}\left(\Omega^*_{x,i},\Omega^*_{x',i'}\right) \\ &= \sum_{0\le m<k_{x,i}}\ \sum_{0\le m'<k_{x',i'}}\int_0^\beta \mathrm{d}\tau \\ &\quad \times \Bigg[\sum_{j\in\Gamma}U^{(2)}\left(u_{x,i}(\tau+\beta m), u_{x',i'}\left(\tau+\beta m'\right)\right) \\ &\quad \times \mathbf{1}\left(l_{x,i}(\tau+\beta m) = j = l_{x',i'}\left(\tau+\beta m'\right)\right) \\ &\quad + \sum_{(j,j')\in\Gamma\times\Gamma} J\left(\mathrm{d}\left(j,j'\right)\right)V \\ &\quad \times \left(u_{i,x}(\tau+\beta m), u_{i',x'}\left(\tau+\beta m'\right)\right) \\ &\quad \times \mathbf{1}\left(l_{x,i}(\tau+\beta m) = j \neq j' = l_{x',i'}\left(\tau+\beta m'\right)\right)\Bigg].\end{aligned} \tag{69}$$

Next, the functional $B(\mathbf{\Omega}^*_\Lambda)$ takes into account the bosonic character of the model:

$$B\left(\mathbf{\Omega}^*_\Lambda\right) = \prod_{i\in\Lambda}\prod_{x\in\mathbf{x}^*(i)}\frac{z^{k_{x,i}}}{k_{x,i}}. \tag{70}$$

The factor $k^{-1}_{x,i}$ in (70) reflects the fact that the starting point of a loop $\Omega^*_{x,i}$ may be selected among points $u(\beta m, \Omega^*_{x,i})$ arbitrarily.

Next, we define the functional $\mathbf{h}^{\Lambda}(\mathbf{\Omega}^*_{\Lambda} \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$: for $\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda} = \{\overline{\mathbf{x}}^*(j),\ j \in \overline{\Gamma}\setminus\Lambda\}$, again assuming that $\sharp\overline{\mathbf{x}}^*(j) \le \kappa$. Set

$$\mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda} \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right) = \mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda}\right) + \mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda} \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right). \tag{71}$$

Here $\mathbf{h}^{\Lambda}(\mathbf{\Omega}^*_{\Lambda})$ is as in (67) and

$$\mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda} \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right) = \sum_{(i,i')\in\Lambda\times(\overline{\Gamma}\setminus\Lambda)} \sum_{x\in\mathbf{x}^*(i),\overline{x}'\in\overline{\mathbf{x}}^*(i')} \mathbf{h}^{(x,i),(\overline{x}',i')}\left(\Omega^*_{x,i}, \left(\overline{x}', i'\right)\right), \tag{72}$$

where, in turn,

$$\begin{aligned}\mathbf{h}^{(x,i),(\overline{x}',i')}\left(\Omega^*_{x,i}, \left(\overline{x}', i'\right)\right) &= \sum_{0\le m<k_{x,i}} \int_0^{\beta} \mathrm{d}\tau \\ &\quad \times \Big[U^{(2)}\left(u_{x,i}\left(\tau+\beta m\right), x'\right) \\ &\qquad \times \mathbf{1}\left(l_{x,i}\left(\tau+\beta m\right) = i'\right) \\ &\qquad + \sum_{j\in\Gamma} J\left(\mathrm{d}\left(j, i'\right)\right) V\left(u_{i,x}\left(\tau+\beta m\right), x'\right) \\ &\qquad \times \mathbf{1}\left(l_{x,i}\left(\tau+\beta m\right) = j\right)\Big].\end{aligned} \tag{73}$$

As before, the functionals $\mathbf{h}^{\Lambda}(\mathbf{\Omega}^*_{\Lambda})$ and $\mathbf{h}^{\Lambda}(\mathbf{\Omega}^*_{\Lambda} \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ have a natural interpretation as energies of loop configurations.

Finally, as before, the functional $\alpha_{\Lambda}(\mathbf{\Omega}^*_{\Lambda})$ is the indicator that the collection of loops $\mathbf{\Omega}^*_{\Lambda} = \{\Omega^*_{x,i}\}$ does not quit Λ:

$$\alpha_{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda}\right) = \begin{cases} 1, & \text{if } \Omega^*_{x,i}(\tau) \in M\times\Lambda, \\ & \forall i \in \Lambda, x \in \mathbf{x}^*(i),\ 0 \le \tau \le \beta k_{x,i}, \\ 0, & \text{otherwise.}\end{cases} \tag{74}$$

Like above, we invoke known results [11, 15–17] to establish the following statement (again a direct argument can be found in [18]).

Lemma 17. *For all finite $\Lambda \subset \Gamma$ and $z, \beta > 0$ satisfying* (22), *the partition functions $\Xi(\Lambda)$ in* (27) *and $\Xi(\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ in* (27) *and* (28) *admit the representations as converging integrals:*

$$\Xi(\Lambda) = \int_{W^*_{\Lambda}} \mathrm{d}\mathbf{\Omega}^*_{\Lambda} B\left(\mathbf{\Omega}^*_{\Lambda}\right) \alpha_{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda}\right) \exp\left[-\mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda}\right)\right], \tag{75}$$

$$\Xi\left(\Lambda \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right) = \int_{W^*_{\Lambda}} \mathrm{d}\mathbf{\Omega}^*_{\Lambda} B\left(\mathbf{\Omega}^*_{\Lambda}\right) \alpha_{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda}\right) \exp\left[-\mathbf{h}^{\Lambda}\left(\mathbf{\Omega}^*_{\Lambda} \parallel \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right)\right] \tag{76}$$

with the ingredients introduced in (60)–(74).

Again, we emphasis that the non-zero contribution to the integral in (75) can only come from loop configurations $\mathbf{\Omega}^*_{\Lambda} = \{\Omega^*_{x,i},\ x \in \mathbf{x}^*(i),\ i \in \Lambda\}$ such that $\forall$ vertex $j \in \Lambda$ and $\tau \in [0, \beta]$, the total number of pairs $(u_{x,i}(\tau+m\beta), l_{x,i}(\tau+m\beta))$ with $0 \le m < k_{x,i}$, and $l(\tau + m\beta) = j$ does not exceed κ.

Remark 18. The integrals in (75) and (76) represent examples of partition functions which will be encountered in the forthcoming sections. See (96), (97), (101), (103), (105), (107), (111), and (113) below. A general form of such a partition function treated as an integral over a set of loop configurations rather than a trace in a Hilbert space is given in (96) and (97).

2.3. The Representation for the RDM Kernels. Let Λ^0, Λ be finite sets, $\Lambda^0 \subset \Lambda \subset \Gamma$. The construction developed in Section 2.2 also allows us to write a convenient representation for the integral kernels of the RDMs $\mathbf{R}^{\Lambda^0}_{\Lambda}$ (see (31)) and $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$. In accordance with Lemma 11 and the definition of $\mathbf{R}^{\Lambda^0}_{\Lambda}$ in (31), the operator $\mathbf{R}^{\Lambda^0}_{\Lambda}$ acts as an integral operator in $\mathscr{H}(\Lambda^0)$:

$$\left(\mathbf{R}^{\Lambda^0}_{\Lambda}\phi\right)\left(\mathbf{x}^{*0}\right) = \int_{M^{*\Lambda^0}} \prod_{j\in\Lambda^0} \prod_{y\in\mathbf{y}^*(j)} \nu(\mathrm{d}y)\, \mathbf{F}^{\Lambda^0}_{\Lambda}\left(\mathbf{x}^{*0}, \mathbf{y}^{*0}\right) \phi\left(\mathbf{y}^{*0}\right), \tag{77}$$

where

$$\begin{aligned}\mathbf{F}^{\Lambda^0}_{\Lambda}\left(\mathbf{x}^{*0}, \mathbf{y}^{*0}\right) &:= \int_{M^{*\Lambda\setminus\Lambda^0}} \prod_{j\in\Lambda\setminus\Lambda^0} \prod_{z\in\mathbf{z}^*(j)} \nu(\mathrm{d}z) \\ &\quad \times \mathbf{F}_{\Lambda}\left(\mathbf{x}^{*0} \vee \mathbf{z}^*_{\Lambda\setminus\Lambda^0}, \mathbf{y}^{*0} \vee \mathbf{z}^*_{\Lambda\setminus\Lambda^0}\right) \\ &:= \frac{\widehat{\Xi}^{\Lambda^0}_{\Lambda}\left(\mathbf{x}^{*0}, \mathbf{y}^{*0}; \Lambda\setminus\Lambda^0\right)}{\Xi(\Lambda)}.\end{aligned} \tag{78}$$

We employ here and below the notation $\mathbf{x}^{*0}$ and $\mathbf{y}^{*0}$ for particle configurations $\mathbf{x}^*_{\Lambda^0} = \{\mathbf{x}^*(i),\ i \in \Lambda^0\}$ and $\mathbf{y}^*_{\Lambda^0} = \{\mathbf{y}^*(j),\ j \in \Lambda^0\}$ over Λ^0. Next, $\mathbf{x}^{*0} \vee \mathbf{z}^*_{\Lambda\setminus\Lambda^0}$; $\mathbf{y}^{*0} \vee \mathbf{z}^*_{\Lambda\setminus\Lambda^0}$ denotes the concatenated configurations over Λ.

Similarly, the RDM $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ is determined by its integral kernel $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}(\mathbf{x}^{*0}, \mathbf{y}^{*0})$, again admitting the representation

$$\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}\left(\mathbf{x}^{*0}, \mathbf{y}^{*0}\right) := \frac{\widehat{\Xi}^{\Lambda^0}_{\Lambda}\left(\mathbf{x}^{*0}, \mathbf{y}^{*0}; \Lambda\setminus\Lambda^0 \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}\right)}{\Xi(\Lambda)}. \tag{79}$$

As in [1], we call $\mathbf{F}^{\Lambda^0}_{\Lambda}$ and $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda}}$ the RDM kernels (in short, RDMKs). The focus of our interest is the numerators $\widehat{\Xi}^{\Lambda^0}_{\Lambda}(\mathbf{x}^{*0}, \mathbf{y}^{*0}, \Lambda\setminus\Lambda^0)$ and $\widehat{\Xi}^{\Lambda^0}_{\Lambda}(\mathbf{x}^{*0}, \mathbf{y}^{*0}; \Lambda\setminus\Lambda^0 \mid \overline{\mathbf{x}}^*_{\overline{\Gamma}\setminus\Lambda})$ in (78) and (79). To introduce the appropriate representation for these quantities, we need some additional definitions.

Definition 19. Repeating (57)-(58), symbol $\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}$ denotes the disjoint union $\bigcup_{\gamma^0}\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ over matchings γ^0 between $\mathbf{x}^{*0}$ and $\mathbf{y}^{*0}$. Accordingly, element $\overline{\mathbf{\Omega}}^{*0} = \overline{\mathbf{\Omega}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0} \in \overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ yields a collection of paths $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)} \in \overline{W}^*_{(x,i),\gamma^0(x,i)}$ lying in $M\times\Gamma$. Each path $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}$ has time lengths $\beta k_{(x,i),(y,j)}$, begins at (x,i), and ends up at $(y,j)=\gamma^0(x,i)$, where $x\in\mathbf{x}^*(i)$, $y\in\mathbf{y}^*(j)$. Like above, we will use for $\overline{\mathbf{\Omega}}^{*0}$ the term a path configuration over Λ^0. Repeating (63)-(64), we obtain the measures $\mathbb{P}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ on $\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ and $\mathbb{P}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}$ on $\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}$.

The assertion of Lemma 20 below again follows directly from known results, in conjunction with calculations of the partial trace $\mathrm{tr}_{\mathscr{H}_{\Lambda\setminus\Lambda^0}}$ in $\mathscr{H}_\Lambda$. The meaning of new ingredients in (79)–(82) is explained below.

Lemma 20. *The quantity* $\widehat{\Xi}^{\Lambda^0}_\Lambda(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0)$ *emerging in* (78) *is set to be* 0 *when* $\sharp\mathbf{x}^{*0}\neq\sharp\mathbf{y}^{*0}$. *On the other hand, for* $\sharp\mathbf{x}^{*0}=\sharp\mathbf{y}^{*0}$,

$$\begin{aligned}\widehat{\Xi}^{\Lambda^0}_\Lambda&\left(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0\right)\\&=\int_{\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}}\overline{\mathbb{P}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}\left(\mathrm{d}\overline{\mathbf{\Omega}}^{*0}\right)\overline{B}\left(\overline{\mathbf{\Omega}}^{*0}\right)\\&\quad\times\alpha_\Lambda\left(\overline{\mathbf{\Omega}}^{*0}\right)\mathbf{1}\left(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0}\right)\\&\quad\times\exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\right)\right]\widehat{\Xi}^{\Lambda^0}_\Lambda\left(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\right),\end{aligned}\tag{80}$$

where

$$\begin{aligned}\widehat{\Xi}^{\Lambda^0}_\Lambda&\left(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\right)\\&=\int_{W^*_{\Lambda\setminus\Lambda^0}}\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}B\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)\\&\quad\times\alpha_\Lambda\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)\mathbf{1}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\in\mathscr{F}^{\Lambda^0}\right)\\&\quad\times\exp\left[-\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0}\right)\right].\end{aligned}\tag{81}$$

Similarly, the quantity $\widehat{\Xi}^{\Lambda^0}_\Lambda(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ *from* (79) *vanishes when* $\sharp\mathbf{x}^{*0}\neq\sharp\mathbf{y}^{*0}$. *For* $\sharp\mathbf{x}^{*0}=\sharp\mathbf{y}^{*0}$,

$$\begin{aligned}\widehat{\Xi}^{\Lambda^0}_\Lambda&\left(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\\&=\int_{\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}}\overline{\mathbb{P}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}\left(\mathrm{d}\overline{\mathbf{\Omega}}^{*0}\right)\overline{B}\left(\overline{\mathbf{\Omega}}^{*0}\right)\\&\quad\times\alpha_\Lambda\left(\overline{\mathbf{\Omega}}^{*0}\right)\mathbf{1}\left(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0}\right)\\&\quad\times\exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\right]\\&\quad\times\widehat{\Xi}^{\Lambda^0}_\Lambda\left(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right),\end{aligned}\tag{82}$$

where

$$\begin{aligned}\widehat{\Xi}^{\Lambda^0}_\Lambda&\left(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\\&=\int_{W^*_{\Lambda\setminus\Lambda^0}}\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}B\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)\alpha_\Lambda\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)\\&\quad\times\mathbf{1}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\in\mathscr{F}^{\Lambda^0}\right)\\&\quad\times\exp\left[-\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\right].\end{aligned}\tag{83}$$

These representations hold $\forall z,\beta>0$ *and finite* $\Lambda^0\subset\Lambda\subset\Gamma$.

Let us define the functionals $\overline{B}(\overline{\mathbf{\Omega}}^{*0})$, $\alpha_\Lambda(\overline{\mathbf{\Omega}}^{*0})$, $\mathbf{1}(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0})$, $\mathbf{1}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\in\mathscr{F}^{\Lambda^0})\mathbf{h}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0})$, $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0})$, $\mathbf{h}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0}\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$, and $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (79)–(83). (The functionals $B(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0})$ and $\alpha_\Lambda(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0})$ are defined as (70) and (74), respectively, replacing Λ with $\Lambda\setminus\Lambda^0$.)

To this end, let $\overline{\mathbf{\Omega}}^{*0}=\overline{\mathbf{\Omega}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}\in\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ be a path configuration represented by a collection of paths $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\in\overline{W}^*_{(x,i),\gamma^{*0}(x,i)}$ ($\overline{\Omega}^*_{x,i}$ in short), with end points (x,i) and $(y,j)=\gamma^0(x,i)$, of time-length $\beta k_{(x,i),(y,j)}$. The functional $\overline{B}(\overline{\mathbf{\Omega}}^{*0})$ is given by

$$\overline{B}\left(\overline{\mathbf{\Omega}}^{*0}\right)=\prod_{i\in\Lambda}\prod_{x\in\mathbf{x}^*(i)}z^{k_{(x,i),(y,j)}}.\tag{84}$$

The functional $\alpha_\Lambda(\overline{\mathbf{\Omega}}^{*0})$ is again an indicator:

$$\alpha_\Lambda\left(\overline{\mathbf{\Omega}}^{*0}\right)=\begin{cases}1, & \text{if } \Omega^*_{(x,i),\gamma^0(x,i)}(\tau)\in M\times\Lambda,\\ & \forall i\in\Lambda, x\in\mathbf{x}^*(i),\\ & 0\le\tau\le\beta k_{(x,i),(y,j)},\\ 0, & \text{otherwise.}\end{cases}\tag{85}$$

Now let us define the indicator function $\mathbf{1}(\cdot\in\mathscr{F}^{\Lambda^0})$ in (80)–(83). The factor $\mathbf{1}(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0})$ equals one if and only if every path $\overline{\Omega}^*_{(x,i),(y,j)}$ from $\overline{\mathbf{\Omega}}^{*0}$, of time-length $\beta k_{(x,i),(y,j)}$, starting at $(x,i)\in M\times\Lambda^0$ and ending up at $(y,j)=\gamma^0(x,i)\in M\times\Lambda^0$ remains in $M\times(\Lambda\setminus\Lambda^0)$ at the intermediate times βl for $l=1,\dots,k_{(x,i),(y,j)}-1$:

$$\overline{\Omega}^*_{(x,i),(y,j)}(l\beta)\notin M\times\Lambda^0,\quad\forall l=1,\dots,k_{(x,i),(y,j)}-1,\tag{86}$$

(when $k_{(x,i),(y,j)}=1$, this is not a restriction).

Furthermore, suppose that $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}=\mathbf{\Omega}_{\mathbf{x}^*_{\Lambda\setminus\Lambda^0}}$ is a loop configuration over $\Lambda\setminus\Lambda^0$, with the initial/end configuration $\mathbf{x}^*_{\Lambda\setminus\Lambda^0}=\{\mathbf{x}^*(i),\ i\in\Lambda\setminus\Lambda^0\}$, represented by a collection of loops $\Omega^*_{x,i}$, $i\in\Lambda\setminus\Lambda^0$, $x\in\mathbf{x}^*(i)$. Then $\mathbf{1}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\in\mathscr{F}^{\Lambda^0})=1$ if and only if each loop $\Omega^*_{x,i}$ of time-length $\beta k_{x,i}$, beginning and finishing at $(x,i) \in M \times (\Lambda \setminus \Lambda^0)$, does not enter the set $M \times \Lambda^0$ at times βl for $l = 1, \ldots, k_{x,i} - 1$:

$$\Omega^*_{x,i}(l\beta) \notin M \times \Lambda^0, \quad \forall l = 1, \ldots, k_{x,i} - 1 \tag{87}$$

(again, if $k_{x,i} = 1$, this is not a restriction).

The functional $\mathbf{h}^{\Lambda^0}(\overline{\boldsymbol{\Omega}}^{*0})$ in (80) gives the energy of the path configuration $\overline{\boldsymbol{\Omega}}^{*0}$ and is introduced similarly to (67), *mutatis mutandis*. Next, the functional $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \mid \overline{\boldsymbol{\Omega}}^{*0})$ in (81) represents the energy of the loop configuration $\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}$ in the potential field generated by the path configuration $\overline{\boldsymbol{\Omega}}^{*0}$:

$$\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \mid \overline{\boldsymbol{\Omega}}^{*0}\right) = \mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}\right) + \mathbf{h}\left(\overline{\boldsymbol{\Omega}}^{*0} \parallel \boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}\right). \tag{88}$$

Here, the summand $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0})$ yields the energy of the loop configuration $\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}$; again confer (67). Further, the term $\mathbf{h}(\overline{\boldsymbol{\Omega}}^{*0} \parallel \boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0})$ yields the energy of interaction between $\overline{\boldsymbol{\Omega}}^{*0}$ and $\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}$: for a path/loop configurations $\overline{\boldsymbol{\Omega}}^{*0} = \{\overline{\Omega}^*_{x,i}\} \in \overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0},\gamma^0}$ and a $\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} = \{\Omega^*_{x',i'}\} \in W^*_{\Lambda\setminus\Lambda^0}$ we set

$$\mathbf{h}\left(\overline{\boldsymbol{\Omega}}^{*0} \parallel \boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}\right) = \sum_{(i,i')\in\Lambda^0\times(\Lambda\setminus\Lambda^0)} \sum_{x\in\mathbf{x}^*(i),x'\in\mathbf{x}^*(i')} \mathbf{h}^{(x,i),(x',i')}\left(\overline{\Omega}^*_{x,i}, \Omega^*_{x',i'}\right). \tag{89}$$

Here, for a path $\overline{\Omega}^*_{x,i} = \overline{\Omega}^*_{(x,i),\gamma^0(x,i)}$, of time-length $\beta k_{(x,i),\gamma^0(x,i)}$, and a loop $\Omega^*_{x',i'}$, of time-length $\beta k_{x,i}$,

$$\begin{aligned}
&\mathbf{h}^{(x,i),(x',i')}\left(\overline{\Omega}^*_{x,i}, \Omega^*_{x',i'}\right) \\
&= \sum_{0\le m<k_{(x,i),\gamma^0(x,i)}} \sum_{0\le m'<k_{x',i'}} \int_0^\beta \mathrm{d}\tau \\
&\quad \times \Bigg[\sum_{j\in\Gamma} U^{(2)}\left(u\left(\tau+\beta m; \overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right), u\left(\tau+\beta m'; \Omega^*_{x',i'}\right)\right) \\
&\quad \times \mathbf{1}\left(l_{(x,i),\gamma^0(x,i)}(\tau+\beta m) = j = l_{x',i'}\left(\tau+\beta m'\right)\right) \\
&\quad + \sum_{j,j'\in\Gamma\times\Gamma} J\left(\mathrm{d}\left(j,j'\right)\right) V\left(u_{i,x}(\tau+\beta m), u_{i',x'}\left(\tau+\beta m'\right)\right) \\
&\quad \times \mathbf{1}\left(l_{(x,i),\gamma^0(x,i)}(\tau+\beta m) = j \neq j' = l_{x',i'}\left(\tau+\beta m'\right)\right)\Bigg].
\end{aligned} \tag{90}$$

Here, in turn, we employ the shortened notation for the positions and indices of the sections $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}(\tau+\beta m)$ and $\Omega^*_{x',i'}(\tau+\beta m')$ of $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}$ and $\Omega^*_{x',i'}$ at times $\tau+\beta m$ and $\tau+\beta m'$, respectively:

$$\begin{aligned}
u_{i,x}(\tau+\beta m) &= u\left(\tau+\beta m; \overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right), \\
l_{i,x}(\tau+\beta m) &= l\left(\tau+\beta m; \overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right), \\
u_{i',x'}\left(\tau+\beta m'\right) &= u\left(\tau+\beta m'; \Omega^*_{x',i'}\right), \\
l_{i',x'}\left(\tau+\beta m'\right) &= l\left(\tau+\beta m'; \Omega^*_{x',i'}\right).
\end{aligned} \tag{91}$$

Further, the functional $\mathbf{h}^{\Lambda^0}(\overline{\boldsymbol{\Omega}}^{*0} \mid \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (82) is determined as in (71)–(73), with $\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}$ instead of $\Omega^*_{x,i}$. Next, for $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \mid \overline{\boldsymbol{\Omega}}^{*0} \vee \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (83), we set

$$\begin{aligned}
&\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \mid \overline{\boldsymbol{\Omega}}^{*0} \vee \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) \\
&\quad = \mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0}\right) + \mathbf{h}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \parallel \overline{\boldsymbol{\Omega}}^{*0} \vee \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right).
\end{aligned} \tag{92}$$

Here again, the summand $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0})$ is determined as in (67). Next, the term $\mathbf{h}^{\Lambda\setminus\Lambda^0}(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \parallel \overline{\boldsymbol{\Omega}}^{*0} \vee \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ is defined similarly to (72)-(73):

$$\begin{aligned}
&\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\boldsymbol{\Omega}^*_{\Lambda\setminus\Lambda^0} \parallel \overline{\boldsymbol{\Omega}}^{*0} \vee \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) \\
&\quad = \sum_{(i,i')\in(\Lambda\setminus\Lambda^0)\times\Lambda^0} \sum_{x\in\mathbf{x}^*(i),x'\in\mathbf{x}^*(i')} \mathbf{h}^{(x,i),(x',i')}\left(\Omega^*_{x,i}, \overline{\Omega}^*_{x',i'}\right) \\
&\quad\quad + \sum_{i\in\Lambda\setminus\Lambda^0} \sum_{x\in\mathbf{x}^*(i),\overline{x}'\in\overline{\mathbf{x}}^*(i')} \mathbf{h}^{(x,i),(\overline{x}',i')}\left(\Omega^*_{x,i}, \left(\overline{x}', i'\right)\right)
\end{aligned} \tag{93}$$

with

$$\begin{aligned}
&\mathbf{h}^{(x,i),(x',i')}\left(\Omega^*_{x,i}, \overline{\Omega}^*_{x',i'}\right) \\
&= \sum_{0\le m<k_{x,i}} \sum_{0\le m'<k_{(x',i'),\gamma^0(x',i')}} \times \int_0^\beta \mathrm{d}\tau \\
&\quad \times \Bigg[\sum_{j\in\Gamma} U^{(2)}\left(u_{x,i}(\tau+\beta m), u_{x',i'}\left(\tau+\beta m'\right)\right) \\
&\quad\quad \times \mathbf{1}\left(l_{x,i}(\tau+\beta m) = j = l_{x',i'}\left(\tau+\beta m'\right)\right) \\
&\quad\quad + \sum_{(j,j')\in\Gamma\times\Gamma} J\left(\mathrm{d}\left(j,j'\right)\right) V \\
&\quad\quad\quad \times \left(u_{i,x}(\tau+\beta m), u_{x',i'}\left(\tau+\beta m'\right)\right) \\
&\quad\quad \times \mathbf{1}\left(l_{x,i}(\tau+\beta m) = j \neq j' = l_{x',i'}\left(\tau+\beta m'\right)\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{h}^{(x,i),(\overline{x}',i')}\left(\Omega^*_{x,i},\left(\overline{x}',i'\right)\right)\\
&\quad=\sum_{0\le m<k_{x,i}}\int_0^\beta \mathrm{d}\tau\left[U^{(2)}\left(u_{x,i}\left(\tau+\beta m\right),\overline{x}'\right)\right.\\
&\qquad\times\mathbf{1}\left(l_{x,i}\left(\tau+\beta m\right)=i'\right)\\
&\qquad+\sum_{j\in\Gamma}J\left(\mathrm{d}\left(j,i'\right)\right)V\left(u_{i,x}\left(\tau+\beta m\right),\overline{x}'\right)\\
&\qquad\left.\times\mathbf{1}\left(l_{x,i}\left(\tau+\beta m\right)=j\right)\right].
\end{aligned}\tag{94}$$

As before, the functionals $\mathbf{h}^\Lambda(\mathbf{\Omega}^*_\Lambda)$ and $\mathbf{h}^\Lambda(\mathbf{\Omega}^*_\Lambda \parallel \overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ have a natural interpretation as energies of loop configurations.

Repeating the above observation, non-zero contributions to the integral in (80) come only from pairs $(\overline{\mathbf{\Omega}}^{*0},\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0})$ such that $\forall j\in\Gamma$ and $\tau\in[0,\beta]$, the total number of pairs $(u(\tau+\beta m;\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}),l(\tau+\beta m;\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}))$ with $0\le m<k_{(x,i),\gamma^0(x,i)}$, $i\in\Lambda^0$ and $x\in\mathbf{x}^*(i)$ incident to the paths of the configuration $\overline{\mathbf{\Omega}}^{*0}$ and pairs $(u(\tau+\beta m';\Omega^*_{x',i'}),l(\tau+\beta m';\Omega^*_{x',i'}))$ with $0\le m'<k_{(x',i')}$, $i'\in\Lambda\setminus\Lambda^0$ and $x'\in\mathbf{x}^*(i')$ incident to the loops of the configuration $\overline{\mathbf{\Omega}}^*_{\Lambda\setminus\Lambda^0}$ does not exceed κ. Similarly, non-zero contributions to the integral in (82) come only from pairs $(\overline{\mathbf{\Omega}}^{*0},\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0})$ such that the above inequality holds when we additionally count points $\overline{x}\in\overline{\mathbf{x}}^*(j)$.

The integral $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0})$ defined in (81) can be considered as a particular (although important) example of a partition function in the volume $\Lambda\setminus\Lambda^0$ with a boundary condition $\overline{\mathbf{\Omega}}^{*0}$. Note the presence of the subscript Λ indicating that the loops contributing to $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0})$ can jump within volume Λ only (owing to the indicator functional α_Λ). On the other hand, the presence of the indicator functional $\mathbf{1}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\in\mathscr{F}^{\Lambda^0})$ in the integral (reflected in the upperscript Λ^0 and the roof sign in the notation $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0})$) indicates a particular restriction on the jumps of the loops, forbidding them to visit set Λ^0 at intermediate times βm. This is true also for the integral $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (83): it is a particular example of a partition function in the volume $\Lambda\setminus\Lambda^0$ with a boundary condition $\overline{\mathbf{\Omega}}^{*0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}$.

Other useful types of partition functions are $\Xi_{\Gamma^0}(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1})$ and $\Xi_{\Gamma^0}(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\vee\overline{\mathbf{x}}^*_{\Gamma^2})$ where the sets of vertices $\widetilde{\Lambda}$, Λ^0, Γ^0, Γ^1, and Γ^2 satisfy

$$\begin{gathered}
\widetilde{\Lambda}\subset\Gamma^0\subseteq\Gamma,\qquad \Gamma^1,\Gamma^2\subset\Gamma\setminus\widetilde{\Lambda},\\
\Gamma^1\cap\Gamma^2=\emptyset,\qquad \Lambda^0\subset\Gamma\setminus\left(\widetilde{\Lambda}\cup\Gamma^1\cup\Gamma^2\right)
\end{gathered}\tag{95}$$

and $\sharp\widetilde{\Lambda},\sharp\Lambda^0<+\infty$. Accordingly, $\overline{\mathbf{\Omega}}^{*0}$ is a (finite) configuration over Λ^0, $\mathbf{\Omega}^*_{\Gamma^1}$ a (possibly infinite) loop configuration over Γ^1, and $\overline{\mathbf{x}}^*_{\Gamma^2}$ a (possibly infinite) particle configuration over Γ^2. The partition functions $\Xi_{\Gamma^0}(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1})$ and $\Xi_{\Gamma^0}(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\vee\overline{\mathbf{x}}^*_{\Gamma^2})$ are given by

$$\begin{aligned}
&\Xi_{\Gamma^0}\left(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\right)\\
&\quad=\int_{W^*_{\widetilde{\Lambda}}}\mathrm{d}\mathbf{\Omega}^*_{\widetilde{\Lambda}}\alpha_{\Gamma^0}\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\right)B\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\right)\\
&\qquad\times\exp\left[-\mathbf{h}^{\widetilde{\Lambda}}\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\right)\right],
\end{aligned}\tag{96}$$

$$\begin{aligned}
&\Xi_{\Gamma^0}\left(\widetilde{\Lambda}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\vee\overline{\mathbf{x}}^*_{\Gamma^2}\right)\\
&\quad=\int_{W^*_{\widetilde{\Lambda}}}\mathrm{d}\mathbf{\Omega}^*_{\widetilde{\Lambda}}\alpha_{\Gamma^0}\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\right)B\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\right)\\
&\qquad\times\exp\left[-\mathbf{h}^{\widetilde{\Lambda}}\left(\mathbf{\Omega}^*_{\widetilde{\Lambda}}\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}\vee\overline{\mathbf{x}}^*_{\Gamma^2}\right)\right]
\end{aligned}\tag{97}$$

with the indicator α_{Γ^0} as in (74). These partition functions, feature loop configurations $\mathbf{\Omega}^*_{\widetilde{\lambda}}$ formed by loops $\Omega^*_{x,i}$, $i\in\widetilde{\Lambda}$, which start and finish in $\widetilde{\Lambda}$, are confined to Γ^0 and move in a potential field generated by $\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Gamma^1}$, where $\overline{\mathbf{\Omega}}^{*0}=\{\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\}$ and $\mathbf{\Omega}_{\Gamma^1}=\{\Omega^*_{x,i},\ x\in\mathbf{x}^*(i),\ i\in\Gamma^1\}$ or $\mathbf{\Omega}^*_{\Gamma^1}\vee\overline{\mathbf{x}}^*_{\Gamma^2}$, where $\overline{\mathbf{x}}^*_{\Gamma^2}=\{\overline{\mathbf{x}}^*(i),\ i\in\Gamma^2\}$. (The latter can be understood as the concatenation of the loop configuration $\mathbf{\Omega}^*_{\Gamma^1}$ over Γ^1 and the loop configuration over Γ^2 formed by the constant trajectories sitting at points $\overline{x}\in\overline{\mathbf{x}}^*(i)$, $i\in\Gamma^2$.) In (96) we assume that, $\forall\tau\in[0,\beta]$ and $j\in\Gamma$, the number

$$\begin{aligned}
&\sharp\left\{(x,i,m):i\in\widetilde{\Lambda},\ l\left(\tau+m\beta;\Omega^*_{x,i}\right)=j,\ 0\le m<k_{x,i}\right\}\\
&\quad+\sharp\left\{(x,i):i\in\Lambda^0,\ l\left(\tau+m\beta;\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right)\right.\\
&\qquad\left.=j,\ 0\le m<k_{(x,i),\gamma^0(x,i)}\right\}\\
&\quad+\sharp\left\{(x,i,m):i\in\Gamma^1,\right.\\
&\qquad\left.l\left(\tau+m\beta;\Omega^*_{x,i}\right)=j,\ 0\le m<k_{x,i}\right\}
\end{aligned}\tag{98}$$

does not exceed κ. Analogously, in (97) it is assumed that the same is true for the above number plus the cardinality $\sharp\overline{\mathbf{x}}^*(j)$.

Such "modified" partition functions will be used in forthcoming sections.

2.4. The FK-DLR Measure $\boldsymbol{\mu}_\Lambda$ in a Finite Volume. The Gibbs states $\boldsymbol{\varphi}_\Lambda$ and $\boldsymbol{\varphi}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ give rise to probability measures $\boldsymbol{\mu}_\Lambda$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ on the sigma algebra $\mathfrak{W}_\Lambda$ of subsets of W^*_Λ. The sigma algebra $\mathfrak{W}_\Lambda$ is constructed by following the structure of the space W^*_Λ (a disjoint union of Cartesian products); confer Definition 16. The measures $\boldsymbol{\mu}_\Lambda$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ are determined by their Radon-Nykodym derivative p_Λ and $p_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ relative to the measure $\mathrm{d}\mathbf{\Omega}^*_\Lambda$:

$$\begin{aligned} p_\Lambda(\mathbf{\Omega}^*_\Lambda) &:= \frac{\boldsymbol{\mu}_\Lambda(\mathrm{d}\mathbf{\Omega}^*_\Lambda)}{\mathrm{d}\mathbf{\Omega}^*_\Lambda} \\ &= \frac{1}{\Xi(\Lambda)}\alpha_\Lambda(\mathbf{\Omega}^*_\Lambda)B(\mathbf{\Omega}^*_\Lambda) \\ &\quad\times\exp\left[-\mathbf{h}^\Lambda(\mathbf{\Omega}^*_\Lambda)\right],\quad \mathbf{\Omega}^*_\Lambda\in W^*_\Lambda, \\ p_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{\Omega}^*_\Lambda) &:= \frac{\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathrm{d}\mathbf{\Omega}^*_\Lambda)}{\mathrm{d}\mathbf{\Omega}^*_\Lambda}\frac{1}{\Xi\left(\Lambda\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)} \\ &= \alpha_\Lambda(\mathbf{\Omega}^*_\Lambda)B(\mathbf{\Omega}^*_\Lambda) \\ &\quad\times\exp\left[-\mathbf{h}^\Lambda\left(\mathbf{\Omega}^*_\Lambda\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\right],\quad \mathbf{\Omega}^*_\Lambda\in W^*_\Lambda. \end{aligned} \tag{99}$$

Given $\Lambda^0\subset\Lambda$, the sigma algebra $\mathfrak{W}_{\Lambda^0}$ is naturally identified with a sigma subalgebra of $\mathfrak{W}_\Lambda$. The restrictions of $\boldsymbol{\mu}_\Lambda$ to $\mathfrak{W}_{\Lambda^0}$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ are denoted by $\boldsymbol{\mu}^{\Lambda^0}_\Lambda$ and $\boldsymbol{\mu}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$; these measures are determined by their Radon-Nikodym derivatives $p^{\Lambda^0}_\Lambda(\mathbf{\Omega}^*_{\Lambda^0}) := \boldsymbol{\mu}^{\Lambda^0}_\Lambda(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^0})/\mathrm{d}\mathbf{\Omega}^*_{\Lambda^0}$ and $p^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{\Omega}^*_{\Lambda^0}) := \boldsymbol{\mu}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^0})/\mathrm{d}\mathbf{\Omega}^*_{\Lambda^0}$.

The first key property of the measures $\boldsymbol{\mu}_\Lambda$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ is expressed in the so-called FK-DLR equation. We state it as Lemma 21 below; its proof repeats a standard argument used in the classical case for establishing the DLR equation in a finite volume $\Lambda\subset\Gamma$.

Lemma 21. *For all $z,\beta>0$ satisfying* (22), *and $\Lambda^0\subset\Lambda'\subset\Lambda$, the probability density $p^{\Lambda^0}_\Lambda$ admits the form:*

$$p^{\Lambda^0}_\Lambda(\mathbf{\Omega}^*_{\Lambda^0}) = \int_{W^*_{\Lambda\setminus\Lambda'}} q^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)\boldsymbol{\mu}^{\Lambda\setminus\Lambda'}_\Lambda\left(\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right), \tag{100}$$

where

$$q^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right) = \exp\left[-\mathbf{h}^{\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)\right] \times\frac{\Xi_\Lambda\left(\Lambda'\setminus\Lambda^0\mid\mathbf{\Omega}^*_{\Lambda^0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)}{\Xi_\Lambda\left(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)}, \tag{101}$$

and the conditional partition functions $\Xi_\Lambda(\Lambda'\setminus\Lambda^0\mid\mathbf{\Omega}^*\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ and $\Xi_\Lambda(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ are determined as in (96).

Similarly, for $p^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ one has:

$$p^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{\Omega}^*_{\Lambda^0}) = \int_{W^*_{\Lambda\setminus\Lambda'}} q^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) \times\boldsymbol{\mu}^{\Lambda\setminus\Lambda'}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\left(\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right), \tag{102}$$

where

$$\begin{aligned} &q^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) \\ &\quad= \exp\left[-\mathbf{h}^{\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\right] \\ &\quad\times\frac{\Xi_\Lambda\left(\Lambda'\setminus\Lambda^0\mid\mathbf{\Omega}^*_{\Lambda^0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)}{\Xi_\Lambda\left(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)} \end{aligned} \tag{103}$$

and the conditional partition functions $\Xi_\Lambda(\Lambda'\setminus\Lambda^0\mid\mathbf{\Omega}^*_{\Lambda^0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ and $\Xi_\Lambda(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ are determined as in (96).

As in [1], (100) and (102) mean that the conditional densities $p^{\Lambda^0}_\Lambda(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ and $p^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ relative to σ-algebra $\mathfrak{W}^{\Lambda\setminus\Lambda'}$ coincide, respectively, with $q^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$, and $q^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$, for $\boldsymbol{\mu}^{\Lambda\setminus\Lambda'}_\Lambda$—and $\boldsymbol{\mu}^{\Lambda\setminus\Lambda'}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$—a.a. $\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\in W^*_{\Lambda\setminus\Lambda'}$ and a.a. $\mathbf{\Omega}^*_{\Lambda^0}\in W^*_{\Lambda^0}$.

As in [1], we call the expressions $q^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ and $q^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$, as well as the expressions $\widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ and $\widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ appearing below, the (conditional) RDM functionals (in brief, the RDMFs). The same name will be used for the quantity $q^{\Lambda^0}_{\Lambda\setminus\Lambda'}(\mathbf{\Omega}^*_{\Lambda^0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ from (110)-(111) and the quantity $\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda}(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$ from (112)-(113).

The second property is that the RDMKs $\mathbf{F}^{\Lambda^0}_\Lambda(\mathbf{x}^{*0},\mathbf{y}^{*0})$ and $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ are related to the measures $\boldsymbol{\mu}_\Lambda$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda'}}$. Again, the proof of this fact is done by inspection.

Lemma 22. *The RDMK $\mathbf{F}^{\Lambda^0}_\Lambda(\mathbf{x}^{*0},\mathbf{y}^{*0})$ is expressed as follows:* $\forall\Lambda^0\subset\Lambda'\subset\Lambda$,

$$\begin{aligned} &\mathbf{F}^{\Lambda^0}_\Lambda\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right) \\ &\quad= \int_{\overline{W}^*_{\mathbf{x}^0,\mathbf{y}^0}} \overline{\mathbb{P}}^*_{\mathbf{x}^0,\mathbf{y}^0}\left(\mathrm{d}\overline{\mathbf{\Omega}}^{*0}\right)\alpha_\Lambda\left(\overline{\mathbf{\Omega}}^{*0}\right) \\ &\quad\times\overline{B}\left(\overline{\mathbf{\Omega}}^{*0}\right)\mathbf{1}\left(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0}\right) \\ &\quad\times\int_{W^*_{\Lambda\setminus\Lambda'}}\boldsymbol{\mu}^{\Lambda\setminus\Lambda'}\left(\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)\mathbf{1}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\in\mathscr{F}^{\Lambda^0}\right) \\ &\quad\times\widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right), \end{aligned} \tag{104}$$

where

$$\begin{aligned} &\widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right) \\ &\quad= \exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)\right] \\ &\quad\times\frac{\widehat{\Xi}^{\Lambda^0}_\Lambda\left(\Lambda'\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)}{\Xi_\Lambda\left(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)}. \end{aligned} \tag{105}$$

Similarly,

$$\begin{aligned}\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right) &= \int_{\overline{W}^*_{\mathbf{x}^0,\mathbf{y}^0}} \overline{\mathbb{P}}^*_{\mathbf{x}^0,\mathbf{y}^0}\left(\mathrm{d}\overline{\mathbf{\Omega}}^{*0}\right)\alpha_\Lambda\left(\overline{\mathbf{\Omega}}^{*0}\right) \\ &\quad \times \overline{B}\left(\overline{\mathbf{\Omega}}^{*0}\right)\mathbf{1}\left(\overline{\mathbf{\Omega}}^{*0}\in\mathscr{F}^{\Lambda^0}\right) \\ &\quad \times \int_{W^*_{\Lambda\setminus\Lambda'}} \boldsymbol{\mu}^{\Lambda\setminus\Lambda'}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\left(\mathrm{d}\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\right)\mathbf{1}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\in\mathscr{F}^{\Lambda^0}\right) \\ &\quad \times \widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right),\end{aligned}\tag{106}$$

where

$$\begin{aligned}\widehat{q}^{\Lambda^0}_{\Lambda\setminus\Lambda'}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'},\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right) &= \exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\right] \\ &\quad \times \frac{\widehat{\Xi}^{\Lambda^0}_\Lambda\left(\Lambda'\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)}{\Xi_\Lambda\left(\Lambda'\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)}.\end{aligned}\tag{107}$$

Here the partition functions $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda'\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'})$ *and* $\widehat{\Xi}^{\Lambda^0}_\Lambda(\Lambda'\setminus\Lambda^0\mid\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^*_{\Lambda\setminus\Lambda'}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ *are determined in* (81) *and* (83).

Remark 23. Summarizing the above observations, the measures $\boldsymbol{\mu}_\Lambda$ and $\boldsymbol{\mu}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ are concentrated on the subset in W^*_Λ formed by loop configurations $\mathbf{\Omega}^*_\Lambda$ such that $\forall\tau\in[0,\beta]$, the section $\mathbf{\Omega}^*_\Lambda(\tau)$ has $\le\kappa$ particles at each vertex $i\in\Lambda$.

3. The Class of Gibbs States $\mathfrak{G}$ for the Fock Space Model

3.1. Definition of Class $\mathfrak{G}$. In this section we apply the idea from [1] to define the class of states $\mathfrak{G}=\mathfrak{G}_{z,\beta}$ for the model introduced in Section 2 and state a number of results. These results will hold under condition (34) which is assumed from now on. As in [1], the definition of a state $\boldsymbol{\varphi}\in\mathfrak{G}$ is based on the notion of an FK-DLR probability measure $\boldsymbol{\mu}$ on the space W^*_Γ; the class of these measures will be also denoted by $\mathfrak{G}$.

Definition 24. Space W^*_Γ is the (infinite) Cartesian product $\times_{i\in\Gamma}W^*_{\{i\}}$ (cf. (60)); its elements are loop configurations $\mathbf{\Omega}^*_\Gamma=\{\mathbf{\Omega}^*(i),\ i\in\Gamma\}$ over Γ. A component $\mathbf{\Omega}^*(i)$ is a finite loop configuration (possibly, empty), with an initial/final particle configuration $\mathbf{x}^*(i)\subset M$. Formally, $\mathbf{\Omega}^*(i)$ is a finite collection of loops $\Omega^*_{x,i}$, of time-length $\beta k_{x,i}$ where $k_{x,i}=1,2,\ldots$, starting and finishing at a point $(x,i)\in M\times\Gamma$. For reader's convenience, we repeat (37) for the case under consideration:

$$\begin{gathered}\Omega^*_{x,i}:\tau\in[0,\beta k_{x,i}]\longmapsto\left(\widetilde{x}\left(\Omega^*_{x,i},\tau\right),\widetilde{i}\left(\Omega^*_{x,i},\tau\right)\right)\in M\times\Gamma, \\ \Omega^*_{x,i}\ \text{is càdlág};\quad \Omega^*_{x,i}(0)=\Omega^*_{x,i}(\beta k_{x,i}-)=(x,i), \\ \Omega^*_{x,i}\ \text{has finitely many jumps on }[0,\beta k_{x,i}]; \\ \text{if a jump occurs at time }\tau,\ \text{then} \\ \mathrm{d}\left[\widetilde{i}\left(\Omega^*_{x,i},\tau-\right),\widetilde{i}\left(\Omega^*_{x,i},\tau\right)\right]=1.\end{gathered}\tag{108}$$

By $\mathfrak{W}=\mathfrak{W}_\Gamma$ we denote the σ-algebra in W^*_Γ generated by cylindrical events. Given a subset $\overline{\Gamma}\subset\Gamma$ (finite or infinite), we denote by $\mathfrak{W}^{\overline{\Gamma}}=\mathfrak{W}^{\overline{\Gamma}}_\Gamma$ the σ-subalgebra of $\mathfrak{W}$ generated by cylindrical events localized in $\overline{\Gamma}$. Given a probability measure $\boldsymbol{\mu}=\boldsymbol{\mu}_\Gamma$ on $(W^*_\Gamma,\mathfrak{W}_\Gamma)$, we denote by $\boldsymbol{\mu}^{\overline{\Gamma}}=\boldsymbol{\mu}^{\overline{\Gamma}}_\Gamma$ the restriction of $\boldsymbol{\mu}$ on $\mathfrak{W}^{\overline{\Gamma}}$.

Definition 25. The class $\mathfrak{G}$ under consideration is formed by measures $\boldsymbol{\mu}$ which satisfy the following equation: $\forall$ finite $\Lambda\subset\Gamma$ and $\Lambda^0\subseteq\Lambda$, the probability density:

$$p^{\Lambda^0}\left(\mathbf{\Omega}^{*0}\right)=p^{\Lambda^0}_{\boldsymbol{\mu}}\left(\mathbf{\Omega}^{*0}\right):=\frac{\boldsymbol{\mu}^{\Lambda^0}_\Gamma\left(\mathrm{d}\mathbf{\Omega}^{*0}\right)}{\boldsymbol{\nu}\left(\mathrm{d}\mathbf{\Omega}^{*0}\right)},\quad \mathbf{\Omega}^{*0}\in W^*_{\Lambda^0},\tag{109}$$

is of the form

$$p^{\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda^0}\right)=\int_{W^*_{\Gamma\setminus\Lambda}} q^{\Lambda^0}_{\Gamma\setminus\Lambda}\left(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right)\boldsymbol{\mu}^{\Gamma\setminus\Lambda}\left(\mathrm{d}\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right),\tag{110}$$

where

$$\begin{aligned}q^{\Lambda^0}_{\Gamma\setminus\Lambda}\left(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right) &= \exp\left[-\mathbf{h}^{\Lambda^0}\left(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right)\right] \\ &\quad\times\frac{\Xi_\Gamma\left(\Lambda\setminus\Lambda^0\mid\mathbf{\Omega}^{*0}\vee\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right)}{\Xi_\Gamma\left(\Lambda\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\right)},\end{aligned}\tag{111}$$

and the conditional partition functions $\Xi_\Gamma(\Lambda\setminus\Lambda^0\mid\mathbf{\Omega}^{*0}\vee\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$ and $\Xi_\Gamma(\Lambda\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$ are determined as in (96).

As in [1], (110) means that the conditional density $p^{\Lambda^0|\Gamma\setminus\Lambda}(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$, relative to σ-algebra $\mathfrak{W}^{\Gamma\setminus\Lambda}$, coincides with $q^{\Lambda^0}_{\Gamma\setminus\Lambda}(\mathbf{\Omega}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$, for $\boldsymbol{\mu}^{\Gamma\setminus\Lambda}_\Gamma$—a.a. $\mathbf{\Omega}^*_{\Gamma\setminus\Lambda}\in W^{*\beta}_{\Gamma\setminus\Lambda}$ and $\boldsymbol{\nu}_{\Lambda^0}$—a.a. $\mathbf{\Omega}^{*0}\in W^*_{\Lambda^0}$.

Remark 26. The measure $\boldsymbol{\mu}_\Gamma$ inherits the property from Remark 23 and is concentrated on the subset in W^*_Γ formed by (infinite) loop configurations $\mathbf{\Omega}^*_\Gamma$ such that, for all $\tau\in[0,\beta]$, the section $\mathbf{\Omega}^*_\Lambda(\tau)$ has $\le\kappa$ particles at each vertex $i\in\Lambda$.

Given a measure $\boldsymbol{\mu} \in \mathfrak{G}$, we associate with it a normalized linear functional $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\boldsymbol{\mu}}$ on the quasilocal C^*-algebra $\mathfrak{B}$. First, we set

$$\begin{aligned}\mathbf{F}^{\Lambda^0}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right) &= \int_{\overline{W}^*_{\mathbf{x}^0,\mathbf{y}^0}} \overline{\mathbb{P}}^*_{\mathbf{x}^0,\mathbf{y}^0}\left(\mathrm{d}\overline{\boldsymbol{\Omega}}^{*0}\right)\overline{B}\left(\overline{\boldsymbol{\Omega}}^{*0}\right)\mathbf{1}\left(\overline{\boldsymbol{\Omega}}^{*0} \in \mathscr{F}^{\Lambda^0}\right)\\ &\times \int_{W^*_{\Gamma\setminus\Lambda}} \boldsymbol{\mu}^{\Gamma\setminus\Lambda}\left(\mathrm{d}\boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right)\mathbf{1}\left(\boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda} \in \mathscr{F}^{\Lambda^0}\right)\\ &\times \widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda}\left(\overline{\boldsymbol{\Omega}}^{*0} \mid \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right),\end{aligned} \tag{112}$$

where

$$\begin{aligned}\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda}\left(\overline{\boldsymbol{\Omega}}^{*0} \mid \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right) &= \exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\boldsymbol{\Omega}}^{*0} \mid \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right)\right]\\ &\times \frac{\widehat{\Xi}^{\Lambda^0}_{\Lambda}\left(\Lambda\setminus\Lambda^0 \mid \overline{\boldsymbol{\Omega}}^{*0} \vee \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right)}{\Xi_\Gamma\left(\Lambda \mid \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda}\right)}.\end{aligned} \tag{113}$$

This defines a kernel $\mathbf{F}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0})$, $\mathbf{x}^{*0},\mathbf{y}^{*0} \in M^{*\Lambda^0}$, where $\Lambda^0 \subset \Gamma$ is a finite set of sites. It is worth reminding the reader of the presence of the indicator functionals $\mathbf{1}(\cdot \in \mathscr{F}^{\Lambda^0})$ in (112) and (113) (in the integral for $\widehat{\Xi}^{\Lambda^0}_{\Lambda}(\Lambda\setminus\Lambda^0 \mid \overline{\boldsymbol{\Omega}}^{*0} \vee \boldsymbol{\Omega}^*_{\Gamma\setminus\Lambda})$). These indicators guarantee the compatibility property: $\forall$ finite $\Lambda^0 \subset \Lambda^1$,

$$\begin{aligned}\mathbf{F}^{\Lambda^0}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right) &= \int_{M^{*\Lambda^1\setminus\Lambda^0}} \prod_{j\in\Lambda^1\setminus\Lambda^0}\prod_{z\in\mathbf{z}^*(j)} \nu(\mathrm{d}z)\\ &\times \mathbf{F}^{\Lambda^1}\left(\mathbf{x}^{*0}\vee\mathbf{z}^*_{\Lambda^1\setminus\Lambda^0},\mathbf{y}^{*0}\vee\mathbf{z}^*_{\Lambda^1\setminus\Lambda^0}\right).\end{aligned} \tag{114}$$

Next, we identify the operator $\mathbf{R}^{\Lambda^0}$ (a candidate for the RDM in volume Λ^0) as an integral operator acting in $\mathscr{H}_{\Lambda^0}$ by

$$\left(\mathbf{R}^{\Lambda^0}\phi\right)\left(\mathbf{x}^{*0}\right) = \int_{M^{*\Lambda^0}} \mathbf{F}^{\Lambda^0}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right)\phi\left(\mathbf{y}^{*0}\right)\mathrm{d}\mathbf{y}^{*0}. \tag{115}$$

Equation (114) implies that

$$\operatorname{tr}_{\mathscr{H}_{\Lambda^1\setminus\Lambda^0}}\mathbf{R}^{\Lambda^1} = \mathbf{R}^{\Lambda^0}. \tag{116}$$

Definition 27. The functional $\boldsymbol{\varphi} \in \mathfrak{G}$ is identified with the (compatible) family of operators $\mathbf{R}^{\Lambda^0}$. If the operators $\mathbf{R}^{\Lambda^0}$ are positive definite (a property that is not claimed to be automatically fulfilled), we again call it an FK-DLR state in the infinite volume (for given values of activity z and inverse temperature β). To stress the dependence on z and β, we sometimes employ the notation $\mathfrak{G}(z,\beta)$.

3.2. Theorems on Existence and Properties of FK-DLR States. We are now in position to state results about class $\mathfrak{G}$. We assume the conditions on the potentials $U^{(1)}$ and $U^{(2)}$ from the previous section, including the hard-core condition for $U^{(1)}$.

Theorem 28. *For all $z,\beta \in (0,+\infty)$ satisfying* (22)*, any limiting Gibbs state $\boldsymbol{\varphi} \in \mathfrak{G}^0$ (see Theorem 3) lies in $\mathfrak{G}$. Therefore, the class of state $\mathfrak{G}$ s is nonempty.*

Theorem 29. *Under condition* (22)*, any FK-DLR state $\boldsymbol{\varphi} \in \mathfrak{G}$ is $\mathtt{G}$-invariant, in the sense that, $\forall$ finite $\Lambda^0 \subset \Gamma$ and $\forall \mathtt{g} \in \mathtt{G}$, the RDM $\mathbf{R}^{\Lambda^0}$ satisfies* (35)*. Consequently,* (36) *holds true.*

4. Proof of Theorems 3, 6, 28, and 29

4.1. Proof of Theorems 3 and 28. The proof is based on the same approach as that used in [1]. First, given $\Lambda^0 \subset \Gamma$, we establish compactness of the sequence of the RDMKs $\mathbf{F}^{\Lambda^0}_{\Lambda}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ and $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ (see (78)–(83)) as functions of variables $\mathbf{x}^{*0} = \{\mathbf{x}^{*0}(i)\}$, $\mathbf{y}^{*0} = \{\mathbf{y}^{*0}(i)\} \in M^{*\Lambda^0}$, with

$$\sharp\mathbf{x}^{*0}_{\Lambda} = \sharp\mathbf{y}^{*0}_{\Lambda}, \qquad \sharp\mathbf{x}^{*0}(i), \sharp\mathbf{y}^{*0}(i) < \kappa, \quad i \in \Lambda, \tag{117}$$

when $\Lambda \nearrow \Gamma$. Then we use Lemma 1.1 from [1] to derive that the sequence of the RDMs $\mathbf{R}^{\Lambda^0}_{\Lambda}$ and $\mathbf{R}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}$ is compact in the trace-norm operator topology in $\mathscr{H}_{\Lambda^0}$.

To verify compactness of the RDMKs $\mathbf{F}^{\Lambda^0}_{\Lambda}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ and $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ we, again as in [1], use the Ascoli-Arzela theorem, which requires the properties of uniform boundedness and equicontinuity. These properties follow from the following.

Lemma 30. (i) *Under condition* (22) *the RDMKs $\mathbf{F}^{\Lambda^0}_{\Lambda}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ and $\mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ admit the bounds*

$$\begin{aligned}&\mathbf{F}^{\Lambda^0}_{\Lambda}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right), \mathbf{F}^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right)\\ &\quad \le \left[\left(\kappa\sharp\Lambda^0\right)!\right]\left(\widehat{p}_M\right)^{\kappa\sharp\Lambda^0}\Phi^{\sharp\Lambda^0},\end{aligned} \tag{118}$$

where

$$\begin{aligned}\Phi &= \sum_{k\ge1} z^k \exp(k\Theta),\\ \text{with } \Theta &= \kappa\beta\left(\overline{U}^{(1)} + \kappa\overline{U}^{(2)} + \kappa\overline{J}(1)\overline{V}\right).\end{aligned} \tag{119}$$

(Note that $\Phi < \infty$ under the assumption (22)*) Let $p^{(k)}_M$ yields the supremum of the transition function over time $k\beta$ for Brownian motion on the torus M:*

$$p^{(k)}_M = \sup_{x,y\in M} p^{k\beta}(x,y) = p^{k\beta}(0,0), \tag{120}$$

and $\widehat{p}_M = \sup_{k\ge1} p^{(k)}_M$. Finally, the upper-bound values $\overline{U}^{(1)}$, $\overline{U}^{(2)}$, $\overline{J}(1)$, and $\overline{V}$ have been determined in (14), (15), *and* (18).

(ii) *The gradients of the RDMKs* $\mathbf{F}_{\Lambda}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ *and* $\mathbf{F}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0})$ *satisfy* $\forall i\in\Lambda$ *and* $x\in\mathbf{x}^{*0}(i)$, $y\in\mathbf{y}^{*0}(i)$,

$$\begin{aligned}&\left|\nabla_x\mathbf{F}_{\Lambda}^{\Lambda^0}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right)\right|,\left|\nabla_x\mathbf{F}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}^{\Lambda^0}\left(\mathbf{x}^{0},\mathbf{y}^{0}\right)\right|\\ &\left|\nabla_y\mathbf{F}_{\Lambda}^{\Lambda^0}\left(\mathbf{x}^{*0},\mathbf{y}^{*0}\right)\right|,\left|\nabla_y\mathbf{F}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}^{\Lambda^0}\left(\mathbf{x}^{0},\mathbf{y}^{0}\right)\right|\\ &\quad\leq\left(\sharp\Lambda^0\right)\Theta\Phi'\widehat{p}_M\left[\left(\kappa\sharp\Lambda^0\right)!\right]\left(\widehat{p}_M\right)^{\kappa\sharp\Lambda^0}\Phi^{\sharp\Lambda^0},\end{aligned}\tag{121}$$

where

$$\Phi'=\sum_{k\geq 1}kz^k\exp\left(k\Theta\right).\tag{122}$$

(Again, $\Phi'<\infty$ *under the condition* (22).*) The ingredients* $\widehat{p}_M$ *and* $\overline{U}^{(1)}$, $\overline{U}^{(2)}$, $\overline{J}(1)$, *and* $\overline{V}$ *as in statement (i).*

Proof of Lemma 30. First, observe that $\widehat{p}_M<\infty$ on a compact manifold. Bound (119) is established in a direct fashion. First, we majorize the energy

$$\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)=\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\right)+\mathbf{h}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\parallel\overline{\mathbf{\Omega}}^{*0}\right)\tag{123}$$

contributing to the RHS in (80) and (81) and the energy

$$\begin{aligned}&\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\vee\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)\\ &\quad=\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}\right)+\mathbf{h}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\parallel\overline{\mathbf{\Omega}}^{*0}\right)\end{aligned}\tag{124}$$

contributing to the RHS in (82) and (83). This yields the factor

$$\prod_{\overline{\omega}\in\overline{\mathbf{\Omega}}^{*0}}z^{k(\overline{\omega}^*)}\exp\left[k\left(\overline{\omega}^*\right)\Theta\right].\tag{125}$$

Next, we majorize the integral $=\int_{\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}}\overline{\mathbb{P}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}(d\overline{\mathbf{\Omega}}^{*0})$ in (80) and (82); this gives the factor

$$\left(\sharp\Lambda^0\right)\widehat{p}_M\left[\left(\kappa\sharp\Lambda^0\right)!\right]\left(\widehat{p}_M\right)^{\kappa\sharp\Lambda^0}\Phi^{\sharp\Lambda^0}.\tag{126}$$

The aftermath are the ratios (78) and (79) with $\mathbf{x}^{*0}=\mathbf{y}^{*0}=\emptyset$; they do not exceed 1.

Passing to (121), let us discuss the gradients ∇_x only. (The gradients in the entries of ∇_y are included by symmetry.) The gradient in (121), of course, affects only the numerators $\widehat{\Xi}_{\Lambda}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0)$ and $\widehat{\Xi}_{\Lambda}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0};\Lambda\setminus\Lambda^0\mid\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda})$ in (78) and (79). The bounds (121) are done essentially as in [1]. For definiteness, we discuss the case of the RDMK $\mathbf{F}_{\Lambda}^{\Lambda^0}(\mathbf{x}^{*0},\mathbf{y}^{*0})$; the RDMK $\mathbf{F}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}^{\Lambda^0}(\mathbf{x}^{0},\mathbf{y}^{0})$ is treated similarly. There are two contributions into the gradient: one comes from varying the measure $\overline{\mathbb{P}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}(d\overline{\mathbf{\Omega}}^{*0})$, and the other from varying the functional $\exp[-\mathbf{h}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0})-\mathbf{h}^{\Lambda\setminus\Lambda^0}(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0})]$.

The first contribution can again be uniformly bounded in terms of the constant $\widehat{p}_M$. The detailed argument, as in [1], includes a deformation of a trajectory and is done similarly to [1] (the presence of jumps does not change the argument because $\widehat{p}_M$ yields a uniform bound in (43)).

The second contribution yields, again as in [1], an expression of the form:

$$\begin{aligned}&\int_{\overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}}\overline{\mathbb{P}}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}\left(d\overline{\mathbf{\Omega}}^{*0}\right)\\ &\quad\times\sum_{i\in\Lambda^0}\sum_{x\in\mathbf{x}^{*0}(i)}\widetilde{\mathbf{h}}_{x,i}\left(\overline{\Omega}^*_{(x,i),y^0(x,i)},\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\right)\\ &\quad\times\exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\right)-\mathbf{h}^{\Lambda\setminus\Lambda^0}\left(\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0}\mid\overline{\mathbf{\Omega}}^{*0}\right)\right],\end{aligned}\tag{127}$$

where the functional $\widetilde{\mathbf{h}}_{x,i}(\overline{\Omega}^*_{(x,i),y^0(x,i)},\mathbf{\Omega}^*_{\Lambda\setminus\Lambda^0})$ is uniformly bounded. Combining this an upper bound similar to (119) yields the desired estimate for the gradients in (121). □

Hence, we can guarantee that the RDMs $\mathbf{R}_{\Lambda}^{\Lambda^0}$ and $\mathbf{R}_{\Lambda|\overline{\mathbf{x}}^*_{\Gamma\setminus\Lambda}}^{\Lambda^0}$ converge to a limiting RDM $\mathbf{R}^{\Lambda^0}$ along a subsequence in $\Lambda\nearrow\Gamma$. The diagonal process yields convergence for every finite $\Lambda^0\subset\Gamma$. A parallel argument leads to compactness of the measures $\mu_{\Lambda}^{\Lambda^0}$ for any given Λ^0 as $\Lambda\nearrow\Gamma$. We only give here a sketch of the corresponding argument, stressing differences with its counterpart in [1].

In the probabilistic terminology, measures μ_Λ represent random marked point fields on $M\times\Gamma$ with marks from the space $W^*=W_0^*$ where $W_0^*=\bigcup_{k\geq1}W_0^{k\beta}$ and $W_0^{k\beta}$ is the space of loops of time-length $k\beta$ starting and finishing at $0\in M$ and exhibiting jumps, that is, changes of the index. (The space $W^*_{x,i}$ introduced in Definitions 7 and 13 can be considered as a copy of W^* placed at site $i\in\Gamma$ and point $x\in M$.) The measure $\mu_{\Lambda}^{\Lambda^0}$ describes the restriction of μ_Λ to volume Λ^0 (i.e., to the sigma algebra $\mathfrak{W}^*_{\Lambda^0}$) and is given by its Radon-Nikodym derivative $p_{\Lambda}^{\Lambda^0}$ relative to the reference measure $d\mathbf{\Omega}^*_{\Lambda^0}$ on $W^*_{\Lambda^0}$ (cf. (66), (100)). The reference measure is sigma-finite. Moreover, under the condition (22), the value $p_{\Lambda}^{\Lambda^0}(\mathbf{\Omega}^*_{\Lambda^0})$ is uniformly bounded (in both $\Lambda\nearrow\Gamma$ and $\mathbf{\Omega}^*_{\Lambda^0}\in W^*_{\Lambda^0}$). This enables us to verify tightness of the family of measures $\{\mu_{\Lambda}^{\Lambda^0},\Lambda\nearrow\Gamma\}$ and apply the Prokhorov theorem. Next, we use the compatibility property of the limit-point measures $\mu_{\Gamma}^{\Lambda^0}$ and apply the Kolmogorov theorem. This establishes the existence of the limit-point measure μ_Γ.

By construction, and owing to Lemmas 21 and 22, the limiting family $\{\mathbf{R}^{\Lambda^0}\}$ yields a state belonging to the class $\mathfrak{G}$. This completes the proof of Theorems 3 and 28.

4.2. Proof of Theorems 6 and 29. The assertion of Theorem 6 is included in Theorem 29. Therefore, we will focus on the proof of the latter. The proof based on the analysis of the conditional RDMFs $q_{\Gamma\setminus\Lambda}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$ and $\widehat{q}_{\Gamma\setminus\Lambda}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^*_{\Gamma\setminus\Lambda})$ introduced in (111) and (112). For definiteness, we assume that vertex $o\in\Lambda^0$, so that Λ^0 lies in the ball Λ_n for n large enough. As in

[1], the problem is reduced to checking that $\forall z, \beta \in (0,\infty)$ satisfying (22), $\mathtt{g} \in \mathtt{G}$ and finite $\Lambda^0 \subset \Gamma$,

$$\lim_{n\to\infty} \frac{\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}\left(\mathtt{g}\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)}{\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)} = 1; \tag{128}$$

here we need to establish this convergence (128) uniformly in the argument $\mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)} = \{\mathbf{\Omega}^*(i),\ i \in \Gamma\setminus\Lambda(n)\}$ with $\sharp\mathbf{\Omega}^*(i) \le \kappa$ and in $\overline{\mathbf{\Omega}}^{*0}$ outside a set of the $\overline{\mathbb{P}}^{*\beta}_{\mathbf{x}^0,\mathbf{y}^0}$ measure tending to 0 as $n \to \infty$. The latter is formed by path configurations $\overline{\mathbf{\Omega}}^{*0}$ that contain trajectories visiting sites $i \in \Gamma \setminus \Lambda(\overline{r}(n))$ where $\overline{r}(n)$ grows with n; see Lemma 31 below. The action of $\mathtt{g}$ upon a path configuration $\overline{\mathbf{\Omega}}^{*0} = \{\overline{\Omega}^*_{(x,i),\gamma^0(x,i)},\ i \in \Lambda^0,\ x \in \mathbf{x}^{*0}\}$ is defined by

$$\begin{aligned} &\mathtt{g}\overline{\mathbf{\Omega}}^{*0} = \left\{\mathtt{g}\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right\}, \\ &\text{where } \left(\mathtt{g}\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}\right)(\tau) = \mathtt{g}\left(\overline{\Omega}^*_{(x,i),\gamma^0(x,i)}(\tau)\right). \end{aligned} \tag{129}$$

We want to establish that $\forall a \in (1,\infty)$, for any n large enough, the conditional RDMFs satisfy

$$\begin{aligned} &a\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}\left(\mathtt{g}\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right) + a\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}\left(\mathtt{g}^{-1}\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right) \\ &\quad \ge 2\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right). \end{aligned} \tag{130}$$

As in [1], we deduce (130) with the help of a special construction of "tuned" actions $\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}$ on loop configurations $\boldsymbol{\omega}_{\Lambda(n)\setminus\Lambda^0}$ (over which there is integration performed in the numerators

$$\begin{aligned} &\widehat{\Xi}^{\Lambda^0}_{\Gamma}\left(\Lambda(n)\setminus\Lambda^0 \mid \mathtt{g}\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right), \\ &\widehat{\Xi}^{\Lambda^0}_{\Gamma}\left(\Lambda(n)\setminus\Lambda^0 \mid \mathtt{g}^{-1}\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right) \end{aligned} \tag{131}$$

in the expression for $\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}(\mathtt{g}\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)})$ and $\widehat{q}^{\Lambda^0}_{\Gamma\setminus\Lambda(n)}(\mathtt{g}^{-1}\overline{\mathbf{\Omega}}^{*0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)})$). The tuning in $\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}$ is chosen so that it approaches $\mathtt{e}$ (or the d-dimensional zero vector in the additive form of writing), the neutral element of $\mathtt{G}$, while we move from Λ^0 towards $\Gamma\setminus\Lambda(n)$.

Formally, (130) follows from the estimate (132) below: $\forall$ finite $\Lambda^0 \subset \Gamma$, $\overline{\mathbf{\Omega}}^{*0} \in \overline{W}_{\Lambda}(n)$, $\mathtt{g} \in \mathtt{G}$ and $a \in (1,\infty)$, for any n large enough, $\forall \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0} = \{\mathbf{\Omega}^*(i),\ i \in \Lambda(n)\setminus\Lambda^0\}$ and $\mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)} = \{\mathbf{\Omega}^*(i),\ i \in \Gamma\setminus\Lambda(n)\}$ with $\sharp\mathbf{\Omega}^*(i) \le \kappa$,

$$\begin{aligned} &\frac{a}{2}\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\left(\mathtt{g}\overline{\mathbf{\Omega}}^{*0}\right) \vee \left(\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right) \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right] \\ &\quad + \frac{a}{2}\exp\Big[-\mathbf{h}^{\Lambda(n)} \\ &\qquad \times\left(\left(\mathtt{g}^{-1}\overline{\mathbf{\Omega}}^{*0}\right) \vee \left(\mathtt{g}^{-1}_{\Lambda(n)\setminus\Lambda^0}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right) \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\Big] \\ &\ge \exp\left[-\mathbf{h}^{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right]. \end{aligned} \tag{132}$$

In (132), the loop configuration $\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}$ is determined by specifying its temporal section $\{\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\Omega^*_{x,i}(\tau + \beta m),\ i \in \Lambda(n)\setminus\Lambda^0,\ x \in \mathbf{x}^*(i),\ 0 \le m < k_{x,i}\}$. That is, we need to specify the sections:

$$\left(u\left(\tau + \beta m; \mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\Omega^*_{x,i}\right), l\left(\tau + \beta m; \mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\Omega^*_{x,i}\right)\right) \tag{133}$$

for loops $\Omega^*_{x,i}$ constituting $\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}$. To this end we set

$$\begin{aligned} &u\left(\tau + \beta m; \mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\Omega^*_{x,i}\right) = \mathtt{g}^{(n)}_j u\left(\tau + \beta m; \Omega^*_{x,i}\right) \\ &\qquad \text{if } l\left(\tau + \beta m; \mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\Omega^*_{x,i}\right) = j. \end{aligned} \tag{134}$$

In other words, we apply the action $\mathtt{g}^{(n)}_j$ to the temporal sections of all loops $\Omega^*_{x,i}$ located at vertex j at a given time, regardless of position of their initial points (x,i) in $\Lambda(n)\setminus\Lambda^0$.

Observe that (130) is deduced from (132) by integrating in $\mathrm{d}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}$ and normalizing by $\Xi_{\Lambda(n)\setminus\Lambda^0}(\mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)})$; see (80) with $\Lambda' = \Lambda(n)$. (The Jacobian of the map $\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0} \mapsto \mathtt{g}_{\Lambda(n)\setminus\Lambda^0}\mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}$ equals 1.)

Thus, our aim becomes proving (132). The tuned family $\mathtt{g}_{\Lambda(n)\setminus\Lambda^0}$ is composed of individual actions $\mathtt{g}^{(n)}_j \in \mathtt{G}$:

$$\mathtt{g}_{\Lambda(n)\setminus\Lambda^0} = \left\{\mathtt{g}^{(n)}_j,\ j \in \Lambda(n)\setminus\Lambda^0\right\}. \tag{135}$$

Elements $\mathtt{g}^{(n)}_j$ are powers (multiples, in the additive parlance) of element $\mathtt{g} \in \mathtt{G}$ figuring in (128)–(132) (resp., of the corresponding vector $\underline{\theta} \in M$; cf. (9)) and defined as follows. Let $\underline{\theta}^{(n)}_j$ denote the vector from M corresponding to $\mathtt{g}^{(n)}_j$, and we select positive integer values $\overline{r}(n) = \lceil \log(1+n) \rceil$ and set

$$\underline{\theta}^{(n)}_j = \underline{\theta} v(n,j), \tag{136}$$

where

$$v(n,j) = \begin{cases} 1, & \mathtt{d}(o,j) \le \overline{r}(n), \\ \vartheta\left(\mathtt{d}(j,o) - \overline{r}(n), n - \overline{r}(n)\right), & \mathtt{d}(o,j) > \overline{r}(n). \end{cases} \tag{137}$$

In turn, the function ϑ is chosen to satisfy

$$\begin{aligned} \vartheta(a,b) = \mathbf{1}(a \le 0) &+ \frac{\mathbf{1}(0 < a < b)}{Q(b)} \\ &\times \int_a^b z(u)\,\mathrm{d}u, \quad a, b \in \mathbb{R}, \end{aligned} \tag{138}$$

with

$$\begin{aligned} &Q(b) = \int_0^b \zeta(u)\,\mathrm{d}u \sim \log\log b, \\ &\text{where } \zeta(u) = \mathbf{1}(u \le 2) + \mathbf{1}(u > 2)\frac{1}{u\ln u}, \quad b > 0. \end{aligned} \tag{139}$$

Lemma 31. *Given $z, \beta \in (0,\infty)$ satisfying (22) and a finite set Λ^0, there exists a constant $C \in (0,\infty)$ such that $\forall \mathbf{x}^0, \mathbf{y}^0 \in M^{*\Lambda^0}$, the set of path configurations $\overline{\mathbf{\Omega}}^{*0} \in \overline{W}^*_{\mathbf{x}^{*0},\mathbf{y}^{*0}}$ with $\mathbf{h}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0}) < +\infty$ which include trajectories visiting points in $\Gamma\setminus\Lambda(\overline{r}(n))$ has the $\overline{\mathbb{P}}^*_{\mathbf{x}^0,\mathbf{y}^0}$ measure that does not exceed $C/(\lceil(1+r(n))\rceil)!$.*

Proof of Lemma 31. The condition that $\mathbf{h}^{\Lambda^0}(\overline{\mathbf{\Omega}}^{*0}) < +\infty$ implies that the total number of sub trajectories of length β in $\overline{\mathbf{\Omega}}^{*0}$ does not exceed $\kappa \times \sharp\Lambda^0$ which is a fixed value in the context of the lemma. Each such trajectory has a Poisson number of jumps; this produces the factor $1/(\lceil(1 + r(n))\rceil)!$. □

Back to the proof of Theorem 29: let $\mathrm{g}^{-1}_{\Lambda(n)\setminus\Lambda^0}$ be the collection of the inverse elements:

$$\mathrm{g}^{-1}_{\Lambda(n)\setminus\Lambda^0} = \left\{\mathrm{g}_j^{(n)^{-1}},\ j \in \Lambda(n) \setminus \Lambda^0\right\}. \quad (140)$$

The vectors corresponding to $\mathrm{g}_j^{(n)^{-1}}$ are $-\underline{\theta}_j^{(n)} \in M$. We will use this specification for $\mathrm{g}_j^{(n)}$ and $\mathrm{g}_j^{(n)^{-1}}$ for $j \in \Lambda(n)$, or even for $j \in \Gamma$, as it agrees with the requirement that $\mathrm{g}_j^{(n)} \equiv \mathrm{g}$ when $j \in \Lambda^0$ and $\mathrm{g}_j^{(n)} \equiv \mathrm{e}$ for $j \in \Gamma \setminus \Lambda(n)$. Accordingly, we will use the notation $\mathrm{g}_{\Lambda(n)} = \{\mathrm{g}_j^{(n)},\ j \in \Lambda(n)\}$.

Observe that the tuned family $\mathrm{g}_{\Lambda(n)\setminus\Lambda^0}$ does not change the contribution into the energy functional $\mathbf{h}^{\Lambda(n)|\Gamma\setminus\Lambda(n)}$ coming from potentials $U^{(1)}$ and $U^{(2)}$: it affects only contributions from potential V.

The Taylor formula for function V, together with the above identification of vectors $\underline{\theta}_j^{(n)}$, gives

$$\begin{aligned}&\left|V\left(\mathrm{g}_j^{(n)}x, \mathrm{g}_{j'}^{(n)}x'\right) + V\left(\mathrm{g}_j^{(n)^{-1}}x, \mathrm{g}_{j'}^{(n)^{-1}}x'\right) - 2V\left(x, x'\right)\right| \\ &\quad \le C|\underline{\theta}|^2\left|v\left(n, j\right) - v\left(n, j'\right)\right|^2\overline{V}, \quad x, x' \in M.\end{aligned} \quad (141)$$

Here $C \in (0, \infty)$ is a constant, $|\underline{\theta}|$ stands for the norm of the vector $\underline{\theta}$ representing the element g, and the value $\overline{V}$ is taken from (14).

Next, the square $|v(n, j) - v(n, j')|^2$ can be specified as

$$\left|v(n, j) - v(n, j')\right|^2 = \begin{cases} 0, \\ \text{if } \mathtt{d}(j, o), \mathtt{d}(j', o) \le \overline{r}(n), \\ 0, \\ \text{if } \mathtt{d}(j, o), \mathtt{d}(j', o) \ge n \\ \left[\vartheta\left(\mathtt{d}(j, o) - \overline{r}(n), n - \overline{r}(n)\right)\right. \\ \quad \left. -\vartheta\left(\mathtt{d}(j', o) - \overline{r}(n), n - \overline{r}(n)\right)\right]^2, \\ \quad \text{if } \overline{r}(n) < \mathtt{d}(j, o), \mathtt{d}(j', o) \le n, \\ \vartheta\left(\mathtt{d}(j, o) - \overline{r}(n), n - \overline{r}(n)\right)^2, \\ \quad \text{if } \overline{r}(n) < \mathtt{d}(j, o) \le n, \mathtt{d}(j', o) \in]\overline{r}(n), n[\,, \\ \vartheta\left(\mathtt{d}(j', o) - \overline{r}(n), n - \overline{r}(n)\right)^2, \\ \quad \text{if } \overline{r}(n) < \mathtt{d}(j', o) \le n, \mathtt{d}(j, o) \in]\overline{r}(n), n[\,. \end{cases} \quad (142)$$

By using convexity of the function exp and (141), $\forall a > 1$,

$$\begin{aligned}&\frac{a}{2}\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\mathrm{g}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right) \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right] \\ &\quad + \frac{a}{2}\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\mathrm{g}^{-1}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right) \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right] \\ &\ge a\exp\left[-\frac{1}{2}\mathbf{h}^{\Lambda(n)|\Gamma\setminus\Lambda(n)}\right. \\ &\qquad \times\left(\mathrm{g}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right), \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right) \\ &\qquad \left.-\frac{1}{2}\mathbf{h}^{\Lambda(n)}\left(\mathrm{g}^{-1}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0}\right) \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right] \\ &\ge a\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0} \vee \mathbf{\Omega}^*_{\Lambda(n)\setminus\Lambda^0} \mid \mathbf{\Omega}^*_{\Gamma\setminus\Lambda(n)}\right)\right]e^{-C\Upsilon/2},\end{aligned} \quad (143)$$

where

$$\begin{aligned}\Upsilon &= \Upsilon(n, \mathrm{g}) = \beta\kappa^2 \\ &\quad \times \sum_{(j,j')\in\Lambda(n)\times\Gamma} J\left(\mathtt{d}\left(j, j'\right)\right)\left|v(n, j) - v(n, j')\right|^2.\end{aligned} \quad (144)$$

The next observation is that

$$\begin{aligned}\Upsilon &\le 3\beta\kappa^2|\underline{\theta}|^2 \\ &\quad \times \sum_{(j,j')\in\Lambda(n)\times\Gamma} \mathbf{1}\left(\mathtt{d}(j, o) \le \mathtt{d}\left(j', 0\right)\right) J\left(\mathtt{d}\left(j, j'\right)\right) \\ &\qquad \times\left[\vartheta\left(\mathtt{d}(j, o) - \overline{r}(n), n - \overline{r}(n)\right)\right. \\ &\qquad\quad \left.-\vartheta\left(\mathtt{d}\left(j', o\right) - \overline{r}(n), n - \overline{r}(n)\right)\right]^2,\end{aligned} \quad (145)$$

where, owing to the triangle inequality, for all $j, j' : \mathtt{d}(j, o) \le \mathtt{d}(j', o)$

$$\begin{aligned}0 &\le \vartheta\left(\mathtt{d}(j, o) - \overline{r}(n), n - \overline{r}(n)\right) \\ &\quad - \vartheta\left(\mathtt{d}\left(j', o\right) - \overline{r}(n), n - \overline{r}(n)\right) \\ &\le \mathtt{d}\left(j, j'\right)\frac{\zeta\left(\mathtt{d}(j, o) - \overline{r}\right)}{Q\left(n - \overline{r}(n)\right)}.\end{aligned} \quad (146)$$

This yields

$$\begin{aligned}\Upsilon &\le \beta\kappa^2\frac{3|\underline{\theta}|^2}{Q(n - \overline{r}(n))^2} \\ &\quad \times \sum_{(j,j')\in\Lambda(n)\times\Gamma} J\left(\mathtt{d}\left(j, j'\right)\right)\mathtt{d}\left(j, j'\right)^2\zeta(\mathtt{d}(j, 0) - \overline{r}(n))^2 \\ &\le \frac{3|\underline{\theta}|^2}{Q(n - \overline{r}(n))^2}\left[\sup_{j\in\Gamma}\sum_{j'\in\Gamma} J_{j,j'}\mathtt{d}\left(j, j'\right)^2\right] \\ &\quad \times \sum_{j\in\Lambda_{n+r_0}} \zeta\left(\mathtt{d}(j, 0) - \overline{r}(n)\right)^2,\end{aligned} \quad (147)$$

where function ζ is determined in (139).

Therefore, it remains to estimate the sum $\sum_{j\in\Lambda_{n+r_0}}\zeta(\mathrm{d}(j,0)-\overline{r}(n))^2$. To this end, observe that $u\zeta(u)<1$ when $u\in(3,\infty)$. The next remark is that the number of sites in the sphere Σ_n grows linearly with n. Consequently,

$$\begin{aligned}\sum_{j\in\Lambda_{n+r_0}}\zeta(\mathrm{d}(j,o)-\overline{r}(n))^2 &= \sum_{1\le k\le n+r_0}\zeta\left(k-\overline{r}\left(n\right)\right)\\ &\quad\times\sum_{j\in\Sigma_k}\zeta\left(k-\overline{r}\left(n\right)\right)\\ &\le C_0\sum_{1\le k\le n+r_0}\zeta\left(k-\overline{r}\left(n\right)\right)\\ &\le C_1 Q\left(n+r_0-\overline{r}\left(n\right)\right),\\ \Upsilon \le \frac{C}{Q\left(n-\overline{r}\left(n\right)\right)} &\longrightarrow\infty,\quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{148}$$

Therefore, given $a>1$ for n large enough, the term $ae^{-C\Upsilon/2}$ in the RHS of (145) becomes >1. Hence,

$$\begin{aligned}&\frac{a}{2}\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\mathrm{g}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^{*}_{\Lambda(n)\setminus\Lambda^0}\right)\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\right]\\ &\quad+\frac{a}{2}\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\mathrm{g}^{-1}_{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^{*}_{\Lambda(n)\setminus\Lambda^0}\right)\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\right]\\ &\ge\exp\left[-\mathbf{h}^{\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^{*}_{\Lambda(n)\setminus\Lambda^0}\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\right].\end{aligned} \tag{149}$$

Equation (149) implies that the quantity

$$\begin{aligned}&q^{\Lambda^0|\Gamma\setminus\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}_{\Gamma\setminus\Lambda(n)}\right)\\ &\quad=\int_{W_{\Lambda(n)\setminus\Lambda^0}}\mathrm{d}\mathbf{\Omega}^{*}_{\Lambda(n)\setminus\Lambda^0}\\ &\qquad\times\frac{\exp\left[-\mathbf{h}^{\Lambda^0}\left(\overline{\mathbf{\Omega}}^{*0}\vee\mathbf{\Omega}^{*}_{\Lambda(n)\setminus\Lambda^0}\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda^0}\right)\right]}{\Xi_{\Lambda(n)}\left(\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)},\end{aligned} \tag{150}$$

obeys

$$\begin{aligned}&\lim_{n\to\infty}\left[q_\beta^{\Lambda^0|\Gamma\setminus\Lambda(n)}\left(\mathrm{g}\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\right.\\ &\qquad\left.+q_\beta^{\Lambda^0|\Gamma\setminus\Lambda(n)}\left(\mathrm{g}^{-1}\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\right]\\ &\quad\ge 2\lim_{n\to\infty}q_\beta^{\Lambda^0|\Gamma\setminus\Lambda(n)}\left(\overline{\mathbf{\Omega}}^{*0}\mid\mathbf{\Omega}^{*}_{\Gamma\setminus\Lambda(n)}\right)\end{aligned} \tag{151}$$

uniformly in boundary condition $\boldsymbol{\omega}_{\Gamma\setminus\Lambda(n)}$. Integrating (151) $\mathrm{d}\mu_\Gamma^{\Gamma\setminus\Lambda(n)}(\boldsymbol{\omega}_{\Gamma\setminus\Lambda(n)})$ yields (132).

Acknowledgments

This work has been conducted under Grant 2011/20133-0 provided by the FAPESP, Grant 2011.5.764.45.0 provided by The Reitoria of the Universidade de São Paulo, and Grant 2012/04372-7 provided by the FAPESP. The authors express their gratitude to NUMEC and IME, Universidade de São Paulo, Brazil, for the warm hospitality.

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