A QUANTUM MODEL OF COMMUNICATION SYSTEMS

A Dissertation

Presented To

the Faculty of the School of Engineering and Applied Science

University of Virginia

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Science (Electrical Engineering)

By

Melvin D. Aldridge

June 1968

APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

the requirements for the degree of

Doctor of Science (Electrical Engineering)

____

Author

Approved:

____

Faculty Adviser

____

Dean, School of
Engineering and
Applied Science

June 1968

ABSTRACT

The problem of including quantum effects within existing methods of communication system analysis is investigated. A mathematical model is postulated for describing narrow bandwidth systems. Utilizing the Glauber P representation, the model permits analysis of both real field detection (measurement of electric and magnetic field magnitudes) and photon detection. The resulting equation which describes photon detection is shown to be consistent with previous work which has been proven experimentally. Unique aspects of the postulated model are its ability to describe both random and deterministic modulation functions in conventional terms, and its ability to demonstrate the inapplicability of the "photon channel" model for describing present-day systems.

An analysis of quantum effects at the transmitter shows that modulation of the radiated field must take place by one of two distinctly different processes. Wave modulation is defined as a process which conveys information in the amplitude and phase of the electromagnetic field. Photon modulation is defined as a process which conveys information in the exact number of photons per pulse of the transmitted field (i.e., the "photon channel" of Stern, Gordon, and others). A wave modulated

signal is shown to describe present-day modulation schemes and, moreover, to yield random photon fluctuations. It is concluded a wave modulated signal is not capable of describing a photon modulated signal. In contrast, a photon modulated signal is shown to possess a completely random phase fluctuation thus demonstrating the inability of the "photon channel" to describe presently used wave modulation schemes. For wave modulation the Glauber P function is shown to be identical (with a change of variable) to the classical joint probability density of the quadrature components of a narrowband sinusoid used in conventional analysis. Equations are derived which "transform" classical probability distributions into distributions which include quantum effects.

The "partitioning noise" studied by Hagfors and Bowen is found to constitute a source of noise only in the "photon channel." The "partitioning effect" is shown to merely preserve the Poisson character of photon counts in wave modulated fields and therefore does not introduce a noise in present-day systems.

For a wave modulated system with real field detection it is shown that photon noise can be included in conventional analysis by adding to the normal input noise, a noise density of $hf_c/2$ Watts/Hertz where h is Planck's constant and f_c is the system carrier frequency. Use of photon heterodyne detection requires the addition of an input noise density of hf_c Watts/Hertz. The photon noise is exactly additive Gaussian for real field detection and approximately additive Gaussian for photon heterodyne detection. Detection by counting the received photons is shown to yield neither Gaussian nor Poissonian statistics. However, with an "ideal measurement process," when the effective received noise density n_c is much less than hf_c, the variance of the counts are found to approach that of a Poisson distribution. Conversely, when n_c is much larger than hf_c, the variance approaches that derivable from a classical analysis.

A channel capacity equation for a wave modulated system is found which is equal to those derived by Lachs and Jelsma. The equation is shown to give capacities less than that derived by Gordon.

ACKNOWLEDGEMENT

The author wishes to acknowledge Doctors Foster, Siegel, and McAlpine for their helpful suggestions and criticisms and the NASA Langley Research Center for permitting this work to be submitted as a dissertation.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

$A(t)$	amplitude of a sinusoid at time t
C	channel capacity
$E(\vec{r},t)$	electric field intensity for one linear polarization at position $\vec{r}$ and time t
$\mathcal{E}$	energy
$G(\omega)$	filter transfer function
N	average number of photons per measurement
$P(\ )$	probability distribution
$P(,)$	joint probability distribution between two variables
$P(\|)$	conditional probability distribution
$P(,;\tau)$	joint probability distribution between two variables whose time separation is τ seconds
$P(\alpha)$	Glauber P function
P	average power
$\frac{S}{N}$	input signal-to-noise ratio to receiver
$\frac{S_o}{N_o}$	receiver output signal-to-noise ratio
$V(it_m)$	output of photon heterodyne detector at time it_m
W	bandwidth

W_F	bandwidth of filter which operates directly on the field before measurement at the receiver
W_m	bandwidth of field measurement process at the receiver
W_o	bandwidth of filter which appears after the field measurement at the receiver
a	area of antenna aperture
C_n	probability amplitude
$e^{( )}$,exp( )	exponential
f_c	system carrier frequency
f_I	intermediate frequency of heterodyne receiver
g(t)	filter impulse response
h	Planck's constant
$\hbar$	$h/2\pi$
i	dummy variable
j	$\sqrt{-1}$
k	dummy variable
m	dummy variable
n	number of photons
p	momentum

$p_{Qp}(p)$	probability distribution of momentum over an ensemble of quantum harmonic oscillators (the result of measuring momentum with perfect accuracy)
$p_{Qq}(q)$	probability distribution of position over an ensemble of quantum harmonic oscillators (the result of measuring momentum with perfect accuracy)
$p_T(t)$	momentum of oscillator (at time t) which is equivalent to one sample of transmitted signal
p_T	$p_T(t = 0)$
q	position
$q_T(t)$	position of oscillator (at time t) which is equivalent to one sample of transmitted signal
q_T	$q_T(t = 0)$
t	time
t_T	time between samples of transmitted signal
t_m	time between samples of received signals
t_o	integration time of the photon counter
x	quadrature component in-phase with cosine function

y	quadrature component in-phase with sine function
α	$\frac{1}{\sqrt{2\hbar\omega_c}}(\omega_c q + jp)$
α_I	imaginary part of α
α_R	real part of α
$\delta(t)$	Dirac delta function
$\delta_{n,m}$	Kronecker delta function
η_c	spectral density of input thermal noise
η_{RF}	effective input noise density for real field detection
$\theta(t)$	phase of a sinusoid at time t
ν	power attenuation coefficient due to free space attenuation
σ^2	variance over an ensemble
$\sigma_n^2 \mid A_s$	variance of n knowing A_s
τ	dummy time variable
ϕ_n	phase of probability amplitude C_n
ω_c	$2\pi f_c$

The following subscripts denote certain conditions as follows:

c	derivation from a classical analysis (no quantum effects)
G	derivation from Glauber P function through a change of variable

LO	local oscillator in photon heterodyne receiver
m	field measurement process
N	variable which describes noise received from external sources
Q	derivation from an analysis which includes quantum effects
R	variable which describes total received signal and noise
s	received signal
T	transmitted signal
g	variable as derived by Gordon in Reference 5
n	measurement of photon number
p	measurement of oscillator momentum
q	measurement of oscillator position

notation:

$\langle \ \rangle$	ensemble average
$\langle ,(\ ) \rangle$	conditional ensemble average
$\overline{\ \ \ \ }$	time average
Δ	root-mean-square deviation
*	impulse representation of a sample

CHAPTER I

INTRODUCTION

Invention of the laser has aroused considerable interest in the use of the electromagnetic spectrum above conventional microwaves. Potential improvements due to highly collimated beams and large bandwidths make very short wavelength radiation attractive for both space and earth communication applications. D. Gabor was the first to point out that use of radio frequencies in the infrared and visible regions would require a consideration of photon noise or the quantum properties of radiation (Reference 1). The study reported herein investigates the general problem of including photon noise within existing methods of communication system analysis.

Review of Previous Work

Following Gabor's original work in 1950, additional studies on quantum effects in communication systems were not reported until 1960. All of the works published thus far fall into one of the two broad categories discussed below.

References 1 through 13 share the characteristic of having derivations (and possibly results) that cannot be related to conventional concepts of amplitude and phase modulation. Most of these works are concerned only with the derivation of a channel capacity equation which includes quantum effects. The usual approach implicitly assumes that any channel can be considered to convey information by the exact number of photons transmitted in each frequency-time cell. This model has been called a "photon channel." The derivations do not reveal how quantum effects can be included in the analysis of conventional communication systems. The most referenced work in this category is Gordon's second paper (Reference 5). She and Hagfors have taken issue with certain aspects of Gordon's work. She (Reference 14, page 4) noted that Gordon's use of only one sample per frequency-time cell does not obey the correspondence principle since it is well known that two samples per cell are required in classical analyses. Hagfors (Reference 7, page 2) has shown that Gordon's work neglects the effects of free space attenuation which introduces a "partitioning noise" in the channel. Bowen (Reference 13) has investigated "partitioning effects" for the case of large free space attenuation. Neither Hagfors or Bowen have discussed the significance of this noise in conventional amplitude and phase modulated systems.

References 14 through 18 utilize approaches where conventional concepts of amplitude and phase as well as the number of photons evolve from the quantum theory of free fields. In 1964 C. Y. She and G. Lachs independently made the first attempts in this direction (References 14 and 15). Simultaneously L. Jelsma was pursuing a similar path (Reference 16). Also based on quantum field approaches, Helstrom and Karp have analyzed specific optical detection problems (References 17 and 18). All of References 14 through 18 either directly utilize or are related to the Glauber P representation of the "coherent state" (References 19-22). The P representation was developed for describing the quantum theory of coherence for optical fields and has been the subject of several quantum theoretical discussions (for example, References 23-25). The relationship of the Glauber P function to a classical probability density appears to be the key to developing methods for directly including photon noise within conventional analysis. Previous neglect of this relationship has created confusion relative to statements made by Glauber. For example, Minkowski, et. al. (Reference 26) have cricized She's work for his use of the P function because Glauber does not equate it to a probability density.

She, Lachs, and Jelsma did not discuss the description of modulated fields. Lachs has subsequently developed a multi-mode description of modulated fields (Reference 27) which has been applied to the analysis of a frequency modulated system (Reference 28). The mathematics of the multi-mode field are cumbersome and as developed by Lachs appear to be useful only for describing deterministic modulating wave forms.

In addition to the problems outlined above, most of the works referenced in this section are so strongly couched in the language and mathematics of quantum theory that the separation of conventional concepts and those unique to the quantum properties of radiation are not well defined. Moreover, no attempt has been made to relate the consistencies or differences between the "photon channel" and quantum free field approaches.

Statement of the Problem

In view of the previous work, the study presented herein is addressed to several specific problems. The general objective is to develop techniques which permit an inclusion of quantum effects directly in existing methods of communication system analysis. This requires the development of a model which is compatible with the requirements of quantum theory but still maintains the conventional concepts of amplitude and phase modulation, correlation functions, and bandwidths. The role of the modulation process at the transmitter must be studied to determine what meaning, if any, the "photon channel" holds in describing conventional modulation schemes. The effects of free space attenuation is studied to determine how the "partitioning noise" studied by Hagfors and Bowen affects the work shown herein. Since the Glauber P representation offers the only quantum theoretical method of describing a field with both classical and quantum concepts, the relevance of the P function to a classically derived probability density must be studied. Finally, the results of this paper will be compared to Gordon's work and his use of the "photon channel."

CHAPTER II

A COMMUNICATION SYSTEM MODEL

To include the quantum properties of radiation in modern communication theory, a model must be available which is describable by the analytical techniques of communication theory and compatible with the requirements of quantum field theory. This chapter presents the development of a model which will be altered in the next chapter to include quantum properties. A unique time sampled description of the field permits conventional concepts of temporal fluctuations, bandwidths, and correlations functions to be carried into the quantum analysis. The inclusion of quantum effects at both the transmitter and receiver will be possible.

The Basic Model

Consider the block diagram shown in Figure 1. The modulated transmitter radiates a field of which the phase or amplitude is being controlled by the information source. A receiving aperture collects a small portion of the transmitted field as well as fields arising from sources of thermal noise. The total received field is measured and demodulated to give an output of the received information.

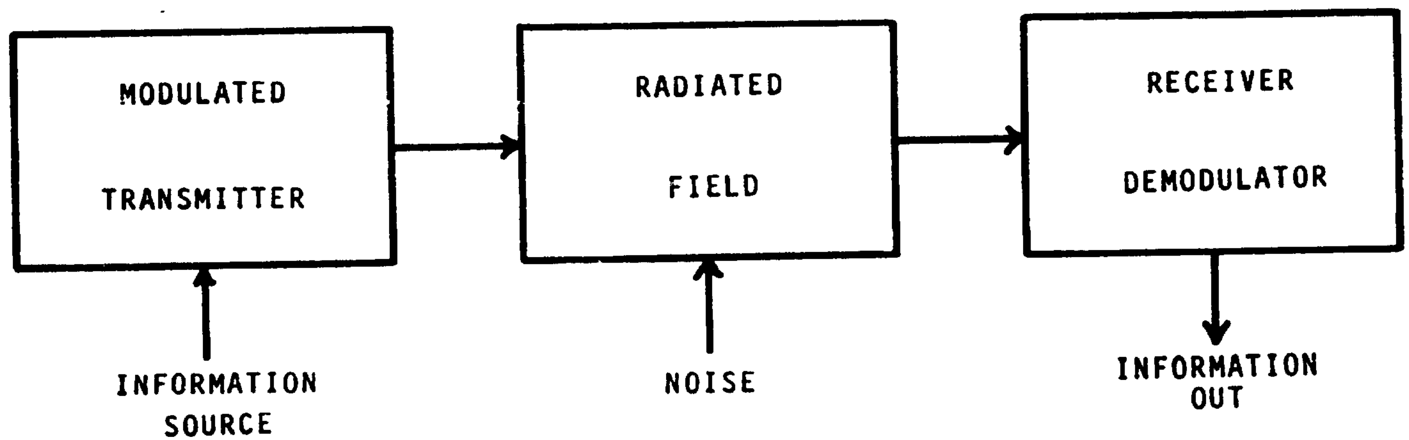

Figure 1.- Basic communication system model.

Temporal fluctuations of the field's amplitude and phase are determined by the information source and the technique of modulation. Many modulation methods are known and have been analyzed extensively to determine the resulting characteristics of the radio frequency wave. Therefore, a mathematical description of the fluctuating radiated field will be adequate for the purposes of this paper since this implicitly includes the information source.

Since quantum effects must be considered at both the transmitter and receiver, a description of the field for both cases is presented.

Description of the Transmitted Field

Consider the linearly polarized electric field $E_T(\vec{r},t)$ which is prepared in the transmitting aperture a_T. The field in the aperture can be considered equivalent to the voltage

$$E_T(t) = \iint_{a_T} E_T(\vec{r},t)\,da_T \qquad (2\text{-}1)$$

Assuming the process which prepared $E_T(t)$ is bandlimited to W_T Hertz and centered at f_c the carrier frequency where

$$W_T \ll f_c \qquad (2\text{-}2)$$

$$E_T(t) = x_T(t)\cos\omega_c t - y_T(t)\sin\omega_c t \qquad (2\text{-}3)$$

where $x_T(t)$ and $y_T(t)$ are the quadrature modulating components. Equation (2-3) can be written in the form

$$E_T(t) = A_T(t)\cos[\omega_c t + \theta_T(t)] \qquad (2\text{-}4)$$

where

$$A_T(t) = \sqrt{x_T^2(t) + y_T^2(t)}$$

$$\theta_T(t) = \tan^{-1}\frac{y_T(t)}{x_T(t)}$$

Since W_T is centered at f_c, it follows that $x_T(t)$ and $y_T(t)$ are bandlimited from zero to $\frac{W_T}{2}$ Hertz. Therefore (Reference 45)

$$x_T(t) = \sum_k x_T(kt_T)\,\frac{\sin\frac{W_T}{2}(t - kt_T)}{\frac{W_T}{2}(t - kt_T)} \qquad (2\text{-}5a)$$

$$y_T(t) = \sum_k y_T(kt_T)\,\frac{\sin\frac{W_T}{2}(t - kt_T)}{\frac{W_T}{2}(t - kt_T)} \qquad (2\text{-}5b)$$

$$E_T(t) = \sum_k \left\{ \left[x_T(kt_T)\cos\omega_c t - y_T(kt_T)\sin\omega_c t \right] \frac{\sin\frac{W_T}{2}(t - kt_T)}{\frac{W_T}{2}(t - kt_T)} \right\} \tag{2-6}$$

where

$$t_T = \frac{1}{W_T} \tag{2-7}$$

Equation (2-6) can be translated into the equivalent circuit shown in Figure 2. If

$$G_T(\omega) = 1 \quad \text{for} \; -\frac{\omega_T}{2} < \omega < \frac{\omega_T}{2}$$

$$= 0 \qquad |\omega| > \frac{\omega_T}{2} \tag{2-8}$$

and

$$x_T^*(t) = \frac{1}{W_T}\sum_k x_T(kt_T)\,\delta(t - kt_T)$$

then

$$x_T'(t) = \int_{-\infty}^{\infty} x_T^*(\tau) g_T(t - \tau) d\tau$$

where $g_T(t)$ is the impulse response of $G_T(\omega)$ or*

*This development obviously yields an impractical result since no filter output can exist before the input impulse occurs. In reality, a filter of the form (2-8) requires an infinite number of energy storage elements and thus introduces an infinitely long time delay. The time delay is neglected here for convenience.

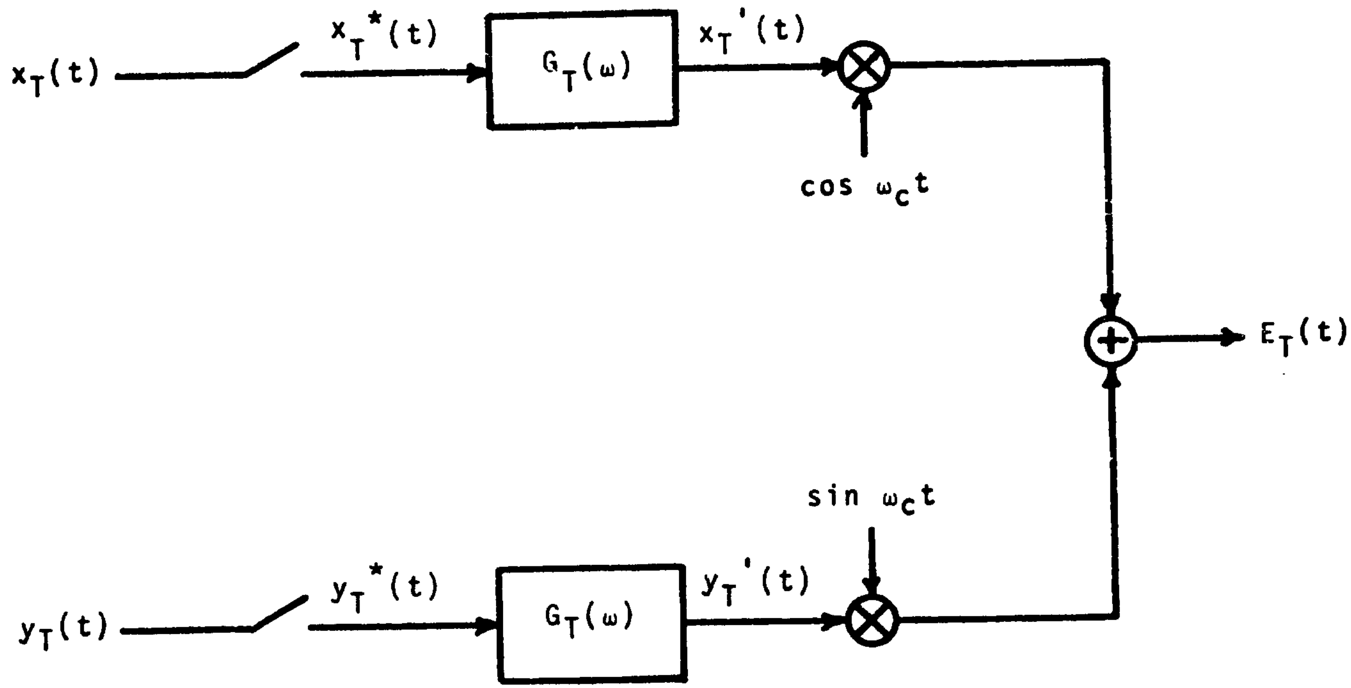

Figure 2.- Analog circuit of transmitter.

$$g_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_T(\omega)e^{j\omega t}\,d\omega$$

$$= W_T \frac{\sin \frac{\omega_T}{2}t}{\frac{\omega_T}{2}t}$$

Therefore

$$x_T'(t) = \int_{-\infty}^{\infty}\sum_k x_T(kt_T)\delta(\tau - kt_T)\,\frac{\sin \frac{\omega_T}{2}(t-\tau)}{\frac{\omega_T}{2}(t-\tau)}\,d\tau$$

$$= \sum_k x_T(kt_T)\,\frac{\sin \frac{\omega_T}{2}(t - kt_T)}{\frac{\omega_T}{2}(t - kt_T)}$$

which agrees with (2-5a). An identical derivation applies to $y_T(t)$ and the $E_T(t)$ in Figure 2 is equal to (2-6).

Description of the Received Field

The voltage equivalent to the received electric field $E_R(\vec{r},t)$ in the receiving aperture a_R is

$$E_R(t) = \iint_{a_R} E_R(\vec{r},t)\,da_R \qquad (2\text{-}9).$$

Assuming the field measurement process is bandlimited to W_m Hertz centered at f_c and

$$W_m << f_c \qquad (2\text{-}10)$$

it follows that

$$E_R(t) = x_R(t)\cos\omega_c t - y_R(t)\sin\omega_c t \qquad (2\text{-}11)$$

$$= A_R(t)\cos[\omega_c t + \theta_R(t)]$$

where

$$A_R(t) = \sqrt{x_R^2(t) + y_R^2(t)}$$

$$\theta_R(t) = \tan^{-1}\frac{y_R(t)}{x_R(t)} .$$

Because W_m is centered at f_c, then

$$E_R(t) = \sum_k \left\{ \left[x_R(kt_m)\cos\omega_c t - y_R(kt_m)\sin\omega_c t \right] \frac{\sin\frac{\omega_m}{2}(t - kt_m)}{\frac{\omega_m}{2}(t - kt_m)} \right\} \qquad (2\text{-}12)$$

where

$$t_m = \frac{1}{W_m} \qquad (2\text{-}13).$$

Since noise fields may not be spatially coherent, the definition of $E_R(t)$ with the integration (2-9) inplicitly assumes that a specification of $E_R(t)$ includes any spatial coherence affects over the receiving aperture. Atmospheric effects may cause the received signal to also not possess perfect spatial coherence. However, for the purposes of this paper such effects will be assumed negligible.

A circuit analog to (2-12) is shown in Figure 3. This circuit can be considered a model of the physical process which measures the incident electromagnetic field to establish its amplitude and phase. If

$$G_m(\omega) = 1 \qquad -\frac{\omega_m}{2} < \omega < \frac{\omega_m}{2}$$
$$= 0 \qquad |\omega| > \frac{\omega_m}{2} \qquad (2\text{-}14)$$

Then

$$x_R'(t) = 2\int_{-\infty}^{\infty} E_R(\tau) g_m(t - \tau)\cos\omega_C\tau\, d\tau$$

where $g_m(t)$ is the impulse response of $G_m(\omega)$, or

$$g_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_m(\omega)e^{j\omega t}\, d\omega$$

$$= W_m \frac{\sin\frac{\omega_m}{2}t}{\frac{\omega_m}{2}t}$$

Therefore

$$x_R'(t) = \int_{-\infty}^{\infty}[x_R(\tau) + x_R(\tau)\cos 2\omega_C\tau - 2y_R(\tau)\sin\omega_C\tau\cos\omega_C\tau]$$

$$W_m \frac{\sin\frac{\omega_m}{2}(t - \tau)}{\frac{\omega_m}{2}(t - \tau)} d\tau$$

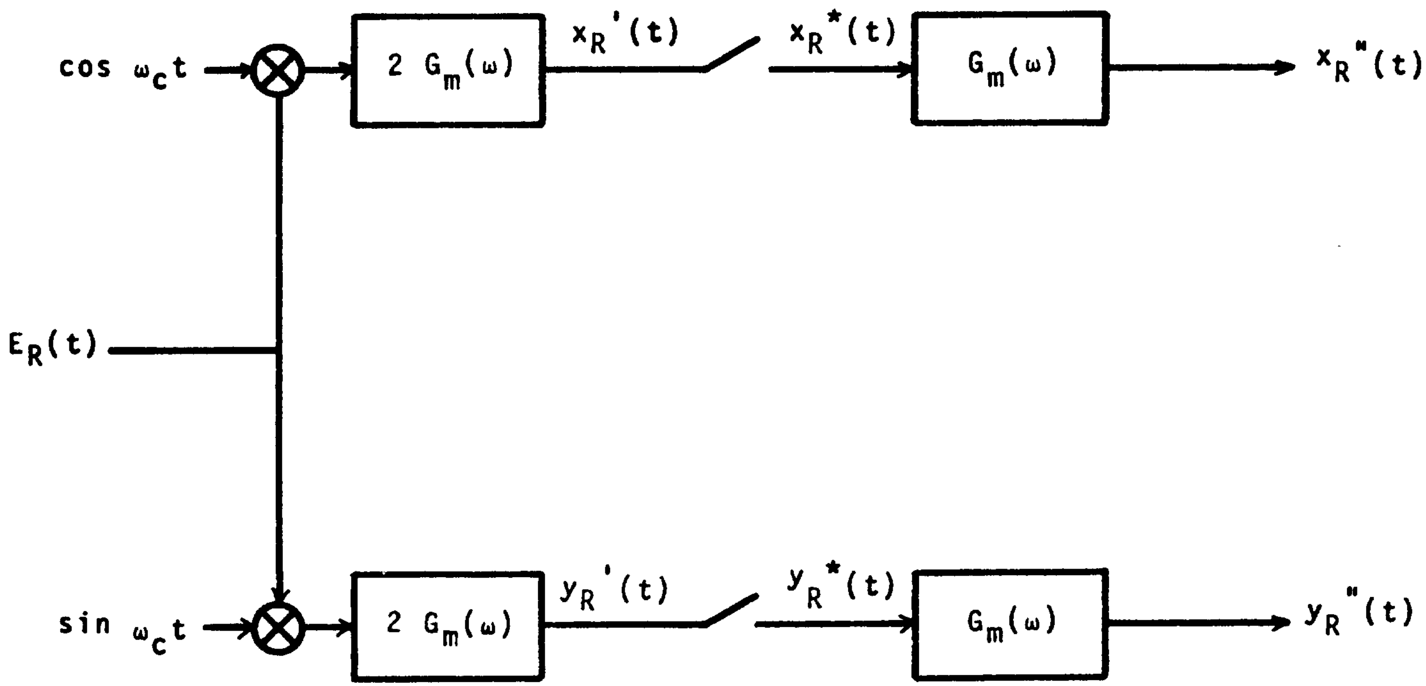

Figure 3.- Analog circuit of receiver measurement process.

and since (2-10) holds, it follows that

$$x_R'(t) = W_m \int_{-\infty}^{\infty} x_R(\tau) \frac{\sin \frac{\omega_m}{2}(t - \tau)}{\frac{\omega_m}{2}(t - \tau)} d\tau \qquad (2\text{-}15)$$

Therefore $x_R'(t)$ is seen to be the result of averaging over states of $x_R(t)$ at other than time t. Since

$$\int_{-\infty}^{\infty} \frac{\sin \frac{\omega_m}{2} t}{\frac{\omega_m}{2} t} dt = t_m \qquad (2\text{-}16)$$

it is evident that the convolution expresses an averaging over the equivalent time period t_m. Taking the sampler to yield impulses

$$x_R^*(t) = \frac{1}{W_m} \sum_k x_R'(kt_m)\delta(t - kt_m)$$

then

$$\begin{aligned} x_R''(t) &= \int_{-\infty}^{\infty} x_R^*(\tau) W_m \frac{\sin \frac{\omega_m}{2}(t - \tau)}{\frac{\omega_m}{2}(t - \tau)} d\tau \\ &= \sum_k x_R(kt_m) \frac{\sin \frac{\omega_m}{2}(t - kt_m)}{\frac{\omega_m}{2}(t - kt_m)} \end{aligned}$$

An identical derivation applies to $y_R''(t)$ proving the equivalence of the analog circuit to (2-12) and the process which measures the classical field. In addition to the measuring process, the receiver can include bandpass filters before and after the measurement process as shown in Figure 4. Examples of premeasurement filters are optical and waveguide filters which operate directly on the electromagnetic field.

Statistical Description of the Samples

If the quadrature components are ergodic, Costas (Reference 29) has shown that

$$\overline{x_T{}^k} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau x_T{}^k(t)dt = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} x^k(it_T) \qquad (2\text{-}17a)$$

$$\overline{y_T{}^k} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau y_T{}^k(t)dt = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} y^k(it_T) \qquad (2\text{-}17b)$$

where k and m are positive integers. Similar equations can be written for correlation functions and for the received quadrature components $x_R(t)$ and $y_R(t)$. From the ergodic assumption, the moments $\overline{x_T{}^k}$ and $\overline{y_T{}^k}$ (and their correlation functions) can be described by the joint probability density $P_T(x_T, y_T)$ over an ensemble of transmitters. That is

$$\overline{x_T{}^k} = \iint x_T{}^k P_T(x_T, y_T) dx_T dy_T \qquad (2\text{-}18a)$$

$$\overline{y_T{}^k} = \iint y_T{}^k P_T(x_T, y_T) dx_T dy_T \qquad (2\text{-}18b)$$

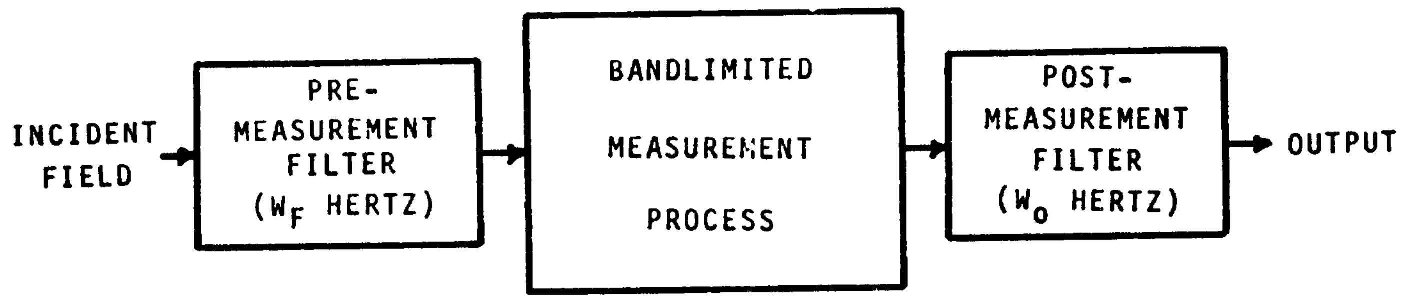

Figure 4.- Block diagram of total receiver.

where each transmitter in the ensemble is modulated by the proper function to yield the required $P_T(x_T,y_T)$. Similar arguments can be applied to the measurement process at the receiver (where the received noise is assumed to possess ergodic quadrature components) to yield

$$\overline{x_R^k} = \iint x_R^k P_R(x_R,y_R)dx_R dy_R \qquad (2\text{-}19a)$$

$$\overline{y_R^k} = \iint y_R^k P_R(x_R,y_R)dx_R dy_R \qquad (2\text{-}19b)$$

where $P_R(x_R,y_R)$ includes both the received signal and noise. The effects of noise can be isolated by considering

$$P_R(x_R,y_R) = \iint P_R[(x_R,y_R)|(x_s,y_s)]P_s(x_s,y_s)dx_s dy_s$$

where $P_s(x_s,y_s)$ is the distribution of the received signal. Then

$$\overline{x_R^k} = \int_{x_s}\int_{y_s}\left[\int_{x_R}\int_{y_R} x_R^k P_R[(x_R,y_R)|(x_s,y_s)]dx_R dy_R\right]P_s(x_s,y_s)dx_s dy_s$$

$$= \int_{x_s}\int_{y_s}\langle x_R^k,(x_s,y_s)\rangle P_s(x_s,y_s)dx_s dy_s \qquad (2\text{-}20)$$

where $\langle x_R^k,(x_s,y_s)\rangle$ is the conditional average of x_R^k knowing what signal was transmitted. In the event a deterministic modulation function is transmitted (such as a sinusoid), the ergodic assumption can again be applied to yield

$$\overline{x_R^k} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle x_R^k, [x_s(t), y_s(t)] \rangle dt \qquad (2\text{-}21).$$

Identical derivations apply to $\overline{y_R^k}$ and correlation functions.

The behavior of $P_T(x_T, y_T)$ for describing modulated signals is an important aspect of the model being developed herein. The distribution of x_T and y_T over an ensemble is indicative of their behavior in time in an actual system. For example, in an amplitude modulated signal, only the wave amplitude is varied while the phase remains constant in time. The magnitude of the phase is unimportant and is merely a function of an arbitrary time reference. Amplitude modulation then requires the quantity $\sqrt{x_T^2(t) + y_T^2(t)}$ be a function of time while the ratio $y_T(t)/x_T(t)$ must be constant with an arbitrary magnitude. Over the ensemble, $P_T(x_T, y_T)$ must reflect this requirement. That is, in polar coordinates

$$P_T'(A_T, \theta_T) = P_T'(A_T)\delta(\theta_T - \theta_T') \qquad (2\text{-}22)$$

where

$$A_T = \sqrt{x_T^2 + y_T^2}$$

$$\theta_T = \tan^{-1} \frac{y_T}{x_T}$$

$$\delta(\theta_T - \theta_T') = \text{Dirac delta function}$$

and θ_T' is an arbitrary constant. This contrasts to a set of experimental systems where θ_T' normally is random over 2π radians.

In the case of phase or frequency modulation the ratio $\frac{y_T(t)}{x_T(t)}$ becomes time variant and $\sqrt{x_T^2(t) + y^2(t)}$ remains constant. However, an arbitrary time or phase reference still must exist so that changes in phase are equal to those of the information source throughout all time. Therefore, in general, either amplitude or phase modulation or their combination requires both x_T and y_T be well defined over the ensemble used to describe time averages.

The same conclusions can be applied to the received signal distribution $P_s(x_s,y_s)$ since the only difference between the $P_s(x_s,y_s)$ and $P_T(x\ y_T)$ is their relative amplitudes A_s and A_T.

CHAPTER III

QUANTIZATION OF THE CLASSICAL MODEL

In this chapter a discussion of the requirements of quantum theory is presented and a method for including these requirements in the classical model developed in the preceding chapter is postulated. The resulting model is shown in this and the next chapter to be consistent with the quantum requirements and experimentally proven equations for photoelectric detection. Quantum effects introduced at the transmitter are investigated. The Glauber P representation is introduced and the classical character of the P function determined.

The model postulated in this chapter is unique in several ways. It is capable of handling both random and deterministic modulation functions in conventional terms. Application of the model to the transmitting process offers the first plausible explanation for the inadequancies of the "photon channel" for describing conventional amplitude and phase modulation systems. For the case of conventional systems, the Glauber P function is shown to be equal (with a change of variable) to the classical probability density $P_R(x_R,y_R)$ when describing the "ideal" receiving process.

The Requirements of Quantum Theory

It is well known that electromagnetic radiation can exhibit properties which are indicative of both wave and particle phenomena. Particle properties are probably most easily demonstrated by photoelectric or photon counting experiments. Extensive theoretical descriptions of the statistics of photon counting distributions have been developed using both semiclassical and purely quantum mechanical methods (References 30-33). Recently these results have been given experimental verification (References 34-36). These works have shown how the wave properties of radiation are exhibited when the photons are detected. Less understood is the converse problem. How are the particle properties of radiation exhibited when the wave phenomena are detected (hereafter called real field detection)? Detection of received signals by an induced antenna current characterizes the latter problem and is important in determining quantum effects in millimeter and submillimeter wave systems.

There has been little discussion in the literature on a quantum mechanical description of the real field detection process. Heisenberg (Reference 37) and Bohr and Rosenfeld (References 38, 39, see also 40) have considered the case of taking one isolated measurement of the electric and magnetic fields. These works use hyphotetical measuring systems to demonstrate how the quantum theory of a free field measurement is consistent with the quantum properties of the charged matter which must be used to effect a real field detection. The referenced works shed little light on how photon noise affects practical real field measuring systems.

In view of the lack of understanding of the real field detection process it will be necessary to postulate a model for the complete description of the transmitted and received electromagnetic field. The model will be shown to yield the same photon counting distribution first derived by Mandel (Reference 30) and later proven experimentally (References 34-36). It will also agree with Bohr and Rosenfeld's description of one ideal real field measurement.

In order to establish the necessary properties of the model to be postulated, the general requirements imposed by quantum theory are outlined in the remainder of this section. In the ensuing discussion the terms preparation, measurement, and system state are defined differently than in conventional quantum theory. A preparation process is defined as an attempt to establish a system in a desired state. However, after completion of the preparation process the system may not exist in the desired state because of the uncertainty principle (see discussion which follows). One is able only to predict in a probabilistic sense the actual resulting state. In contrast, a measurement process is defined as an attempt to determine the state in which a system exists prior to the beginning of the measurement process. However, the accuracy of the measurement is limited by the uncertainty principle and one can only guess in a probabilistic sense what the true system state was prior to the measurement. Conventional quantum theory does not give separate emphasis to the preparation process since it is argued that the two are identical in principle and therefore should not be distinguished. However, Margenau has argued the contrary (Reference 41) and recently Prugovecki has viewed the distinction as fundamental (Reference 42). She and Heffner (Reference 43) have shown how the distinction exists in the mathematics which describe a conservative quantum harmonic oscillator. In this paper a consideration of the preparation process is necessary to account for quantum effects introduced by the transmitter, a point neglected by previous studies. The state of a system is defined by two conjugate quantities rather than only one quantity as in conventional theory. Such a definition is not new and has been utilized by She (References 14 and 43). The newly defined state still permits one to consider the conventionally defined state as discussed in the next paragraph.

As noted above, the results of a preparation or measurement process must not violate the Heisenberg uncertainty principle. This principle accounts for a natural limit on man's ability to simultaneously prepare or measure certain variable pairs which describe the state of a system. Heisenberg has shown that the limit is approximately (Reference 37)

$$\Delta p \Delta q \sim h \qquad (3\text{-}1a)$$

$$\Delta \mathcal{E} \Delta t \sim h \qquad (3\text{-}1b)$$

where

Δ root-mean-square deviation

q position

p momentum

$\mathcal{E}$ energy

t time

h Planck's constant ($\hbar = \frac{h}{2\pi}$)

The uncertainties or root-mean-square (rms) deviations in (3-1) are of a statistical nature over an ensemble of systems and are the result of the unavoidable and unpredictable interaction between the preparation or measuring process and the system. For example, consider an attempt to prepare each ensemble member into the same state defined by the variable pair (q_0, p_0). By Heisenberg's arguments it is physically impossible to prepare all members into the same state (q_0, p_0), but rather the best nature will permit is expressed by (3-1). This does not preclude the possibility, at least in principle, of attempting to prepare all members into identical states of either p_0 or q_0 (i.e., conventionally defined states). From (3-1a) such an operation would yield an infinite rms deviation of q or p respectively. This is an unrealistic result which has long been recognized (Reference 44). Although a consideration of only one variable has been successful for most quantum mechanical problems, the developments which follow in this paper will use harmonic oscillator states which require both p and q be well defined over an ensemble.

In many cases one does not wish to prepare all ensemble members into identical states (q_0, p_0). For example, the statistical description of a noisy system can give rms deviations larger than those in (3-1). However, the excess uncertainty over h is, in principle, controllable by man while that expressed by (3-1) is not controllable and must be an inherent part of quantum theory. For example, one can control the amount of noise emitted from a thermal noise source via control of its physical temperature, but no such control is possible over the uncertainty of (3-1). In the sequel, sources of uncertainty will be referred to as "classical" or "quantum" in orgin, the latter noting the uncontrollable uncertainties introduced by the unpredictable effects of the preparation or measurement process.

Following an attempt to prepare a system into some state an attempt can be made to measure the resulting state. However, the measurement process is also subject to the uncertainty principle making it impossible, under any circumstances, for one's knowledge to exceed that allowed by (3-1). Again this does not preclude the possibility of making a perfectly accurate measurement of either p or q for the example given above. However, such an operation yields no knowledge of q or p respectively. It should be noted that the uncertainty relation (3-1) has been shown to be consistent with the quantum theory of free fields (References 39 and 40).

Dirac (Reference 44) has not only required measurements to be taken instantaneously, but he has noted that the unpredictable interaction between the measuring process and the system being measured requires a new wave function to describe the system immediately after the measurement. The form of the new wave function is important in establishing how the unpredictable interference of the measuring process affects the accuracy of all subsequent measurements. She (Reference 14) is apparently the only worker to have considered this problem in detail. Specifically, he analyzed the case of measuring the position and momentum of a lossy harmonic oscillator which possessed a Lorentz shaped bandwidth. When the measurement was performed with the most accuracy possible (i.e., with minimum uncertainty), She showed that after the measurement the oscillator must be in a "coherent state" (see page 41 for definition). Utilizing the new system wave function (as defined by the "coherent state") a second measurement was performed. The accuracy of the second measurement was found to be degraded by the first measurement. Moreover, the effect of the first on the second measurement was found to decay as the impulse response of a Lorentz shaped filter, i.e., the effect was dependent on the time interval between the two measurements. It follows that the effect of a measurement on the accuracy of subsequent measurements can be represented by an appropriate impulse excitation (in the variable being measured) of the lossy oscillator.

A measurement of field energy at any time instant (received field power averaged over a finite time period) must yield only values which are integral multiples of

hf_c where f_c is the frequency of radiation. This accounts for the well known postulate of Planck and permits the description of photon counting experiments (hereafter called photon detection).

A complete description of the field preparation and measurement processes must utilize a mathematics which is self consistent with both the classical wave properties of fields and the nonclassical particle properties. This, of course, is the general goal of quantum theory. The problem is one of finding a quantum mathematical description within which classical characteristics can be easily identified as opposed to purely quantum phenomena. The importance of this requirement is demonstrated with the following example. Consider a field which is prepared (transmitted) with a classical state of well defined amplitude and phase. At the receiver, the field measurement can take place by either the classical real field detection process or the quantum mechanical photon detection process. In either case the classical characteristics of the transmitted signal must be identifiable in the description of either detection process in order to determine the quantum effects in the system.

The requirement of self consistency of quantum descriptions with classical phenomena is embodied in Ehrenfest's theorem and the correspondence principle. Ehrenfest's theorem requires that over an ensemble of systems which include quantum effects, the average value of quantities with classical counterparts must equal the corresponding average values over an equivalent classical ensemble (one which does not include any quantum effects). The correspondence principle requires that in the limit $h \to o$ the quantum statistical distribution of quantities with classical counterparts must equal the corresponding classical distribution. An example of a quantity which has no classical counterpart and to which the above requirements cannot be applied is photon number.

The Postulated Model

Now that the description of a classical model and the basic requirements or quantum theory have been established, quantum properties can be included.

The classical model already meets three of the quantum requirements. That is: (1) the transmitted and received field can be described at discrete time instants without loss of information; (2) preparation or measurement processes can be considered to occur instantaneously in the form of impulse functions; and (3) in the absence of any preparation or measurement process the system evolves in time in a casual manner. In addition, the statistical properties of the transmitted and received fields are expressed in an ensemble sense thus establishing the ensemble needed to define the classical properties in a quantum description of the model.

The remaining quantum requirements are fulfilled bv postulating that the bandlimited preparation or measurement of each term in the summations (2-6) and (2-12) is equivalent to the preparation or measurement of a quantum mechanical oscillator of frequency f_c. The ensemble of systems used to define the classical distributions $P_T(x_T,y_T)$ and $P_R(x_R,y_R)$ can be extended to the postulated model by considering an equivalent ensemble of quantum harmonic oscillators. Equivalence of the two ensembles is established by Ehrenfest's theorem and the correspondence principle. If $P_{QR}(x_R,y_R)$ is the distribution defining the results of measuring x_R and y_R over the quantum ensemble, by Ehrenfest's theorem

$$\iint x_R P_{QR}(x_R,y_R)dx_R dy_R = \iint x_R P_R(x_R,y_R)dx_R dy_R \quad (3\text{-}2a)$$

$$\iint y_R P_{QR}(x_R,y_R)dx_R dy_R = \iint x_R P_R(x_R,y_R)dx_R dy_R \quad (3\text{-}2b)$$

By the correspondence principle

$$P_R(x_R,y_R) = \lim_{h\to o} P_{QR}(x_R,y_R) \quad (3\text{-}3)$$

The extension of the ergodic assumption to the quantum ensemble in the postulated model implicitly assumes that the unpredictable effect of a quantum mechanical preparation or measuring process on the communication system does not affect subsequent preparation or measurement operations. This is consistent with the following: (1) the preparation and measurement processes are taken to be bandlimited operations; (2) samples are considered only at the Nyquist rates (2-7) and (2-13); and (3) the quantum effects of a preparation or measurement process are instantaneous at the time of the sample and can be represented by an impulse of appropriate strength (in the variable being measured) as concluded from She's work. Since the $\frac{\sin x}{x}$ impulse response of $G_T(\omega)$ and $G_m(\omega)$ is zero at all other sampling instants, it follows that the unpredictable effect of a preparation or measurement process does not affect subsequent preparation or measurement processes. This is obviously an idealized case resulting from the use of an unrealizable bandwidth limited preparation or measurement process. The results are in any case consistent with the work of She for sequential measurements and that of Bohr and Rosenfeld for the case of one isolated measurement. Moreover, for the case of photon detection the resulting model will be shown to yield the photon counting distribution first derived by Mandel and subsequently proven experimentally. Therefore, within the constraints of the model postulated herein, the visualization of each term of (?-6) and (2-12) as a physically independent, zero bandwidth oscillator is permissible. Fulfillment of the remaining quantum requirements will be demonstrated in the sequel.

Quantum Effects at the Transmitter

The first step in transmitting information in a communication system is that of preparing the electromagnetic field to be transmitted. By the model postulated in the previous section this is equivalent to an attempt to prepare an ensemble of quantum harmonic oscillators into states determined by the information to be transmitted and the method of modulation. Classically, only the amplitude and phase of the emitted wave can be controlled by the modulation process. When quantum effects are introduced the energy levels of the mechanical oscillators become quantized. That is

$$\mathcal{E}_Q = n\hbar\omega, \qquad (3\text{-}4)$$

where $\mathcal{E}_Q$ is the energy level corresponding to n photons of the radiated field and is the amount of energy associated with one sample prepared by the transmitter. At least within the framework of the postulated model, the number of photons transmitted in each sample becomes a new variable which could be modulated by the information source. This will be called "photon modulation" in contrast to the modulation of the field's wave properties which will be called "wave modulation."

The term "photon modulation" as defined above is more restrictive than may first appear. To illustrate, one may wish to consider a coherent light beam modulated by the opening and closing of a shutter as a type of photon modulation since the flux of photons emitted from the transmitter has indeed been modulated. However, in this example, one cannot predict with good accuracy the exact number of photons emitted during each sample whereas the wave amplitude can (as will be shown later in this section). It follows that the transmitted information is contained not in the number of photons but in the wave amplitude. Photon modulation, as used herein, defines only the case where the transmitted information is contained in the exact number of photons emitted (not the emission rate).

As outlined in the introduction, the work of Gordon and others implicitly assume that in any channel the number of photons in each field sample describes the transmitted information. This obviously is photon modulation as defined above.

Before presenting the quantum properties of the prepared (transmitted) field samples, it will be convenient to make a change of variable. First consider a classical mechanical oscillator which is equivalent to one term in (2-6). The equations of motion of the oscillator as a function of position q_T and momentum p_T at time zero are

$$p_T(t) = p_T \cos \omega_c t - \omega_c q_T \sin \omega_c t \qquad (3\text{-}5a)$$

$$q_T(t) = q_T \cos \omega_c t + \frac{p_T}{\omega_c} \sin \omega_c t \qquad (3\text{-}5b)$$

It follows that the classical oscillator energy is

$$\mathcal{E}_T = \frac{1}{2}(p_T{}^2 + \omega_c{}^2 q_T{}^2) \qquad (3\text{-}6)$$

which is the energy available for the one prepared sample of the transmitted field being described by the oscillator. Since the dimension of (2-6) is voltage and the effective averaging time per sample is t_T, an equivalent oscillator can be defined as

$$p_T = x_T \sqrt{t_T} \qquad (3\text{-}7a)$$

$$q_T = \frac{y_T}{\omega_c} \sqrt{t_T} \qquad (3\text{-}7b)$$

$$\mathcal{E}_T = \frac{1}{2}(x_T^{\;2} + y_T^{\;2})t_T \tag{3-8}$$

Consider an ensemble of unit mass quantum harmonic oscillators which have been prepared in an arbitrary state. Ensemble averages or the position and momentum are given by reference 46 as (the subscript T is dropped from q and p for convenience)

$$\langle q\rangle = \sqrt{\frac{2\hbar}{\omega_c}}\sum_n \sqrt{n+1}\,|C_{n+1}C_n|\cos(\omega_c t + \phi_{n+1} - \phi_n) \tag{3-9}$$

$$\langle p\rangle = -\sqrt{2\hbar\omega_c}\sum_n \sqrt{n+1}\,|C_{n+1}C_n|\sin(\omega_c t + \phi_{n+1} - \phi_n) \tag{3-10}$$

$$\langle q^2\rangle = \frac{\hbar}{\omega_c}\Big[N + \frac{1}{2} + \sum_n \sqrt{(n+1)(n+2)}\,|C_{n+2}C_n|\cos(2\omega_c t + \phi_{n+2} - \phi_n)\Big] \tag{3-11}$$

$$\langle p^2\rangle = \hbar\omega_c\Big[N + \frac{1}{2} - \sum_n \sqrt{(n+1)(n+2)}\,|C_{n+2}C_n|\cos(2\omega_c t + \phi_{n+2} - \phi_n)\Big] \tag{3-12}$$

where $|C_n|^2$ is the probability distribution of the number of photons or

$$P_n(n) = |C_n|^2$$

where

$$N = \sum_n n P_n(n)$$

Richter, et. al., have shown that <p> and <q> attain their maximum amplitude only when (Reference 46)

$$\phi_{n+1} - \phi_n = \phi \qquad (3\text{-}13)$$

for all n, and

$$P_n(n) = \frac{N^n}{n!} e^{-N} \qquad (3\text{-}14)$$

which is a Poisson distribution with average value of N. When (3-13) and (3-14) hold, the ensemble averages become

$$\langle q \rangle = \sqrt{\frac{2\hbar N}{\omega_c}} \cos(\omega_c t + \phi) \qquad (3\text{-}15)$$

$$\langle p \rangle = -\sqrt{2\hbar\omega_c N} \sin(\omega_c t + \phi) \qquad (3\text{-}16)$$

$$\langle q^2 \rangle = \frac{2\hbar N}{\omega_c} \cos^2(\omega_c t + \phi) + \frac{\hbar}{2\omega_c} \qquad (3\text{-}17)$$

$$\langle p^2 \rangle = 2\hbar\omega_c N \sin^2(\omega_c t + \phi) + \frac{\hbar\omega_c}{2} \qquad (3\text{-}18)$$

These describe an ensemble of harmonic oscillators which are in near synchronism. The lack of perfect synchronism exists because of the uncertainties

$$(\Delta q)^2 = \sigma_q^{\,2} = \langle q^2 \rangle - \langle q \rangle^2 = \frac{\hbar}{2\omega_c} \qquad (3\text{-}19)$$

$$(\Delta p)^2 = \sigma_p^{\,2} = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar\omega_c}{2} \qquad (3\text{-}20)$$

which conform to the minimum uncertainty relation

$$\Delta q \Delta p = \frac{\hbar}{2} \qquad (3\text{-}21)$$

Therefore, when an attempt is made to prepare each ensemble member into identical states of amplitude and phase, not only does the position and momentum possess the minimum uncertainty (3-21), but the number of photons possess a Poisson distribution. By the postulated model this corresponds to an attempt to transmit a wave of which the amplitude and phase remain constant in time. Similarly the statistical distributions over the ensemble of oscillators correspond to unpredictable fluctuations in time of the variables concerned. The fluctuations of position and momentum are normally negligible since the transmitted wave is many times larger. However, the photon fluctuations are very significant and can be measured for the case of single mode laser radiation (c.f. References 34 and 35). Photon fluctuations cannot be made negligible since the variance is equal to the average value, thus any increase in the transmitted signal proportionally increases the variance of photon counts. Therefore, from the definitions of photon and wave modulation given earlier two conclusions follow: (1) the above situation describes a wave modulated signal, and (2) a wave modulated signal cannot be used to describe a photon modulated signal.

Consider the photon modulation counterpart to the above example. Each time sample of the field must contain the same number of photons N. Then

$$|C_n| = \delta_{n,N}$$

$$P_n(n) = \delta^2_{n,N} \qquad (3\text{-}22)$$

where $\delta_{n,N}$ is the Kronecker delta function. From (3-9) to (3-12)

$$\langle p\rangle = \langle q\rangle = 0 \qquad (3\text{-}23)$$

$$\langle p^2\rangle = \hbar\omega_c(N + \tfrac{1}{2}) \qquad (3\text{-}24)$$

$$\langle q^2\rangle = \frac{\hbar}{\omega_c}(N + \tfrac{1}{2}) \qquad (3\text{-}25)$$

These averages describe an ensemble of harmonic oscillators where the phase ϕ is equally distributed over 2π radians. Therefore a photon modulated signal possesses a random phase modulation which is independent of the transmitted signal level. That is, a signal produced under the constraints of photon modulation (as defined earlier) possesses an unpredictable phase from one sample to the next and therefore is not capable of describing a wave modulated signal. The latter conclusion follows from the requirements of amplitude and phase modulated signals discussed at the end of Chapter II. Wave modulation describes all known continuous wave, modulation schemes. Only a pulsed modulation process which controls the number of photons per pulse could fulfill the photon modulation requirements. It appears that all present day schemes are basically wave modulation since no physical process for generating a photon modulated field has been demonstrated. The remainder of this paper will be devoted primarily to wave modulation. A discussion of a photon modulated system will be given in Chapter V to place the work shown herein into proper perspective with the "photon channel" model.

The Glauber P Representation

Before proceeding to an investigation of the magnitude of quantum effects in a wave modulated communication system, the Glauber P function must be introduced and its relationship to classical functions determined.

The "coherent state" was named by R. J. Glauber in his application of the minimum uncertainty state of the harmonic oscillator to the study of the quantum theory of optical coherence (Reference 19-22). Both Glauber and Sudarshan (Reference 47) recognized the usefulness of a diagonal representation with the "coherent state" for describing some general field states. Glauber developed an extensive mathematics around the diagonal form which is now known as the Glauber P representation.

Lachs has shown that by using the P representation, the probability distributions for position, momentum, and photon number for an ensemble of unit mass oscillators are (Reference 48)

$$p_p(p) = \iint P(\alpha') \frac{1}{\sqrt{\pi\hbar\omega_c}} \exp\left[\frac{-(p - \sqrt{2\hbar\omega_c}\,\alpha_I')^2}{\hbar\omega_c}\right] d^2\alpha' \quad (3\text{-}26)$$

$$p_q(q) = \iint P(\alpha') \frac{1}{\sqrt{\pi\hbar/\omega_c}} \exp\left[\frac{-(q - \sqrt{\frac{2\hbar}{\omega_c}}\,\alpha_R')^2}{\hbar/\omega_c}\right] d^2\alpha' \quad (3\text{-}27)$$

$$P_n(n) = \iint P(\alpha) \frac{|\alpha|^{2n}}{n!} \exp(-|\alpha|^2) d^2\alpha \quad (3\text{-}28)$$

where $P(\alpha)$ Glauber's P function (a shorthand notation for the joint function of α_R and α_I)

$$\alpha = \frac{1}{\sqrt{2\hbar\omega_c}}(\omega_c q + jp) = \alpha_R + j\alpha_I \tag{3-29}$$

$$d^2\alpha = d\alpha_R \, d\alpha_I$$

She and Heffner's work can be extended as shown in Appendix I to give the joint probability density

$$P_Q(q,p) = \iint P(\alpha') \, \frac{\exp}{2\pi\hbar}\left[- \frac{(q - \sqrt{\frac{2\hbar}{\omega_c}}\,\alpha_R')^2}{2\hbar/\omega_c} - \frac{(p - \sqrt{2\hbar\omega_c}\,\alpha_I')^2}{2\hbar\omega_c}\right] d^2\alpha' \tag{3-30}$$

The P representation is important since it yields the joint probability density $P_Q(q,p)$ which by Ehrenfest's theorem and the correspondence principle must contain classical properties of the ensemble of oscillators as well as the quantum properties. If the classical distribution $P_c(q,p)$ used in describing conventional wave modulated systems can be identified within the P function, a set of equations will result which "transform" the classical probability density into distributions which contain the quantum effects.

The mathematical properties of $P(\alpha)$ have been the subject of much discussion (References 23-25). These works have centered on the purely quantum theoretical properties of $P(\alpha)$ rather than the possible classical meaning it may possess. This particular aspect of the development and use of the "coherent state" has been a stumbling block to a simple interpretation of the equations (3-26) through (3-30).

Glauber has noted that, in general, $P(\alpha)$ is not a probability density. Moreover, he allows $P(\alpha)$ to be interpretable as a probability density only in an approximate sense and never as an equality. Quoting from Glauber (Reference 21, page 2776)

> "The function $P(\alpha)$ might then be thought of as playing a role analogous to a probability density for the distribution of values α over the complex plane. Such an interpretation may... be justified at times. In general, however, it is not possible to interpret the function $P(\alpha)$ as a probability distribution in any precise way.... When the function $P(\alpha)$ tends to vary little over...large ranges of the parameter α ..., $P(\alpha)$ will then be interpretable approximately as a probability density."

To study the classical properties of $P(\alpha)$ one can use the correspondence principle and find $\lim_{h \to o} P_Q(q,p)$.

To do this it is convenient to make the change of variable (3-29) and express $P(\alpha)$ as a function of q and p. It follows

$$P(\alpha)\, d\alpha_R d\alpha_I = P_G(q,p) d_q d_p \tag{3-31}$$

$$d\alpha_R = \sqrt{\frac{\omega_c}{2\hbar}}\, d_q$$

$$d\alpha_I = \frac{1}{\sqrt{2\hbar\omega_c}}\, d_p$$

Then

$$d^2\alpha = \frac{1}{2\hbar}\, d_q d_p$$

$$P(\alpha)\Bigg|_{\substack{\alpha_R = \sqrt{\frac{\omega_c}{2\hbar}}\, q \\ \alpha_I = \frac{1}{\sqrt{2\hbar\omega_c}}\, p}} = 2\, P_G(q,p)$$

Equations (3-26) to (3-28) and (3-30) become

$$p_{Qp}(p) = \iint P_G(q',p') \frac{1}{\sqrt{\pi\hbar\omega_c}} \exp\left[-\frac{(p-p')^2}{\hbar\omega_c}\right] d_q' d_p' \tag{3-32}$$

$$p_{Qq}(q) = \iint P_G(q',p') \frac{1}{\sqrt{\pi\hbar/\omega_c}} \exp\left[-\frac{(q-q')^2}{\hbar/\omega_c}\right] d_q' d_p' \tag{3-33}$$

$$P_Q(q,p) = \iint P_G(q',p') \frac{\exp}{2\pi\hbar}\left[-\frac{(q-q')^2}{2\hbar/\omega_c} - \frac{(p-p')^2}{2\hbar\omega_c}\right] d_q' d_p' \tag{3-34}$$

$$P_n(n) = \iint P_G(q',p') \frac{\left(\frac{\omega_c q^2 + p^2}{2\hbar\omega_c}\right)^n}{n!} \exp\left[- \frac{\omega_c q^2 + p^2}{2\hbar\omega_c}\right] d_q{}'d_p{}' \qquad (3\text{-}35)$$

By the correspondence principle

$$p_{cp}(p) = \lim_{h \to o} p_{Qp}(p) \qquad (3\text{-}36)$$

$$p_{cq}(q) = \lim_{h \to o} p_{Qq}(q) \qquad (3\text{-}37)$$

$$P_c(q,p) = \lim_{h \to o} P_Q(q,p) \qquad (3\text{-}38)$$

Since the limits on the integrals are over all possible p and q, they are independent of h. If $P_G(q,p)$ is also independent of h, then

$$p_{cp}(p) = \int P_G(q,p)\, dq \qquad (3\text{-}39a)$$

$$p_{cq}(q) = \int P_G(q,p)\, dp \qquad (3\text{-}39b)$$

$$P_c(q,p) = P_G(q,p) \qquad (3\text{-}39c)$$

where the relation

$$\lim_{\sigma \to o} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x - x_1)^2}{2\sigma^2}\right] = \delta(x - x_1)$$

was used. The fact that $P_G(q,p)$ must be independent of h for the constraints placed on the quantum description can be seen by the following argument. By the correspondence principle a calculation of the second moment of p and q must yield the classical second moment plus a quantum uncertainty proportional to h. The model used herein has already taken the ensemble to be one of minimum quantum uncertainty. Therefore any dependency of $P_G(q,p)$ on h could not affect the magnitude (3-21), but only the ratio of uncertainties in p and q. Previous analysis (Reference 49) using the maximization of entropy as a criterion for a least biased estimate has shown that there is an equal balance of uncertainty between p and $\omega_c q$ when the only constraint is the product (3-19). Furthermore, Lachs (Reference 15) has shown this condition satisfies the energy-time uncertainty relation (3-1b). It follows that $P_G(q,p)$ is independent of h since this condition satisfies the above requirements in equations (3-32) and (3-33).

Therefore, the Glauber P function $P_G(q,p)$ becomes equal to the classical distribution $P_C(q,p)$ when describing a field which is known in classical terms and is being prepared and measured by ideal processes.

The substitution of $P_c(q,p)$ into (3-32) to (3-35) yields the set of desired equations which transform the probability density of $P_c(q,p)$ to those which include quantum effects.

Part of the arguments used here are similar to those of Mandel and Wolf (Reference 25). However, they required only the equality of averages as in Ehrnferst's theorem to show that when an all positive $P(\alpha)$ exists, there must also exist a classical function which gives the same averages. They failed to point out that $P(\alpha)$ possesses a definite classical meaning as required by the correspondence principle as discussed above.

It should be emphasized that the classical or quantum character of $P(\alpha)$ is determined only by the preparation process and has nothing to do with the eventual measurement of the field. The measurement process may also be of either a classical or quantum character depending on the type of field-matter interaction process used at the receiver. In the conventional device, an electron current is induced in synchronism with the received electric field. The induced sinusoidal variations are filtered, amplified, possibly heterodyned, and then subjected to some form of amplitude, phase, or power detection. This is equivalent to a measurement of p and q and is described by equation (3-34). The operation will be referred to as a "real field receiver" or "real field detection." In contrast, the "photon receiver of detector" responds only to the discrete photons in the received field. This is a measurement of n and is described by (3-35). The photon receiver still responds to wave properties since (3-35) is a function of $P_G(q,p)$.

CHAPTER IV

WAVE MODULATED SYSTEMS

Amplitude and phase modulated communication systems were shown in the preceding chapter to be wave modulated systems. Furthermore, the Glauber P representation was found to provide a description of wave modulated systems where $P_G(q,p)$ can be equated to the classical $P_c(q,p)$. In this chapter an investigation of free space attenuation effects is followed by a derivation of the equations which describe both real field and photon detection situations. The effects of photon noise relative to classical Gaussian noise are determined and a channel capacity equation is found and compared to Gordon's work.

The Effects of Free Space Attenuation

Before discussing the effects of free space attenuation on the results of the preceding chapter, a significant difference between Equations (3-32), (3-33), and (3-34) must be recognized. Consider the case where the desired classical field is of definite amplitude and phase, or

$$P_c(q,p) = \delta(q - q_s)\delta(p - p_s) \qquad (4\text{-}1)$$

which gives in (3-32), (3-33), and (3-34)

$$p_{Qp}(p) = \frac{1}{\sqrt{\pi\hbar\omega_c}} \exp\left[-\frac{(p - p_s)^2}{\hbar\omega_c}\right] \tag{4-2}$$

$$p_{Qq}(q) = \frac{1}{\sqrt{\pi\hbar/\omega_c}} \exp\left[-\frac{(q - q_s)^2}{\hbar/\omega_c}\right] \tag{4-3}$$

$$P_Q(q,p) = \frac{1}{2\pi\hbar} \exp\left[-\frac{(q - q_s)^2}{2\hbar/\omega_c} - \frac{(p - p_s)^2}{2\hbar\omega_c}\right] \tag{4-4}$$

Equations (4-2) and (4-3) yield

$$\sigma_p^2 = \frac{\hbar\omega_c}{2} = (\Delta p)^2 \tag{4-5}$$

$$\sigma_q^2 = \frac{\hbar}{2\omega_c} = (\Delta q)^2$$

which is the minimum uncertainty (3-23). However, (4-4) yields

$$\begin{aligned} \sigma_p^2 &= \hbar\omega_c = (\Delta p)^2 \\ \sigma_q^2 &= \hbar/\omega_c = (\Delta q)^2 \end{aligned} \tag{4-6}$$

or

$$\Delta q \Delta p = \hbar \tag{4-7}$$

which is twice the minimum uncertainty (3-21). As shown by She and Heffner (Reference 43) the uncertainty principle applies to both the preparation and measurement processes. Therefore (3-32) and (3-33) must apply when a preparation into a "coherent state" is followed by a perfectly accurate measurement of either p or q alone (which eliminates any possible knowledge of the other variable). Equation (3-34) must apply then when both p and q are measured following the preparation thus giving twice the minimum uncertainty as shown in (4-7).

Note that (3-34) applies when the oscillator is dissipationless between the preparation and the measurement processes. In the case of a communication system, the magnitude of the received field is much less than the total prepared field at the transmitter because of free space attenuation.

Since the field evolves in time in a classical manner after the preparation process, it follows that the uncertainties of quantum origin introduced at the transmitter will be attenuated in the same manner as classical variations of the field. By assuming either the transmitted signal is sufficiently large or that the free space attenuation is sufficiently great, the quantum noise introduced by the wave modulation process at the transmitter will be negligibly small at the receiver. Therefore, it follows that for real field detection of a wave modulated field, the statistical distribution is given by the joint probability density

$$P_Q'(q,p) = \frac{1}{2\pi\hbar}\iint P_c(q',p')\exp\left[-\frac{(q-q')^2}{\hbar/\omega_c} - \frac{(p-p')^2}{\hbar\omega_c}\right]dq'dp' \quad (4\text{-}8)$$

where the minimum uncertainty (3-21) holds rather than (4-7).

In the case of photon detection Hagfors has shown quantum mecnanically (Reference 7, page 11) that the number of photons received is affected by free space attenuation as described by the binomial distribution

$$P_n(n|M) = \frac{M!}{n!(M-n)!}\nu^n(1-\nu)^{(M-n)} \quad (4\text{-}9)$$

where $P_n(n|M)$ probability that n photons will be received knowing that exactly M photons were transmitted;

ν power attenuation coefficient.

For the wave modulated system the number of transmitted photons cannot be known with certainty and obeys (3-35) which is a Poisson distribution with a randomly varying average value. LaTourette and Steinberg have shown that a Poisson distribution followed by a bionomial operation gives another Poisson distribution (Reference 50)

$$P_n(n|M) = \frac{(N\nu)^n}{n!} \exp(-N\nu) \qquad (4\text{-}10)$$

where N average number of photons transmitted. Therefore, the distribution (3-35) applies at the receiver with the average values as determined by classical analysis.

It should be noted that the above conclusion is concurrent with the assumption of She and Jelsma (References 14 and 16), namely, the transmitted signal is sufficiently large that quantum effects at the transmitter are negligible. However, She apparently did not discriminate between the uncertainties which arose from preparation at the transmitter and measurement at the receiver until his later work (Reference 43). Therefore, She's original paper (Reference 14) essentially uses Equation (3-34) rather than (4-8).

Analysis of Quantum Effects

Since the time sampled field model utilizes the voltages x_R and y_R to describe the measured quantities it is convenient to make the change of variable

$$p = x_R\sqrt{t_m} \qquad (4\text{-}11a)$$

$$q = \frac{y_R}{\omega_c}\sqrt{t_m} \qquad (4\text{-}11b)$$

which yields

$$P_{RQ}(x_R, y_R) = \frac{\omega_c}{t_m} P_Q(q,p)\Bigg|_{\substack{q = \frac{y_R}{\omega_c}\sqrt{t_m} \\ p = x_R\sqrt{t_m}}} \qquad (4\text{-}12)$$

$$P_R(x_R, y_R) = \frac{\omega_c}{t_m} P_G(q,p)\Bigg|_{\substack{q = \frac{y_R}{\omega_c}\sqrt{t_m} \\ p = x_R\sqrt{t_m}}} \qquad (4\text{-}13)$$

Substituting these into equations (3-35) and (4-8)

$$P_{RQ}(x_R, y_R) = \iint P_R(x_R', y_R') \frac{\exp}{2\pi(\frac{\hbar\omega_c}{2}W_m)} \left[- \frac{(x_R - x_R')^2}{\hbar\omega_c W_m} - \frac{(y_R - y_R')^2}{\hbar\omega_c W_m} \right] dx_R' \, dy_R' \qquad (4\text{-}14)$$

$$P_n(n) = \iint P_R(x_R, y_R) \frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!} \exp\left[\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right] dx_R \, dy_R \qquad (4\text{-}15)$$

where $t_m = \frac{1}{W_m}$

These two equations express one of the main results of this paper. They transform the probability density $P_R(x_R, y_R)$ derived from classical theory to distributions which include the quantum effects. Equation (4-11) holds for the case of real field detection whereas Equation (4-15) describes the results of photon detection. With Equations (4-14) and (4-15) one can use the equations derived from a classical analysis of the time sampled model to calculate the magnitude of photon noise.

In Figure 4 it was shown how bandlimiting filters may be placed before and after the actual field measurement process. The bandwidth W_F was defined as operating directly on the field prior to its measurement, whereas W_0 operates on the measured values. If W_0 is the minimum bandwidth required by the modulation spectrum, it follows that

$$W_F \geq W_0 \qquad (4\text{-}16).$$

The finite bandwidth W_m of the measurement process should also be

$$W_m \geq W_0 \qquad (4\text{-}17)$$

for the same reason. In the following examples, the choice of relative values for W_F, W_M, and W_0 will be more for convenience than practicality in order to demonstrate the desired properties of the detection processes. However, the use of Equations (4-14) and (4-15) within the framework of the postulated model permits any combination of these bandwidths to be analyzed.

In order to determine the effects of photon or quantum noise relative to conventional noise, take $P_R(x_R, y_R)$ to consist of the received signal $P_s(x_s, y_s)$ and an additive "white" Gaussian noise of power $n_c W_F$ which yields

$$P_R(x_R, y_R) = \iint P_s(x_s, y_s) \frac{\exp}{2\pi n_c W_F} \left[- \frac{(x_R - x_s)^2}{2 n_c W_F} - \frac{(y_R - y_s)^2}{2 n_c W_F} \right] dx_s dy_s \tag{4-18}$$

where n_c is the spectral power density of the incident noise field.

Real field detection.- Substitution of (4-18) into (4-14) yields the double convolution

$$P_{QR}(x_R, y_R) = \int_{x_s}\int_{y_s} P_s(x_s, y_s) \left\{ \int_{x_R'}\int_{y_R'} \frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R' - x_s)^2 - (y_R' - y_s)^2}{2 n_c W_F} \right] \frac{\exp}{2\pi \hbar\omega_c W_m} \left[\frac{-(x_R - x_R')^2 - (y_R - y_R')^2}{2(\frac{\hbar\omega_c}{2} W_m)} \right] dx_R' dy_R' \right\} dx_s dy_s$$

which reduces to the single convolution

$$P_{QR}(x_R,y_R) = \iint P_s(x_s,y_s) \frac{\exp\left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2(n_c W_F + \frac{\hbar\omega_c}{2} W_m)}\right]}{2(n_c W_F + \frac{\hbar\omega_c}{2} W_m)} dx_s dy_x \tag{4-19}$$

Therefore, the output of the measurement process yields the signal, the classical noise of bandwidth W_F, and a Gaussian noise of power $n_c W_m$. Since the quantum noise results from the "interference" of the measurement process with quantities being measured and is representable by an impulse function in the sampling process, it must be "white" or evenly distributed over the bandwidth W_m. Since (4-16) and (4-17) hold, the effective input noise power density is

$$n_{RF} = (n_c + \frac{\hbar\omega_c}{2}) \tag{4-20}$$

Therefore, for the case of real field detection in a wave modulated system, quantum noise enters as an additive Gaussian input noise of density $\hbar\omega_c/2$ watts/Hertz. This conclusion is true only to the extent that (4-17) holds. That is, an upper limit on the total amount of quantum noise that can be introduced by the measurement process is imposed by W_m. Practically W_m must be finite, but if W_m is allowed to become infinitely large an infinite amount of quantum noise power is introduced. This is just another example of the so-called "zero-point field catastrophe" in quantum field theory for which there is no physical explanation. However, the model utilized herein is useable only for describing narrow bandwidth systems and therefore as W_m is increased, the model's ability to describe a modulated signal breaks down much before the "zero-point field catastrophe" becomes important.

Photon detection.- The effects of photon noise cannot be as easily generalized for photon detection as was found possible for real field detection. Not only must two types of photon detection (direct and heterodyne) be analyzed, but the nonlinear character of counting distribution (4-15) causes the relative magnitudes of the bandwidths W_F, W_M, and W_0 to affect the relative magnitude of photon and conventional noises. The effect of bandwidths W_F and W_0 has been investigated by Mandel (Reference 30) for the case of photoelectric detection where the assumption $W_m \to \infty$ is valid for most practical systems. Appendix II shows that the model used herein yields the same counting distribution derived by

Mandel and which has been proven experimentally (References 34 - 36).

In view of the previous work, the purposes of this section are to show some of the important properties of the photon detection process, to demonstrate how the effects of photon noise can be analyzed with the postulated model, and to determine if the effects of photon noise can be included in a conventional analysis in a simple manner.

For the case of additive Gaussian noise, substitution of (4-18) into (4-15) yields at the photon counter output (neglecting W_0 for the moment)

$$P_n(n) = \iint_{x_s y_s} P_s(x_s, y_s) \left\{ \iint_{x_R y_R} \frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R - y_s)^2 - (y_R - y_s)^2}{2 n_c W_F} \right] \frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m} \right)^n}{n!} \exp\left[- \frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m} \right] dx_R dy_R \right\} dx_s dy_s \qquad (4\text{-}21)$$

To understand the behavior of photon noise in (4-21), consider the part of Equation (4-21) in braces. This is tne conditional probability distribution of the number of received photons knowing the classical received signal (x_s, y_s). That is

$$P_n[n|(x_s,y_s)] = \iint \frac{\exp}{2\pi n_c W_F}\left[\frac{-(x_R - y_s)^2 - (y_R - y_s)^2}{2 n_c W_F}\right]$$

$$\frac{\left(\dfrac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!} \exp\left[-\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right] dx_R dy_R \qquad (4\text{-}22)$$

The average and variance of (4-22) are calculated in Appendix III and found to be

$$\langle n, A_s \rangle = \frac{A_s^2}{2\hbar\omega_c W_m} + \frac{n_c W_F}{\hbar\omega_c W_m} \qquad (4\text{-}23)$$

$$\sigma^2_{n|A_s} = 2\frac{n_c W_F}{\hbar\omega_c W_m}\left(\frac{A_s^2}{2\hbar\omega_c W_m}\right) + \left(\frac{n_c W_F}{\hbar\omega_c W_m}\right)^2 + \left[\frac{A_s^2}{2\hbar\omega_c W_m} + \frac{n_c W_F}{\hbar\omega_c W_M}\right] \qquad (4\text{-}24)$$

where $A_s^2 = x_s^2 + y_s^2$

These equations show how the photon detector behaves as a classical square law device with the addition of a "shot noise" (shown in brackets in 4-24). Rice (Reference 51) has analyzed the square law detector and identified the first two terms in (4-24). The first term arises from "mixing" between the sinusoidal field and the noise field, and the second term from "mixing" between spectral components of the noise. The Poisson "shot noise" is well known to yield a "white" noise spectrum.

Two examples of photon detection will be analyzed. First take the case of a direct detection of the field. Consider a system which transmits a signal of constant amplitude A_s for t_F seconds. Detection of A_s is accomplished by counting the number of photons received during the t_F second interval. For convenience assume the "ideal situation exists, that is

$$W_n = W_F \qquad (4\text{-}25)$$

and time synchronism is known. Equations (4-23) and (4-24) apply. To discern the relative effects of photon and classical noise, rewrite (4-24) as

$$\sigma_n^2 = \frac{n_c}{\hbar\omega_c}\left(\frac{S}{N} + 1\right)\left[1 + \frac{n_c}{\hbar\omega_c}\left(\frac{2\frac{S}{N} + 1}{\frac{S}{N} + 1}\right)\right] \qquad (4\text{-}26)$$

where

$$\frac{S}{N} = \frac{A_s^2}{2n_c W_F}$$

and is the receiver input signal to noise ratio.

If Poisson statistics were applicable for describing the photon counts, only the term $\frac{n_c}{\hbar\omega_c}(\frac{S}{N} + 1)$ would appear in (4-26). The additional factor is due to the fluctuating amplitude of the incident field and could be calculated by treating the photon detector as a classical square law detector. The magnitude of this term relative to one, is an indication of the accuracy of utilizing Poisson statistics to analyze direct photon detection. That is, if

$$n_c << \hbar\omega_c \qquad (4-27)$$

Poisson statistics will provide an accurate description of the variance of the photon counts. However if

$$n_c \approx \hbar\omega_c \qquad (4-28)$$

the counting statistics must be found through Equation (4-22). If

$$n_c >> \hbar\omega_c \qquad (4-29)$$

the "shot noise" can be neglected and the photon detector can be treated as a classical square law device where the $\hbar\omega_c$ factor is included.

Therefore, the effect of the "particle" and "wave" properties of radiation is determined by the relative magnitude of the received noise power density n_c to the energy of one quantum at the operating frequency. It is seen that the popular method of analyzing photon counting detection techniques with the Poisson distribution is dependent on the requirement (4-27). More generally, the exact counting distribution is given by (4-22).

If

$$W_o < W_m \tag{4-30}$$

the additional effects of W_o can be included in the analysis by averaging W_m/W_o samples together. Note that the correlation between adjacent samples must be taken into account.

As stated earlier, in cases more practical than the above example ($W_m \to \infty$), the relative magnitude of W_F and W_o can effect the inequalities (4-27) to (4-29). In the case

$$W_F > W_o \tag{4-31}$$

larger η_c will be required for wave effects to be significant in the output.

The following general observation can be made about these results. The photon detector behaves as a classical square law device with an additional noise in the output, i.e., the photon or shot noise. The resulting counting statistics are neither Gaussian nor Poissonian and no simple method has been found for including quantum effects in the analysis of direct photon detection.

Next consider the photon detector when used as a "mixer." In addition to the received signal and noise, an incident local oscillator field must be present for the photon detector to behave as a "mixer." This case is analyzed in Appendix IV and output signal-to-noise ratio in bandwidth W_F about the intermediate frequency is shown to be

$$\frac{S_0}{N_0} = \frac{P_s}{(n_c + \hbar\omega_c)W_F} \qquad (4\text{-}32)$$

where

P_s received signal power

$n_c W_F$ received noise power in bandwidth W_F.

In deriving (4-32) it was assumed the local oscillator power is much larger than P_s and W_F is much less than the local oscillator frequency. It is shown in Appendix IV that $\hbar\omega_c W_F$ is the Poisson "shot noise" arising from the local oscillator field. Since the Poisson distribution approaches a Gaussian distribution for large average values, it follows that the photon noise $\hbar\omega_c W_F$ in (4-32) is approximately additive Gaussian. Therefore quantum effects in photon heterodyne detection enters conventional analysis as an equivalent input noise of density $\hbar\omega_c$ watts/Hertz. These results agree with Oliver (References 52 and 53).

A Channel Capacity Equation

Shannon's channel capacity equation

$$C = W_F \log_2\left(1 + \frac{P_s}{n_c W_F}\right) \qquad (4\text{-}33)$$

is true when $n_c W_F$ is the power for additive Gaussian noise (Reference 54). Since the photon noise was found in (4-19) to be additive Gaussian for real field detection, the capacity including quantum effects is

$$C_Q = W_F \log_2\left[1 + \frac{P_s}{\left(n_c + \frac{\hbar\omega_c}{2}\right) W_F}\right] \qquad (4\text{-}34)$$

This capacity can readily be shown to disagree with Gordon's results. Consider the case for no classical noise n_c and

$$P_s \gg \frac{\hbar\omega_c}{2} W_F \qquad (4\text{-}35)$$

Equation (4-34) becomes

$$C_Q \simeq W_F \log_2\left(\frac{2\, P_s}{\hbar\omega_c W_F}\right) \qquad (4\text{-}36)$$

and under the same conditions, Gordon has shown his capacity equation yields (Reference 5)

$$C_g \quad W_F\left[\log_2\left(\frac{F_s}{\hbar\omega_c W_F}\right) + \log_2 e\right]$$
$$\cdot\ W_F\left[\log_2\left(\frac{2P_s}{\hbar\omega_c W_F}\right) + \log_2 \frac{e}{2}\right] \qquad (4\text{-}37)$$

Therefore

$$C_g \simeq C_Q + 0.44W_F \qquad (4\text{-}38)$$

which shows that the capacity derived by Gordon is larger than the capacity derived herein.

Lachs and Jelsma (References 15 and 16) have arrived at (4-34) by similar arguments while She obtained a different result (Reference 14). The reason for She's disagreement was noted on page 54.

CHAPTER V

PHOTON MODULATED SYSTEMS

As discussed in the Introduction, many of the investigations on quantum effects in communication systems assume that information capacity of a field is related to the specification of numbers of photons, i.e., photon modulation as defined in Chapter III. Gordon (References 5 and 8) and Deryugin and Kurashove (Reference 9) have argued (without proof) that their approach is independent of the modulation scheme in establishing a limit on channel capacity. However, the work in the preceding chapter has already shown that Gordon's capacity is too large for the description of conventional amplitude and phase modulated systems. The purpose of this chapter is to point out in what sense photon modulation has any physical meaning in contrast to conventional wave modulation.

The Photon Modulated Source

The basic properties of the photon modulated field was investigated in Chapter III. Those results will be reviewed as applied to the model developed in Chapters II and III. Rather than controlling the quadrature variables $x_T(kt_T)$ and $y_T(kt_T)$ in each term of (2-17), photon modulation controls the number of photons of each field sample. In the bandwidth W_T, only one sample (of the number of photons) every t_T seconds is required to describe the photon modulated field. This contrasts to the two samples (x_T and y_T) required for describing the wave modulated field. She (Reference 14) was the first note this discrepancy in Gordon's work, i.e., its failure to obey the correspondence principle. The work in Chapter III has shown that this difference exists because photon modulation (as defined herein) has no classical counterpart and cannot, under any circumstances, describe a wave modulated field. No physical means has been demonstrated which can prepare a field containing a prescribed number of photons.

The quantum model utilized herein implicitly requires all photons to be the same frequency f_c. Gordon (References 5 and 6) also imposed the same requirement. However, through the use of the energy-time uncertainty relation (3-1b), Bowen (Reference 11) has extended the "photon channel" to include frequency uncertainty effects due to the channel's finite bandwidth. He notes that the energy of each emitted photon must be uncertain by the amount

$$\Delta \mathcal{E} = hW_T$$

where W_T is the bandwidth of the transmission process as defined in Chapter II. This yields in (3-1b)

$$W_T \Delta t \approx 1$$

which agrees with (2-7) if Δt and the time between samples t_T are equated. This is a reasonable result, since t_T is the effective averaging time of the bandlimited transmission process.

Utilizing the energy-time uncertainty relation, Bowen shows that the results of Gordon (Reference 5) are correct only for the narrow bandwidth condition

$$W_T \ll f_c .$$

Therefore, the description of the "photon channel" by the postulated model is consistent with the same narrow bandwidth restriction (2-2) required for describing wave modulated systems.

The Effects of Free Space Attenuation

Hagfors has analyzed the effects of free space attenuation on a photon modulated signal. Using a quantum theoretical analysis he showed that if exactly M photons were transmitted, then the probability of receiving n photons is the binomial distribution (Reference 7)

$$P_n(n|m) = \frac{m!}{n!(m-n)!}\nu^n(1-\nu)^{m-n} \qquad (5\text{-}1)$$

where ν is the power attenuation coefficient due to free space attenuation. Hagfors concludes that free space attenuation introduces a "partitioning noise" which was not included in Gordon's work. Therefore, the actual capacity of a photon channel will be less than that predicted by Gordon. The work in Chapter III has shown that the "partitioning noise" does not affect the wave modulated system, but merely preserves the Poisson character of the photon distribution.

Channel Capacity

From the arguments presented in this paper, it is concluded that Gordon's capacity derivations (Reference 5) apply only to a photon modulated system in which no power loss is incurred between transmitter and receiver. Later works by Gordon and others (References 8 and 11) have noted that the results apply only to a lossless channel. However, no one seems to have recognized that basic differences exist between wave and photon modulated systems.

Hagfors (Reference 7) and Bowen (Reference 13) have attempted to find the capacity for a general photon channel which includes partition noise (but neglects external noise sources). Hagfors encountered mathematical difficulties which prevented a general solution. He did analyze a ternary system and demonstrated how the partition noise has a strong effect on the optimum statistics for the transmitted signal. Utilizing the partition function formalism of statistical mechanics, Bowen was able to study the asymptotic condition of large attenuation. He found that for a large rate of received photons the channel capacity approaches one-half that derived by Gordon for the condition of no thermal noise.

CHAPTER VI

CONCLUDING REMARKS

The problem of including quantum effects within existing methods of communication system analysis has been investigated. In contrast to previous works, a unique model was postulated for including the quantum properties of radiation at both the transmitter and receiver. The model was shown to be consistent with certain requirements of quantum theory and to yield a description of photon detection which agrees with previously derived and experimentally proven equations.

From the differing characteristics of fields prepared by the transmitter it was concluded that modulation of the radiated field must take place by one of two distinctly different processes. Wave modulation was defined as a process which conveys information in the amplitude and phase of the electromagnetic field. Photon modulation was defined as a process which conveys information in the exact number of photons per sample of the transmitted field (i.e., the "photon channel" as defined by Stern, Gordon, and others). A wave modulated signal was found to describe present day modulation schemes and, moreover, to yield random photon fluctuations.

It was concluded a wave modulated signal is not capable of describing a photon modulated signal and therefore cannot describe a "photon channel." In contrast, a photon modulated signal was found to possess a completely random phase fluctuation thus showing the inability of the "photon channel" to describe presently used wave modulation schemes. No physical process has been demonstrated which can produce photon modulated electromagnetic field.

For a description of wave modulated signals with the postulated model, the Glauber P function was shown to be identical (with a change of variable) to the classical joint probability density of the quadrature components of a narrowband sinusoid used in conventional analyses. Using this fact, equations were derived which "transform" classical probability distributions into distributions which include quantum effects.

The analysis of "partitioning noise" developed by Hagfors using the "photon channel" model was applied to the postulated model. The partitioning effect was shown to yield a noise only in photon modulation systems. For a wave modulated signal, the effect merely preserved the Poisson character of the photon distribution during free space attenuation and thus demonstrated the consistency of wave and particle pictures of radiation phenomena.

For a wave modulated system with real field detection it was shown that photon noise can be included in conventional analysis by adding to the normal input noise, a noise density of $hf_c/2$ Watts/Hertz where h is Planck's constant and f_c is the system carrier frequency. Use of photon heterodyne detection was found to require the addition of an equivalent input "white" noise of density hf_c Watts/Hertz. The photon noise was shown to be exactly additive Gaussian for real field detection and approximately additive Gaussian for photon heterodyne detection. Detection by counting the received photons was shown to yield neither Gaussian nor Poissonian statistics. However, with an "ideal measurement process," when the effective received noise density η_c is much less than hf_c, the variance of the counts were found to approach that of a Poisson distribution. Conversely, when η_c is much larger than hf_c, the variance approached that derivable from a classical analysis.

A channel capacity equation for a wave modulated system was found and shown to give capacities less than that derived by Gordon.

REFERENCES

1. Gabor, D., "Communication Theory and Physics," Phil. Mag., Vol. 41, 1950, pp. 1161-1187.

2. Stern, T. E., "Some Quantum Effects in Information Channels," IRE Transactions on Information Theory, September 1960, pp. 435-440.

3. Lasher, G. J., "A Quantum-Statistical Treatment of the Channel Capacity Problem of Information Theory," Advances in Quantum Electronics, J. R. Singer, Ed., Columbia University Press, 1961.

4. Gordon, J. P., "Information Capacity of a Communication Channel in the Presence of Quantum Effects," Advances in Quantum Electronics, J. R. Singer, Ed., Columbia University Press, 1961.

5. Gordon, J. P., "Quantum Effects in Communications Systems," Proc. IRE, September 1962, pp. 1898-1908.

6. O'Neal, J. B., Jr., "A Theory of Quantum Communications," SR No. 2, Engineering and Industrial Experiment Station, University of Florida, December 1963.

7. Hagfors, T., "Information Capacity and Quantum Effects in Propagation Circuits," TR No. 344, Massachusetts Institute of Technology Lincoln Laboratory, January 1964.

8. Gordon, J. P., "Noise at Optical Frequencies; Information Theory," Quantum Electronic and Coherent Light, P. A. Miles, Ed., Academic Press, 1964.

9. Deryugin, I. A., and Kurashov, V. N., "Quantum Effects in Information Transmission Systems," NASA TTF-476, April 1967 (translation from Kvantovaya Elektronika, Proceedings of the Republic Seminar on Quantum Electronics, M. P. Lisitsa, Ed., Kiev, pp. 292-319, 1966).

10. Lebedev, D. S., and Levitin, L. B., "Information Transmission by Electromagnetic Field," Information and Control, Vol. 9, 1966, pp. 1-22.

11. Bowen, J. I., "On the Capacity of a Noiseless Photon Channel," IEEE Transactions on Information Theory, Vol. IT-13, April 1967, pp. 230-236.

12. Takahasi, H., "Information Theory of Quantum-Mechanical Channels," in Advances in Communication Systems, Vol. 1, A. V. Balakrishnan, Ed., Academic Press, 1965, pp. 227-310.

13. Bowen, J. I., "The Effect of Attenuation on the Capacity of a Photon Channel," IEEE Transactions of Information Theory, Vol. IT-14, January 1968, pp. 44-50.

14. She, C. Y., "Quantum Descriptions on Communication Theory," TR No. 0415-1, Stanford Electronics Laboratories, May 1964, (She has developed the same results from a different system model as reported in: "Quantum Electrodynamics of a Communication Channel," IEEE Transactions on Information Theory, Vol. IT-14, January 1968, pp. 32-37).

15. Lachs, G., "On the Quantum Mechanical Aspects of the Waveshape of Electromagnetic Radiation," TR No. 7, Syracuse University Research Institute, September 1964.

16. Jelsma, L. F., "Quantum Field Theoretical Models of Communication Systems," TR No. Q 52, University of Delaware, June 1965 (AD 628-711).

17. Helstrom, C. W., "Quantum Limitations on the Detection of Coherent and Incoherent Signals," IEEE Transactions on Information Theory, Vol. IT-11, October 1965, pp. 482-490.

18. Karp, S., "A Statistical Model for Radiation with Applications to Optical Communications," University of Southern California, January 1967.

19. Glauber, R. J., "Photon Correlations," Phys. Rev. Let., Vol. 10, 1 Feb. 1963, pp. 84-86.

20. Glauber, R. J., "The Quantum Theory of Optical Coherence," Phy. Rev., Vol. 130, 14 June 1963, pp. 2529-2539.

21. Glauber, R.J., "Coherent and Incoherent States of the Radiation Field," Phy. Rev., Vol. 131, 15 Sept. 1963, pp. 2766-2788.

22. Glauber, R.J., "Optical Coherence and Photon Statistics" in Quantum Optics and Electronics, DeWitt, C., Blandin, A., and C. Cohen-Tannoudji, Editors, Gordon and Breach, New York, 1965.

23. Cahill, K. L., "Coherent-State Representations for the Photon Density Operator," Phy. Rev., Vol. 138, 21 June 1963, pp. B1566-B1575.

24. Klauder, J. R., McKenna, J., and Currie, D. G., "On 'Diagonal' Coherent-State Representations for Quantum-Mechanical Density Matrices," J. Math. Phys., Vol. 6, May 1964, pp. 734-739.

25. Mandel, L., and Wolf, E., "Photon Statistics and Classical Fields," Phy. Rev., Vol. 149, 30 Sept. 1966, pp. 1033-1037.

26. Minkowski, J. M., Gundersdorf, C. J., and Gabriele, T. L., "Study of Problem Areas in Optical Communications," Technical Report AFAL-TR-66-214, The John Hopkins University, August 1966, (AD 805607).

27. Lachs, G., "Quantum Statistics of Multiple-Mode, Superposed Coherent and Chaotic Radiation," J. App. Phy., Vol. 38, August 1967, pp. 3439-3449.

28. Ruggieri, N. F., "A Quantum Statistical Analysis of a Frequency Modulated Laser Communication System," The Pennyslvania State University, March 1967.

29. Costas, John P., "Periodic Sampling of Stationary Time Series," TR 156, Research Laboratory of Electronics, Massachusetts Institute of Technology, May 16, 1950.

30. Mandel, L., "Fluctuation of Photon Beams and their Correlations," Proc. Phy. Soc. (London), Volume 72, pp. 1037-1048.

31. Mandel, L., Sudarshan, E. C. G., and Wolf, E., "Theory of Photoelectric Detection of Light Fluctuations," Proc. Phys. Soc., Vol. 84, 1964, pp. 435-444.

32. Glauber, R. J., "Optical Coherence and Photon Statistics" in Quantum Optics and Electronics, DeWitt, C., Blandin, A., and C. Cohen-Tannoudji, Editors, Gordon and Breach, New York, 1965.

33. Paoli, T. L., "Role of the Photon Field Commutator in the Quantum Theory of Intensity Correlations," Phys. Rev., Vol. 163, 25 November 1967, pp. 1348-1353.

34. Johnson, F. A., Jones, R., McLean, T. P., and Pike, E.R., "The Measurement and Analysis of Photon Counting Distributions," Optica Acta, Vol. 14, 1967, pp. 35-40.

35. Fray, S., Johnson, F. A., Jones, R., McLean, T. P., and Pike, E. R., "Photon-Counting Distributions of Modulated Laser Beams," Phys. Rev., Vol. 153, 10 January 1967, pp. 357-359.

36. Pfleegor, R. L., and Mandel, L., "Interference of Independent Photon Beams," Phys. Rev., Vol. 159, 25 July 1967, pp. 1084-1088.

37. Heisenberg, W., The Physical Principles of the Quantum Mechanics, University of Chicago Press, 1930.

38. Rosenfeld, L., "On Quantum Electrodynamics," in Neils Bohr and the Development of Physics, W. Pauli, Ed., McGraw-Hill, 1955.

39. Bohr, N., and Rosenfeld, L., "Field and Charge Measurements in Quantum Electrodynamics," Phys. Rev., Vol. 78, June 15, 1950.

40. Heitler, W., The Quantum Theory of Radiation, Oxford, 1954 (Third Edition).

41. Margenau, H., "Philsophical Problems Concerning the Meaning of the Measurement in Physics," Philsophy of Science, Vol. 25, January 1958.

42. Prugovecki, E., "On a Theory of Measurement of Incompatible Observables in Quantum Mechanics," Canadian Journal of Physics, Vol. 45, June 1967.

43. She, C. Y., and Heffner, H., "Simultaneous Measurement of Noncommuting Observables," Phy. Rev., Vol. 152, 23 December 1966, pp. 1103-1110.

44. Dirac, P. A. M., The Principles of Quantum Mechanics, 1947 (Third Edition).

45. Goldman, S., Information Theory, Prentice-Hall, Inc., 1953.

46. Richter, Von G., Brunner, W., and Paul, H., "Elektrische Feldstarke and Interferenz von Laserstrahlen," Annalen der Physik, Vol. 14, 1964, pp. 239-261.

47. Sudarshan, E. C. G., "Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams," Phy. Rev. Let., Vol. 10, 1 April 1963, pp. 277-279.

48. Lachs, G., "Theoretical Aspects of Mixtures of Thermal and Coherent Radiation," Phy. Rev., Vol. 138, 24 May 1965, pp. B1012-B1016.

49. Dasvarma, R. L., "On a Derivation of the Probability Densities of Position and Momentum for the Harmonic Oscillator in the Ground State," Nat. Acad. of Sci. India, Sec. A., Vol. 36, Part 1, 1966, pp. 166-168.

50. Steinberg, H. A., and LaTourette, J. T., "A Note on Photon-Electron Converters with Reference to a Paper by A. Arcese," Applied Optics, Vol. 3, July 1964, p. 902.

51. Rice, S. O., "Mathematical Analysis of Random Noise Bell System Technical Journal, Vol. 23, 1944, pp. 282-332, Vol. 24, 1945, pp. 46-156.

52. Oliver, B. M., "Signal-to-Noise Ratios in Photoelectric Mixing," (Correspondence) Proc. IRE, December 1961.

53. Haus, H. A., and Townes, C. H., "Comments on 'Noise in Photoelectric Mixing,'" (Correspondence) Proc. IRE, June 1962.

54. Shannon, C. E., "Communication in the Presence of Noise," Proc. IRE, January 1949, pp. 10-21.

APPENDIX I

EXTENSION OF SHE AND HEFFNER'S WORK TO INCLUDE THE GLAUBER P FUNCTION

Using the Dirac notation (Reference 44), the following symbols are defined for this appendix only:

$\vert\rangle$	ket vector
$\langle\vert$	bra vector
a	photon annihilation operator
a^+	photon creation operator
ρ	density operator

In Reference 43, She and Heffner prove that if

$$a|\alpha\rangle = \alpha|\alpha\rangle = \left(\frac{\omega_c q + jp}{\sqrt{2\hbar\omega_c}}\right)|\alpha\rangle$$

$$\langle\alpha|a^+ = \langle\alpha|\alpha^* = \langle\alpha|\left(\frac{\omega_c q - jp}{\sqrt{2\hbar\omega_c}}\right)$$

$$a|\alpha'\rangle = \alpha'|\alpha'\rangle = \left(\frac{\omega_c q' + jp'}{\sqrt{2\hbar\omega_c}}\right)|\alpha'\rangle$$

$$\langle\alpha'|a^+ = \langle\alpha'|\alpha^{*\prime} = \langle\alpha'|\left(\frac{\omega_c q' - jp'}{\sqrt{2\hbar\omega_c}}\right)$$

$$\rho = |\alpha'\rangle\langle\alpha'|$$

then

$$\frac{1}{h}\langle\alpha|\rho|\alpha\rangle = \frac{1}{2\pi\hbar}\exp\left[-\frac{(q - q')^2}{2\hbar/\omega_c} - \frac{(p - p')^2}{2\hbar\omega_c}\right]$$

which they interpret as being a joint probability density describing a system which has been prepared into the state (q',p') and subsequently measured as accurately as permitted by the uncertainty principle.

The above can be extended to include Glauber's P function. From Reference 21, Equation (7.6)

$$\rho = \iint P(\alpha')|\alpha'\rangle\langle\alpha'|d^2\alpha'$$

Therefore,

$$\frac{1}{h}\langle\alpha|\rho|\alpha\rangle = \iint P(\alpha')\frac{|\langle\alpha|\alpha'\rangle|}{2\pi\hbar}d^2\alpha' \qquad \text{(I-1)}$$

where

$$|\langle\alpha|\alpha'\rangle|^2 = \langle\alpha|\alpha'\rangle\langle\alpha'|\alpha\rangle$$
$$= \exp[|\alpha - \alpha'|^2]$$

from Reference 21, Equation (3.33). But from Equation (3-29)

$$|\langle\alpha|\alpha'\rangle|^2 = \exp\left[-\left(\frac{\omega q}{\sqrt{2\hbar\omega_c}} - \alpha_R'\right)^2 - \left(\frac{p}{\sqrt{2\hbar\omega_c}} - \alpha_I'\right)^2\right]$$

$$= \exp\left[-\frac{\left(q - \sqrt{\frac{2\hbar}{\omega_c}}\alpha_R'\right)^2}{2\hbar/\omega} - \frac{\left(p - \sqrt{2\hbar\omega_c}\alpha_I'\right)^2}{2\hbar\omega_c}\right] \tag{I-2}$$

Substituting (I-2) into (I-1) yields (3-30).

APPENDIX II

DERIVATION OF MANDEL'S PHOTON COUNTING DISTRIBUTION

The purpose of this appendix is to derive via the model postulated in Chapter III the photon counting distribution first found by Mandel (Reference 30) and subsequently verified experimentally (References 34-36).

Let the receiver output filter W_0 be an ideal integrator over a time period to where t_0/t_m is an integer k. During t_0, there are k samples taken by the receiver. The total number of photons, n_T, counted during t_0 determines the integrator output. The resulting probability distribution of n_T is found by considering all the possible combinations of counts in each of the k samples that yields n_T.

From the postulated model each sample for photon detection is represented by the energy level of a harmonic oscillator. Glauber has shown that the joint probability distribution of energy levels between many separate oscillators whose states may be statistically dependent is of the form (Reference 19)

$$P_n(n_1,\ldots,n_k) = \iint\cdots\iint P_R[(x_{R1},y_{R1}),\ldots,(x_{Rk},y_{Rk})]$$

$$\prod_{i=1}^{k}\left[\frac{\left(\dfrac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right)^{n_i}}{n_i!}\exp\left[-\left(\frac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right)dx_{Ri}y_{Ri}\right]\right]$$

where $P_R[(),\ldots,()]$ is the joint probability distribution between the k samples as determined from a classical analysis. For convenience, write the above equation in the form

$$P_n(n_1,\ldots,n_k) = \frac{1}{n_T!}\iint\cdots\iint\left\{n_T!\prod_{i=1}^{k}\left[\frac{\dfrac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}}{n!}\right]\right\}$$

$$\exp\left[-\sum_{i=1}^{k}\frac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right]P_R[(x_{R1},y_{R1}),\ldots,(x_{Rk},y_{R1})]$$

$$\prod_{i=1}^{k} dx_{Ri}dy_{Ri}$$

where

$$n_T = \sum_{i=1}^{k} n_i$$

Note that the term in braces is precisely one term of an expansion of the quantity

$$\left[\sum_{i=1}^{k}\left(\frac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right)\right]^{n_T}$$

Since

$$P_{n_T}(n_T) = \sum_m [P_n(n_{1m}, n_{2m}, \ldots, n_{km})]$$

where the summation is over all the possible combinations of $(n_{1m}, n_{2m}, \ldots, n_{km})$ which yield

$$n_T = \sum_{i=1} n_{im}$$

it follows that

$$P_{n_T}(n_T) = \iint \cdots \iint \frac{\left[\sum_{i=1}^{k} \frac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right]^{n_T}}{n_T!} \exp\left[-\sum_{i=1}^{k} \frac{x_{Ri}^2 + y_{Ri}^2}{2\hbar\omega_c W_m}\right]$$

$$P_R[(x_{R1}, y_{R1}), \ldots, (x_{Rk}, y_{Rk})] \prod_{i=1}^{k} dx_{Ri}\, dy_{Ri}$$

By the ergodic assumption this is equal to

$$P_{n_T}(n_T) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau}\frac{\left(\sum_{i=1}^{k}\frac{A_R^2(t_i)}{2\hbar\omega_c}t_m\right)^{n_T}}{n_T!}\exp\left(-\sum_{i=1}^{k}\frac{A_R^2(t_i)}{2\hbar\omega_c}t_m\right)dt$$

where

$$A_R^2(t_i) = x_R^2(t_i) + y_R^2(t_i)$$

$$t_i = t + it_m$$

$$t_m = \frac{1}{W_m}$$

Utilizing the assumption made by Mandel, i.e., the response of the photoelectric process is sufficiently short that

$$W_m >> W_F$$

$$W_m >> W_o$$

and by defining

$$U(t,t_o) = \lim_{t_m \to o}\sum_{i=1}^{k}\frac{A_R^2(t_i)}{2\hbar\omega_c}t_m$$

$$= \int_t^{t+t_o}\frac{A_R^2(t^1)}{2\hbar\omega_c}dt^1$$

yields

$$P_{nT}(n_T) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{[U(t,t_0)]^{n_T}}{n_T!}\exp[-U(t,t_0)]dt$$

which is the distribution derived by Mandel.

APPENDIX III

DERIVATION OF PHOTON DETECTOR EQUATIONS

Repeating equation (4-22)

$$P_n(n|x_s,y_s) = \iint \frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2n_c W_F}\right]$$

$$\frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!} \exp\left[-\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)\right] dx_R dy_R$$

The averages are calculated as follows:

$$<n,(x_s,y_s)> = \sum_n n P_n[n|(x_s,y_s)]$$

$$= \iint \frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2n_c W_F}\right]$$

$$\left\{\sum_n n \frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!} \exp\left[-\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)\right]\right\} dx_R dy_R$$

$$= \frac{1}{2\hbar\omega_c W_m} \iint (x_R^2 + y_R^2)$$

$$\frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2n_c W_F}\right] dx_R dy_R$$

$$= \frac{1}{2\pi\omega_c W_m} [(x_s^2 + n_c W_F) + (y_s^2 + n_c W_F)]$$

$$<n, A_s> = \frac{A_s^2}{2\pi\omega_c W_m} + \frac{n_c W_F}{\pi\omega_c W_m}$$

where $A_s^2 = x_s^2 + y_s^2$

$$<n^2, (x_s, y_s)> = \sum_n n^2 P_n[n|(x_s, y_s)]$$

$$= \iint \frac{\exp}{2\pi n_c W_F} \left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2 n_c W_F}\right]$$

$$\left\{\sum_n n^2 \frac{\left(\frac{x_R^2 + y_R^2}{2\pi\omega_c W_m}\right)^n}{n!} \exp\left[-\left(\frac{x_R^2 + y_R^2}{2\pi\omega_c W_m}\right)\right]\right\} dx_R dy_R$$

$$= \iint \left[\left(\frac{x_R^2 + y_R^2}{2\pi\omega_c W_m}\right)^2 + \left(\frac{x_R^2 + y_R^2}{2\pi\omega_c W_m}\right)\right]$$

$$\frac{\exp}{2\pi n_c W_F}\left[\frac{-(x_R - x_s)^2 - (y_R - y_x)^2}{2 n_c W_F}\right] dx_R dy_R$$

$$= \frac{1}{(2\hbar\omega_c W_m)^2} \iint (x_R^4 + 2x_R^2 y_R^2 + y_R^4)$$

$$\frac{\exp}{2\pi\eta_c W_F} \left[\frac{-(x_R - x_s)^2 - (y_R - y_s)^2}{2\eta_c W_F}\right] dx_R dy_R$$

$$+ \langle n, A_s \rangle$$

$$= \frac{1}{(2\hbar\omega_c W_m)^2}[3(\eta_c W_F)^2 + 6x_s^2 \eta_c W_F + x_s^4$$

$$+ 2(x_s^2 + \eta_c W_F)(y_s^2 + \eta_c W_F) + 3(\eta_c W_F)^2$$

$$+ 6y_s^2 \eta_c W_F + y_s^4] + \langle n, A_s \rangle$$

$$= \frac{1}{(2\hbar\omega_c W_m)^2}[A_s^4 + 8\eta_c W_F A_s^2 + 8(\eta_c W_F)^2] + \langle n, A_s \rangle$$

Then

$$\sigma n^2|A_s = \frac{1}{(2\hbar\omega_c W_m)^2}\left[A_s^4 + 8\eta_c W_F A_s^2 + 8(\eta_c W_F)^2\right.$$

$$+ 2\hbar\omega_c (A_s^2 + 2\eta_c W_F) W_m - \left(A_s^4 + 4A_s^2 \eta_c W_F\right.$$

$$\left.\left. + 4(\eta_c W_F)^2\right)\right]$$

$$\sigma^2_{n|A_s} = 2\left(\frac{n_c W_F}{\hbar\omega_c W_m}\right)\left(\frac{A_s^2}{2\hbar\omega_c W_m}\right) + \left(\frac{n_c W_F}{\hbar\omega_c W_m}\right)^2 + \left[\frac{A_s^2}{2\hbar\omega_c W_m} + \frac{n_c W_F}{\hbar\omega_c W_m}\right]$$

APPENDIX IV

DERIVATION OF PHOTON HETERODYNE EQUATIONS

The purpose of this appendix is to derive the output signal-to-noise ratio for a photon detector when used as a mixer. In addition to the received signal and noise, a local oscillator field must be incident on the detector for mixing to occur. The total received voltage is

$$E_R(t) = E_S(t) + E_N(t) + E_{LO}(t)$$

where $E_{LO}(t)$ is due to the local oscillator. Taking $E_{LO}(t)$ to be sinusoidal with constant amplitude and phase

$$E_{LO}(t) = x_{LO}\cos(\omega_c + \omega_I)t - y_{LO}\sin(\omega_c + \omega_I)t$$

where $f_I = \left(\frac{\omega_I}{2\pi}\right)$ = frequency of the intermediate amplifier. Rewriting in the form (2-11)

$$E_{LO}(t) = A_{LO}[\cos(\omega_I t + \theta_{LO})\cos \omega_c t - \sin(\omega_I t + \theta_{LO})\sin \omega_c t]$$

where

$$A_{LO} = \sqrt{x_{LO}^2 + y_{LO}^2}$$

$$\theta_{LO} = \tan^{-1}\frac{y_{LO}}{x_{LO}}$$

Therefore

$$E_R(t) = [x_s(t) + x_N(t) + A_{LO}\cos(\omega_I t + \theta_{LO})]\cos\omega_c t$$

$$-[y_s(t) + y_N(t) + A_{LO}\sin(\omega_I t + \theta_{LO})]\sin\omega_c t$$

The photon detector will respond to the function

$$\begin{aligned} A_R^2(t) &= x_R^2(t) + y_R^2(t) \\ &= A_s^2(t) + A_N^2(t) + A_{LO}^2 \\ &+ [x_s(t)x_N(t) + y_s(t)y_N(t)] \\ &+2A_{LO}\Big\{[x_s(t) + x_N(t)]\cos(\omega_I t + \theta_{LO}) \\ &+[y_s(t) + y_N(t)]\sin(\omega_I t + \theta_{LO})\Big\} \end{aligned} \qquad \text{(IV-1)}$$

It is assumed that

$$\left(f_I - \frac{W_F}{2}\right) > 2W_F$$

so that only the terms in braces in (IV-1) yield an output in the intermediate frequency bandpass. In the postulated model, the bandwidth of the measuring process W_m must be wide enough to pass both the signal and noise inputs and the local oscillator frequency. That is

$$\frac{W_m}{2} \geq \left(f_I + \frac{W_F}{2}\right)$$

or

$$t_m \leq \frac{1}{2(f_I + \frac{W_F}{2})}$$

since

$$t_m = \frac{1}{W_m}$$

The samples yield an output of bandwidth extending from zero to $W_m/2$ Hertz.

Define the output voltage for the i th field sample to be

$$V(it_m) = \frac{n(it_m)}{t_m}$$

where $n(it_m)$ is the number of photons counted during the i th sample. By the ergodic assumption, it follows that

$$\overline{V(it_m)} = \frac{\langle n \rangle}{t_m}$$

From equation (4-15)

$$\overline{V(it_m)} = \frac{1}{t_m} \sum_{n=0}^{\infty} nP_n(n)$$

$$= \frac{1}{t_m}\iint P_R(x_R,y_R) \sum_n n \frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!} \exp\left[-\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)\right] dx_R dy_R$$

$$= \iint \left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c}\right) P_R(x_R,y_R) dx_R dy_R$$

By (2-21)

$$\overline{V(it_m)} = \frac{1}{2\hbar\omega_c}\ \overline{x_R^2(t) + y_R^2(t)} = \frac{\overline{A_R^2(t)}}{2\hbar\omega_c} .$$

From (IV-1) and assuming the signal and noise are uncorrelated,

$$\overline{V(it_m)} = \frac{P_s}{\hbar\omega_c} + \frac{n_c W_F}{\hbar\omega_c} + \frac{P_{LO}}{\hbar\omega_c}$$

where

$$P_s = \frac{\overline{A_s^2(t)}}{2} \qquad \text{received signal power}$$

$$n_c W_F = \frac{\overline{A_N^2(t)}}{2} \qquad \text{received noise power}$$

$$P_{LO} = \frac{A_{LO}^2}{2} \qquad \text{incident local oscillator power}$$

From the form of (IV-1) it follows that the desired output signal power (excluding noise) is

$$S_o = \frac{1}{2}\overline{\left(\frac{A_{LO}A_s(t)}{\hbar\omega_c}\right)^2} = 2\frac{P_{LO}P_s}{(\hbar\omega_c)^2}$$

Consider now the total output noise power. From (IV-1) it is apparent that the bandlimited noise around f_I will contain the power

$$N_{oc} = \frac{1}{2}\overline{\left(\frac{A_{LO}A_N(t)}{\hbar\omega_c}\right)^2} = 2\frac{P_{LO}n_c W_F}{(\hbar\omega_c)^2}$$

in the same manner as the signal. The photon noise can be found from the second moment of V(t)

$$\overline{V^2(it_m)} = \frac{\langle n^2\rangle}{t_m^2} = \frac{1}{t_m^2}\iint P_R(x_R,y_R)\sum n^2 \frac{\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)^n}{n!}$$

$$\exp\left[-\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c W_m}\right)\right] dx_R dy_R$$

$$= \iint \left[\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c}\right)^2 + \frac{1}{t_m}\left(\frac{x_R^2 + y_R^2}{2\hbar\omega_c}\right)\right] P_R(x_R,y_R)\, dx_R dy_R$$

But from equation (2-21)

$$\overline{V^2(it_m)} = \overline{\left(\frac{x_R^2(t) + y_R^2(t)}{2\hbar\omega_c}\right)^2} + \frac{\overline{V(it_m)}}{t_m}$$

The last term is the photon or shot noise which has a "white" spectrum extending from zero to $\frac{W_m}{2}$ Hertz. Therefore the photon noise density is

$$n_s = \frac{\dfrac{\overline{V(t)}}{\tau_m}}{\dfrac{W_m}{2}} = 2\overline{V(t)} = \frac{2}{\hbar\omega_c}\left(P_s + n_c W_F + P_{LO}\right)$$

The output shot noise in the bandwidth W_F about f_I is $n_s W_F$. The total output noise around f_I becomes

$$N_o = \frac{2}{(\hbar\omega_c)^2}\left[P_{LO}\eta_c W_F + \hbar\omega_c(P_s + n_c W_F + P_{LO})W_F\right]$$

Therefore the output signal-to-noise ratio is

$$\frac{S_o}{N_o} = \frac{\dfrac{2P_{LO}P_s}{(\hbar\omega_c)^2}}{\dfrac{2}{(\hbar\omega_c)^2}\left[P_{LO}\ c W_F + \hbar\omega_c(P_s + n_c W_F + P_{LO})W_F\right]}$$

$$= \frac{P_s}{n_c W_F + \hbar\omega_c\left(1 + \dfrac{P_s}{P_{LO}} + \dfrac{n_c W_F}{P_{LO}}\right)}$$

But in practice

$$P_{LO} \gg P_s$$

$$P_{LO} \gg n_c W_F$$

yielding

$$\frac{S_o}{N_o} = \frac{P_s}{(n_c + \hbar\omega_c)W_F}$$