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A Quasi-Infinite Horizon Nonlinear Model Predictive Control Scheme with Guaranteed Stability*

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Key Words—Nonlinear model predictive control; stability; terminal inequality constraint; terminal cost; quasi-infinite horizon.

Abstract—We present in this paper a novel nonlinear model predictive control scheme that guarantees asymptotic closed-loop stability. The scheme can be applied to both stable and unstable systems with input constraints. The objective functional to be minimized consists of an integral square error (ISE) part over a finite time horizon plus a quadratic terminal cost. The terminal state penalty matrix of the terminal cost term has to be chosen as the solution of an appropriate Lyapunov equation. Furthermore, the setup includes a terminal inequality constraint that forces the states at the end of the finite prediction horizon to lie within a prescribed terminal region. If the Jacobian linearization of the nonlinear system to be controlled is stabilizable, we prove that feasibility of the open-loop optimal control problem at time $t = 0$ implies asymptotic stability of the closed-loop system. The size of the region of attraction is only restricted by the requirement for feasibility of the optimization problem due to the input and terminal inequality constraints and is thus maximal in some sense. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The history of model predictive control (MPC), also referred to as moving horizon control or receding horizon control, began with an attempt to use the powerful computer technology to improve the control of processes that are constrained, multivariable and uncertain (Cutler and Ramaker, 1980; Richalet *et al.*, 1978). In the last decade, many formulations have been developed for linear or nonlinear systems (Garcia *et al.*, 1989; Rawlings *et al.*, 1994; Mayne, 1995; van den Boom, 1996; Lee, 1997), that found successful applications especially in the process industries (Richalet, 1993; Qin and Badgwell, 1996).

In general, the MPC problem is formulated as solving on-line a finite horizon open-loop optimal control problem subject to (linear or nonlinear) system dynamics and constraints involving states and inputs. However, as shown in Bitmead *et al.* (1990), this general form of MPC does not guarantee closed-loop stability, because a finite horizon criterion is not designed to deliver an asymptotic property such as stability. Closed-loop stability can only be achieved by a suitable tuning of design parameters such as prediction horizon, control horizon and weighting matrices. Therefore, Bitmead *et al.* (1990) suggested an infinite horizon method (closely related to LQ control), which, however, results in an optimization problem that can generally be solved only for unconstrained linear systems.

For *linear* systems with constraints, the work of Rawlings and Muske (1993) represents a significant leap forward in the MPC theory. They propose a receding horizon control scheme with infinite prediction horizon and finite control horizon. For both stable and unstable systems, nominal closed-loop stability is guaranteed by the feasibility of the constraints, independent of the choice of performance parameters. For other MPC approaches and stability results see, for example, Genceli and Nikolaou (1993) and Polak and Yang (1993).

Mayne and Michalska have contributed some very important issues on the stability of *nonlinear* receding horizon control. They have shown in Mayne and Michalska (1990) that under some rather strong assumptions, receding horizon control is able to stabilize a class of nonlinear systems with constraints (see also Chen and Shaw, 1982; Keerthi and Gilbert, 1988). The finite horizon constrained optimal control problem is posed as minimizing a standard quadratic objective functional subject to an additional *terminal state equality constraint* requiring the states to be zero at the end of the finite prediction horizon. The strong assumptions are needed to ensure that the optimal value function is continuously differentiable. Those assumptions are relaxed in Michalska and Mayne

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(1991) to ensure merely local Lipschitz continuity of the optimal value function. However, from a computational point of view, an exact satisfaction of the terminal equality constraint requires an infinite number of iterations in the nonlinear case. An approximate satisfaction means that the achieved stability is lost inside the region of approximation. In order to avoid this, they extend their work in Michalska and Mayne (1993) with a *terminal inequality constraint* such that the states are on the boundary of a *terminal region* at the end of a *variable* prediction horizon. They suggest a *dual-mode* receding horizon control scheme with a local linear state feedback controller inside the terminal region and a receding horizon controller outside the terminal region. Closed-loop control with this scheme is implemented by switching between the two controllers, depending on the states being inside or outside the terminal region.

Yang and Polak (1993) present a moving horizon control scheme that deviates from conventional MPC schemes in that the control horizon is also a minimizer and the whole input sequence is implemented. In this scheme *inequality contraction constraints* are added so as to ensure the state vector to contract by a prespecified factor before a new optimization begins. Like in the linear case of this scheme (Polak and Yang, 1993), guaranteed stability is achieved when the existence of a solution to the optimization problem at each time is assumed. However, this is a very strong assumption and cannot be guaranteed in general (Mayne, 1995). In analogy to the linear case (Genceli and Nikolaou, 1993), Genceli and Nikolaou (1995) derive an *end condition* for nonlinear MPC with second-order Volterra models, when the system being controlled is square and stable. The end condition requires the input values at the end of the finite horizon to be equal to the steady-state values corresponding to the setpoint and the steady-state estimates of disturbances. With the end condition, closed-loop stability is achieved under some restrictions not only on prediction and control horizons but also on control move suppressions in the objective functional. This makes an independent specification of control performance difficult. Another method to achieve stability for nonlinear MPC is suggested by Nevistic and Morari (1995), combining state feedback linearization and stability issues of linear MPC with constraints, for feedback linearizable systems. However, because the exact state feedback linearization law is state-dependent and generally nonlinear, the originally linear input constraints are transformed into state-dependent and in general nonlinear constraints. In addition, an originally quadratic cost functional will become an arbitrary nonlinear cost functional in the transformed coordinates.

For *discrete* nonlinear systems subject to constraints, Keerthi and Gilbert (1988) discuss the moving horizon control problem as an approximation of an infinite horizon optimal feedback control problem. They provide sufficient conditions for the existence of a solution to the general nonlinear program and for closed-loop stability, based on a controllability assumption that is however not easy to verify in the nonlinear case. With terminal equality constraints, Meadows *et al.* (1995) propose a comparatively easily implementable formulation and discuss the existence and stability conditions.

In this paper, we introduce a quasi-infinite horizon nonlinear MPC scheme that optimizes on-line an objective functional consisting of a finite horizon cost and a *terminal cost* subject to system dynamics, input constraints and an additional terminal state inequality constraint. The feasibility of the terminal inequality constraint implies that the states at the end of the finite horizon are in a prescribed terminal region. The terminal states are penalized in such a way that the terminal cost bounds the infinite horizon cost of the nonlinear system controlled by a "fictitious" (i.e. not implemented) local linear state feedback. Thus, the proposed nonlinear model predictive controller has a *quasi-infinite prediction horizon*, but the input profile to be determined on-line is only of finite nature. If the Jacobian linearization of the nonlinear system to be controlled is stabilizable, the unique positive-definite, symmetric solution of an appropriate Lyapunov equation can serve as terminal penalty matrix of the terminal cost, and a neighborhood of the origin serving as terminal region can be determined off-line. Closed-loop asymptotic stability is then guaranteed by the feasibility of the open-loop optimal control problem at time $t = 0$. As is usual in MPC, the closed-loop control is calculated by solving the optimization problem on-line at each time, no matter whether the states are inside or outside the terminal region. Thus, no switching between controllers is needed. The local linear state feedback is only used to determine a terminal penalty matrix and a terminal region off-line. The contribution of this paper is thus a computationally attractive formulation of nonlinear MPC for which asymptotic stability can be guaranteed. Compared to other nonlinear MPC approaches that also guarantee stability (terminal equality constraint and dual-mode), this approach appears to be more general and computationally more attractive.

The paper is structured as follows: Section 2 describes the mathematical formulation of the proposed quasi-infinite horizon nonlinear MPC problem. Section 3 gives some preliminary results about a region of attraction and a performance bound of the nonlinear system controlled by a local linear state feedback. Based on these results, a procedure for systematically determining a terminal region and a terminal penalty matrix off-line is summarized. In Section 4, asymptotic stability of the proposed nonlinear MPC scheme is discussed and sufficient stability conditions are given. Simulation results for an unstable constrained system are given in Section 5.

2. A QUASI-INFINITE HORIZON NONLINEAR MODEL PREDICTIVE CONTROL SCHEME

The class of systems to be controlled is described by the following general nonlinear set of ordinary differential equations (ODEs):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

with state vector $\mathbf{x}(t) \in \mathbb{R}^n$, input vector $\mathbf{u}(t) \in \mathbb{R}^m$, and subject to input constraints

$$\mathbf{u}(t) \in U, \quad \forall t \geq 0. \tag{2}$$

It is assumed in this paper that

(A1) $\mathbf{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is twice continuously differentiable and $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Thus, $\mathbf{0} \in \mathbb{R}^n$ is an equilibrium of the system with $\mathbf{u} = \mathbf{0}$.

(A2) $U \subset \mathbb{R}^m$ is compact, convex and $\mathbf{0} \in \mathbb{R}^m$ is contained in the interior of U.

(A3) System (1) has a unique solution for any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ and any piecewise continuous and right-continuous $\mathbf{u}(\cdot): [0, \infty) \to U$.

Assumption $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ is not very restrictive, since if $\mathbf{f}(\mathbf{x}_s, \mathbf{u}_s) = \mathbf{0}$, one can always shift the origin of the system to $(\mathbf{x}_s, \mathbf{u}_s)$. We consider in this paper the state feedback case and thus assume that all states are measurable.

In the following, we describe the problem setup for the quasi-infinite horizon nonlinear MPC scheme introduced in this paper. For a description of the general idea and the principle of nonlinear MPC we refer for example to the excellent papers by Mayne and Michalska (Mayne and Michalska, 1990; Michalska and Mayne, 1993).

We shall first introduce some notations that will be used in this paper. For any vector $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|$ denotes the 2-norm and the weighted norm $\|\mathbf{x}\|_P$ is defined by $\|\mathbf{x}\|_P^2 := \mathbf{x}^T P \mathbf{x}$, where P is an arbitrary Hermitian, positive-definite matrix. For any Hermitian matrix A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the largest and the smallest real part of the eigenvalues of the matrix A, respectively, and $\|A\|$ stands for the induced 2-norm of A. In order to distinguish clearly between the system, that evolves in "real" time, and the system model, used to predict the future "within" the controller and evolving in some fictitious time, we denote the internal variables in the controller by a bar ($\bar{\mathbf{x}}, \bar{\mathbf{u}}$) to indicate that the predicted values need not and will not be the same as the actual values.

For the particular setup considered in this paper, the open-loop optimal control problem at time t with initial state $\mathbf{x}(t)$ is formulated as

$$\min_{\bar{u}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) \tag{3}$$

with

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) = \int_t^{t+T_p} \left(\|\bar{\mathbf{x}}(\tau)(\mathbf{x}(t), t)\|_Q^2 + \|\bar{\mathbf{u}}(\tau)\|_R^2 \right) d\tau + \|\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t)\|_P^2 \tag{4}$$

subject to

$$\dot{\bar{\mathbf{x}}} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}), \qquad \bar{\mathbf{x}}(t; \mathbf{x}(t), t) = \mathbf{x}(t) \tag{5a}$$

$$\bar{\mathbf{u}}(\tau) \in U, \quad \tau \in [t, t + T_p] \tag{5b}$$

$$\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) \in \Omega, \tag{5c}$$

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ denote positive-definite, symmetric weighting matrices; T_p is a finite prediction horizon; $\bar{\mathbf{x}}(\cdot; \mathbf{x}(t), t)$ is the trajectory of equation (5a) driven by $\bar{\mathbf{u}}(\cdot): [t, t + T_p] \to U$ (for simplicity of exposition, the control and prediction horizons are chosen to have identical values in this paper). Note the initial condition in equation (5a): The system model used to predict the future in the controller is initialized by the actual system states $\mathbf{x}(t)$ at "real" time t.

The objective functional (4) consists of a finite horizon standard cost to specify the desired control performance and a terminal cost to penalize the states at the end of the finite horizon. The terminal inequality constraint (5c) will force the states at the end of the finite prediction horizon to be in some neighborhood Ω of the origin, called here *terminal region*. This terminal region Ω will be chosen such that it is invariant for the nonlinear system controlled by a local linear state feedback. The quadratic terminal cost $\|\mathbf{x}(t + T_p; \mathbf{x}(t), t)\|_P^2$ bounds the *infinite* horizon cost of the nonlinear system starting from Ω and controlled by the local linear state feedback, i.e.

$$\|\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t\|_P^2 \geq \int_{t+T_p}^{\infty} (\|\bar{\mathbf{x}}(\tau; \mathbf{x}(t), t)\|_Q^2 + \|\bar{\mathbf{u}}(\tau)\|_R^2) d\tau$$

$$\bar{\mathbf{u}} = K\bar{\mathbf{x}}, \forall \bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) \in \Omega. \tag{6}$$

We will show that this allows us to guarantee closed-loop stability. The positive definite and symmetric *terminal penalty matrix* $P \in \mathbb{R}^{n \times n}$ together with the terminal region Ω is determined off-line such that the invariance property of Ω holds and the input constraints are satisfied in Ω. If we substitute inequality (6) into equation (4), we can conclude that the cost functional to be minimized bounds the infinite horizon cost defined by

$$J^{\infty}(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) := \int_t^{\infty} (\|\mathbf{x}(\tau; \mathbf{x}(t), t)\|_Q^2 + \|\bar{\mathbf{u}}(\tau)\|_R^2)\, d\tau,$$

where $\bar{\mathbf{u}}(\tau) = K\bar{\mathbf{x}}(\tau; \mathbf{x}(t), t)$ for $\tau \geq t + T_p$, i.e. $J^{\infty}(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) \leq J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot))$. In this sense, the prediction horizon of the proposed nonlinear predictive controller expands quasi to infinity, hence the name *quasi-infinite horizon nonlinear MPC scheme.*

An optimal solution to the optimization problem (3)–(5) (existence assumed), will be denoted by $\bar{\mathbf{u}}^*(\cdot; \mathbf{x}(t), t, t + T_p)$: $[t, t + T_p] \to U$ and the corresponding objective value is denoted by $J^*(\mathbf{x}(t), t, t + T_p) := J(\mathbf{x}(t), \bar{\mathbf{u}}^*)$.

The idea behind this setup is to guarantee stability of the closed-loop system with a quasi-infinite horizon objective functional, where the input profile needs to be determined on-line only for a finite horizon. In the sense of MPC, the "open-loop" control can be thought of as having two steps: over a finite horizon, an optimal input profile found by solving the open-loop optimal control problem drives the nonlinear system model into the terminal region; after that, a local linear state feedback control is assumed to steer it to the origin. In the moving horizon implementation, the local linear state feedback controller will *never* be directly applied to the system. Indeed, the input profile found is applied to the system only until the next measurement becomes available. We assume that this will be the case every δ time-units. So δ denotes the "sampling time" with $\delta < T_p$, and the closed-loop control represented by $\mathbf{u}^*(\cdot)$ is defined by

$$\mathbf{u}^*(\tau) := \bar{\mathbf{u}}^*(\tau; \mathbf{x}(t), t, t + T_p), \quad \tau \in [t, t + \delta]. \quad (7)$$

Updated with the new measurement, the above optimization problem will be solved again to find a new input profile. Thus, the closed-loop control is obtained by solving the open-loop optimal control problem on-line at each time, no matter whether the states are inside or outside the terminal region. The linear state feedback is only used to determine a terminal penalty matrix P and a terminal region Ω off-line, as described in the next section.

3. PRELIMINARY RESULTS

By a slight modification of the associated content in Michalska and Mayne (1993), we present preliminary results about a region of attraction and a performance bound of the nonlinear system controlled by a local linear state feedback. These results allow us to outline a procedure to systematically determine a terminal region and a terminal penalty matrix, and will be used to prove closed-loop asymptotic stability of the proposed control scheme. Since a terminal region and a terminal penalty matrix can be calculated off-line, variables without a bar will be used in this section.

We consider the Jacobian linearization of the system (1) at the origin

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad (8)$$

where $A := (\partial \mathbf{f}/\partial \mathbf{x})(\mathbf{0}, \mathbf{0})$ and $B := (\partial \mathbf{f}/\partial \mathbf{u})(\mathbf{0}, \mathbf{0})$. If equation (8) is stabilizable, then a linear state feedback $\mathbf{u} = K\mathbf{x}$ can be determined such that $A_K := A + BK$ is asymptotically stable. Consequently, we can state the following lemma.

Lemma 1. Suppose that the Jacobian linearization of the system (1) at the origin is stabilizable. Then,

(a) the following Lyapunov equation

$$(A_K + \kappa I)^{\mathrm{T}} P + P(A_K + \kappa I) = -Q^* \quad (9)$$

admits a unique positive-definite and symmetric solution P, where $Q^* = Q + K^{\mathrm{T}} R K \in \mathbb{R}^{n \times n}$ is positive definite and symmetric; $\kappa \in [0, \infty)$ satisfies

$$\kappa < -\lambda_{\max}(A_K). \quad (10)$$

(b) There exists a constant $\alpha \in (0, \infty)$ specifying a neighborhood Ω_α of the origin in the form of

$$\Omega_\alpha := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^{\mathrm{T}} P \mathbf{x} \leq \alpha\} \quad (11)$$

such that

(i) $K\mathbf{x} \in U$, for all $\mathbf{x} \in \Omega_\alpha$, i.e., the linear feedback controller respects the input constraints in Ω_α,

(ii) Ω_α is invariant for the nonlinear system (1) controlled by the local linear feedback $\mathbf{u} = K\mathbf{x}$,

(iii) for any $\mathbf{x}_1 \in \Omega_\alpha$, the infinite horizon cost

$$J^{\infty}(\mathbf{x}_1, \mathbf{u}) = \int_{t_1}^{\infty} (\|\mathbf{x}(t)\|_Q^2 + \|\mathbf{u}(t)\|_R^2)\, dt$$

subject to nonlinear system (1), starting from $\mathbf{x}(t_1) = \mathbf{x}_1$ and controlled by the local linear state feedback $\mathbf{u} = K\mathbf{x}$, is bounded from above as follows:

$$J^{\infty}(\mathbf{x}_1, \mathbf{u}) \leq \mathbf{x}_1^{\mathrm{T}} P \mathbf{x}_1. \quad (12)$$

Proof. Since $Q^* > 0$, by the general conditions for the solvability of Lyapunov equations, it is known that the Lyapunov equation (9) has a unique positive definite and symmetric solution, if the real parts of all eigenvalues of $(A_k + \kappa I)$ are negative. Because of the asymptotic stability of A_K, any constant $\kappa \in [0, -\lambda_{\max}(A_K))$ ensures the negativity of the real parts of all eigenvalues of $(A_k + \kappa I)$. Thus, (a) is true.

Since the point $\mathbf{0} \in \mathbb{R}^m$ is in the interior of U, we can always—for any fixed $P > 0$—find a constant $\alpha_1 \in (0, \infty)$, that specifies a region in the form of (11), such that $K\mathbf{x} \in U$, for all $\mathbf{x} \in \Omega_{\alpha_1}$. Thus, the linear feedback control values satisfy the input constraints in Ω_{α_1}.

Let $\alpha \in (0, \alpha_1]$ specify a region in the form of equation (11). Since $\alpha \le \alpha_1$, the input constraints are satisfied in Ω_α and thus (i) is true. In other words, the system can be thought of as being unconstrained in Ω_α.

We differentiate $\mathbf{x}^{\mathrm{T}}P\mathbf{x}$ along a trajectory of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, K\mathbf{x}) \tag{13}$$

and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)^{\mathrm{T}}P\mathbf{x}(t) = \mathbf{x}(t)^{\mathrm{T}}(A_K^{\mathrm{T}}P + PA_K)\mathbf{x}(t) + 2\mathbf{x}(t)^{\mathrm{T}}P\phi(\mathbf{x}(t)), \tag{14}$$

where $\phi(\mathbf{x}) := \mathbf{f}(\mathbf{x}, K\mathbf{x}) - A_K\mathbf{x}$. The term involving $\phi(\mathbf{x})$ in the above equation is bounded above as follows:

$$\mathbf{x}^{\mathrm{T}}P\phi(\mathbf{x}) \le \|\mathbf{x}^{\mathrm{T}}P\| \cdot \|\phi(\mathbf{x})\| \le \|P\| \cdot L_\phi \cdot \|\mathbf{x}\|^2 \le \frac{\|P\| \cdot L_\phi}{\lambda_{\min}(P)} \|\mathbf{x}\|_P^2, \tag{15}$$

where $L_\phi := \sup\{\|\phi(\mathbf{x})\|/\|\mathbf{x}\| \mid \mathbf{x} \in \Omega_\alpha, \mathbf{x} \neq \mathbf{0}\}$. Now we choose an $\alpha \in (0, \alpha_1]$ such that in Ω_α

$$L_\phi \le \frac{\kappa \cdot \lambda_{\min}(P)}{\|P\|}. \tag{16}$$

Then, inequality (15) leads to

$$\mathbf{x}^{\mathrm{T}}P\phi(\mathbf{x}) \le \kappa \cdot \mathbf{x}^{\mathrm{T}}P\mathbf{x}. \tag{17}$$

Substituting inequality (17) into equation (14) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)^{\mathrm{T}}P\mathbf{x}(t) \le \mathbf{x}(t)^{\mathrm{T}}((A_K + \kappa I)^{\mathrm{T}}P + P(A_K + \kappa I))\mathbf{x}(t)$$

that by equation (9) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)^{\mathrm{T}}P\mathbf{x}(t) \le -\mathbf{x}(t)^{\mathrm{T}}Q^*\mathbf{x}(t). \tag{18}$$

Since $P > 0$ and $Q^* > 0$, inequality (18) implies that the region Ω_α defined by equation (11) is invariant for the nonlinear system (1) controlled by the local linear state feedback $\mathbf{u} = K\mathbf{x}$. Moreover, any trajectory of equation (13) starting in Ω_α stays in Ω_α and converges to the origin.

Then, for any $\mathbf{x}_1 \in \Omega_\alpha$, integrating inequality (18) from t_1 to ∞ with initial condition $\mathbf{x}(t_1) = \mathbf{x}_1$ yields the desired result (12). □

It should be pointed out that if we use the notation introduced in Section 2 for internal variables in the controller and set $\mathbf{x}_1 = \bar{\mathbf{x}}(t + T_{\mathrm{p}}; \mathbf{x}(t), t)$, then inequality (12) is equivalent to inequality (6). The solution P of equation (9) and the region Ω_α in the form of equation (11) are able to serve as a terminal penalty matrix and a terminal region. From the above proof, a procedure can be stated to determine a terminal penalty matrix P and a terminal region Ω_α (preferably as large as possible) off-line such that inequality (14) holds true and the input constraints are satisfied:

Step 1. Solve a control problem based on the Jacobian linearization to get a locally stabilizing linear state feedback gain K.

Step 2. Choose a constant $\kappa \in [0, \infty)$ satisfying inequality (10) and solve the Lyapunov equation (9) to get a positive-definite and symmetric P.

Step 3. Find the largest possible α_1 such that $K\mathbf{x} \in U$, for all $\mathbf{x} \in \Omega_{\alpha_1}$.

Step 4. Find the largest possible $\alpha \in (0, \alpha_1]$ such that inequality (16) is satisfied in Ω_α.

Remark 3.1. In Step 4, inequality (16) is not easy to satisfy for an arbitrary large terminal region Ω_α. Due to a typically small value of $\lambda_{\min}(P)/\|P\|$, it is possible that for some systems this inequality can only be met for an extremely small terminal region Ω_α. From the proof of Lemma 1, we know that if inequality (17) holds true for all $\mathbf{x} \in \Omega_\alpha$, then, inequality (18) is also valid. In addition, since $\phi(\mathbf{x})$ satisfies $\phi(\mathbf{x}) \to \mathbf{0}$ and $\|\phi(\mathbf{x})\|/\|\mathbf{x}\| \to 0$ as $\|\mathbf{x}\| \to 0$, there exists a constant $\varepsilon > 0$ such that inequality (17) holds true for all $\mathbf{x}$ with $\|\mathbf{x}\| \le \varepsilon$. Hence, in order to get a less conservative terminal region, we may take a different approach. First, we follow the above procedure until Step 3. Then, we may make iterations of the simple optimization problem

$$\max_x \{\mathbf{x}^{\mathrm{T}}P\phi(\mathbf{x}) - \kappa \cdot \mathbf{x}^{\mathrm{T}}P\mathbf{x} \mid \mathbf{x}^{\mathrm{T}}P\mathbf{x} \le \alpha\} \tag{19}$$

for the chosen κ by reducing α from α_1 until the optimal value of (19) is nonpositive. A discussion on the optimization problem (19) and on finding the maximum α_1 in Step 3 can be found in Michalska and Mayne (1993). If a suitable α is found in this way, it specifies a region Ω_α in the form of (11), in which inequality (17) holds true. In turn, inequality (18) is valid and the results in Lemma 1 hold consequently. This region can then serve as a terminal region.

Remark 3.2. Following the above procedure does not yield a unique terminal region for a given nonlinear system. For the sake of reducing the on-line computational burden, we are interested in determining the largest possible region. This is, however, not an easy task. First, this requires a suitable selection of the stabilizing linear feedback gain K, where many linear control methods can in principle be used. Because of the "optimality" of MPC, the linear optimal control technique (LQR) may be preferentially applied for determining a stabilizing K. Secondly, for a given gain K, an appropriate choice of κ is needed. This will be discussed in Section 5. Moreover, the size of the terminal region depends generally on the nonlinearity of the system to be controlled. The stronger nonlinear the system is, the smaller the terminal region will be. For linear or some mildly nonlinear systems, the size of the terminal region will only be restricted by the input constraints. This will also be shown in the example in Section 5.

Remark 3.3. If there exists no linear feedback controller that can locally asymptotically stabilize the nonlinear system, Ω_α contracts to the origin. Thus, the terminal inequality constraint (5c) reduces to the terminal equality constraint $\mathbf{x}(t + T_p) = \mathbf{0}$, which is well known to lead to stability (Mayne and Michalska, 1990; Rawlings and Muske, 1993). For a generalization of the proposed approach to systems with non-stabilizable Jacobian linearization see Chen and Allgöwer (1997a).

Remark 3.4. If the system to be controlled is linear, i.e. $\phi(\mathbf{x}) = \mathbf{0}$ and $L_\phi = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, then, $\kappa = 0$ satisfies equation (16). In turn, equation (9) becomes the Lyapunov equation for linear systems, and equation (12) is satisfied with equality. That means that the following equality:

$$\int_t^{t+T_p} (\|\mathbf{x}(\tau)\|_Q^2 + \|\mathbf{u}(\tau)\|_R^2)\,d\tau + \|\mathbf{x}(t+T_p)\|_P^2$$

$$= \int_t^{\infty} (\|\mathbf{x}(\tau)\|_Q^2 + \|\mathbf{u}(\tau)\|_R^2)\,d\tau$$

is valid for linear systems. Thus, the model predictive controller has exactly an infinite prediction horizon with only a finite horizon input profile to be determined on-line. For "open-loop" control, the control law beyond the finite horizon would be given by the local linear state feedback $\mathbf{u} = K\mathbf{x}$ (cf. Rawlings and Muske (1993) and Muske (1995), where the control beyond the finite horizon is chosen to be zero). A similar result can be found in Scokaert and Rawlings (1996).

4. ASYMPTOTIC STABILITY RESULTS

According to the principle of MPC, the open-loop optimal control problem given by equations (3)–(5) will be solved repeatedly, updated with new measurements. The closed-loop control $\mathbf{u}^*(\cdot)$ is defined by equation (7), where $\bar{\mathbf{u}}^*(\cdot\,; \mathbf{x}(t), t, t + T_p)$: $[t, t + T_p] \to U$ is a solution to the optimization problem. In this section, we address the stability property of the closed-loop system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}^*(t)). \tag{20}$$

To do this, we use the following standard definitions (e.g. Khalil, 1992) and assume for the moment (later it will be shown) that $\mathbf{x} = \mathbf{0}$ is an equilibrium of equation (20).

Definition 1. The equilibrium point $\mathbf{x} = \mathbf{0}$ of equation (20) is stable if for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$, such that $\|\mathbf{x}(0)\| < \eta(\varepsilon)$ implies $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq 0$.

Definition 2. The equilibrium point $\mathbf{x} = \mathbf{0}$ of equation (20) is asymptotically stable if it is stable and η can be chosen such that $\|\mathbf{x}(0)\| < \eta$ implies $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

For the sake of a clear proof, we use in this section the notation for internal controller variables and distinguish between the predicted values in the controller and the actual ones in the "real" system. Thus, $\bar{\mathbf{x}}(\cdot\,; \mathbf{x}(t), t)$ denotes the predicted trajectory of the nonlinear system starting from the actual state $\mathbf{x}(t)$ and driven by an open-loop control $\bar{\mathbf{u}}(\cdot)$, when the prediction is made in the controller at "real" time t.

4.1. *Feasibility of the optimization problem*

Due to the repeated solution of the optimization problem given by equations (3)–(5), we need its feasibility at each time $t \geq 0$. Feasibility of the optimization problem means that there exists at least one (not necessarily optimal) input profile $\bar{\mathbf{u}}(\cdot): [t, t + T_p] \to U$ such that the generated trajectory of equation (5a) satisfies the terminal inequality constraint (5c), and such that the value of the objective functional (4) is bounded. In the following, we give a lemma on the feasibility of the optimization problem at each time. This lemma follows a standard argument also used for example in Genceli and Nikolaou (1993), Michalska and Mayne (1993) and Rawlings and Muske (1993).

Lemma 2. For the nominal system with perfect state measurement and no disturbances, and for a sufficiently small sampling time $\delta > 0$, the feasibility of the open-loop optimal control problem (3) with equation (4) subject to equation (5) at time $t = 0$ implies its feasibility for all $t > 0$.

Proof. It is assumed for the moment that, at time t, an optimal solution $\bar{\mathbf{u}}^*(\cdot\,; \mathbf{x}(t), t, t + T_p): [t, t + T_p] \to U$ to the optimization problem described by equations (3)–(5) exists and is found. When applied in open loop, this finite horizon optimal input profile drives the system model (5a) from initial state $\mathbf{x}(t)$ into the terminal region Ω along the corresponding open-loop optimal state trajectory $\bar{\mathbf{x}}^*(\cdot; \mathbf{x}(t), t, t+T_p)$ on $[t, t+T_p]$ with $\bar{\mathbf{x}}^*(t+T_p; \mathbf{x}(t), t, t+T_p) \in \Omega$.

In terms of MPC, the *closed-loop* control $\mathbf{u}^*(\cdot)$ from time t to $t+\delta$ is defined by equation (7). Since, by assumption, there are no disturbances and we only consider the nominal system, the state measurement at time $t+\delta$ is then $\mathbf{x}(t+\delta) = \bar{\mathbf{x}}^*(t+\delta; \mathbf{x}(t), t, t+T_p)$. Therefore, for solving the open-loop optimal control problem at time $t+\delta$ with initial condition $\bar{\mathbf{x}}(t+\delta; \mathbf{x}(t+\delta), t+\delta) = \mathbf{x}(t+\delta)$, a candidate input profile $\bar{\mathbf{u}}(\cdot)$ on $[t+\delta, t+\delta+T_p]$ may be chosen with

$$\bar{\mathbf{u}}(\tau) = \begin{cases} \bar{\mathbf{u}}^*(\tau; \mathbf{x}(t), t, t+T_p) & \text{for } \tau \in [t+\delta, t+T_p], \\ K\bar{\mathbf{x}}(\tau; \mathbf{x}(t+\delta), t+\delta) & \text{for } \tau \in [t+T_p, t+\delta+T_p], \end{cases} \quad (21)$$

where K is the local linear state feedback gain used in determining P and Ω off-line (compare Section 3). From Lemma 1, the terminal region Ω is invariant for the nonlinear system model controlled with the linear state feedback. Thus, $\bar{\mathbf{x}}^*(t+T_p; \mathbf{x}(t), t, t+T_p) \in \Omega$ implies $\bar{\mathbf{x}}(t+\delta+T_p; \mathbf{x}(t+\delta), t+\delta) \in \Omega$, due to the choice (21) for the input profile. In addition, since the input constraints are satisfied in Ω, input profile (21) is a feasible but perhaps not optimal solution to the optimization problem at time $t+\delta$. Obviously, this reasoning is also valid, if at time t we start out with a feasible solution only, that needs not be optimal.

For a numerical implementation, the input profile is in general parameterized in a step-shaped manner. Thus, choosing $\bar{\mathbf{u}}(\tau) = K\bar{\mathbf{x}}(\tau; \mathbf{x}(t+\delta), t+\delta)$ for $\tau \in [t+T_p, t+\delta+T_p]$ as in equation (21) is practically impossible. However, we do have $\bar{\mathbf{x}}^*(t+T_p; \mathbf{x}(t), t, t+T_p) \in \Omega$. Then, if we choose $\bar{\mathbf{u}}(\tau) = K\bar{\mathbf{x}}(t+T_p; \mathbf{x}(t+\delta), t+\delta)$ for $\tau \in [t+T_p, t+\delta+T_p]$, from the continuity of the trajectory, we can assume w.l.o.g. that for a small enough $\delta > 0$, the trajectory $\bar{\mathbf{x}}(\cdot; \mathbf{x}(t+\delta), t+\delta)$ on $[t+T_p, t+\delta+T_p]$ stays in Ω. Then, the result can be achieved by induction. □

Remark 4.1. Lemma 2 indicates that the prediction horizon T_p (tuning parameter) needs to be chosen such that the optimization problem (3) with equation (4) subject to equation (5) is feasible at time $t = 0$.

4.2. *Asymptotic stability*

Before the asymptotic stability of the closed-loop system (20) is addressed, we show that the optimal value function is non-increasing. This result is crucial for the stability proof.

Lemma 3. Suppose that the optimization problem is feasible at time $t = 0$. Then, for the unperturbed nominal system, for any $t \geq 0$ and $\tau \in (t, t+\delta]$ the optimal value function satisfies

$$J^*(\mathbf{x}(\tau), \tau, \tau+T_p) \leq J^*(\mathbf{x}(t), t, t+T_p) - \int_t^\tau (\|\mathbf{x}(s)\|_Q^2 + \|\mathbf{u}^*(s)\|_R^2)\,\mathrm{d}s. \quad (22)$$

Proof. From Lemma 2, feasibility of the optimization problem at each time $t > 0$ is guaranteed by its feasibility at time $t = 0$.

If, at time t, a finite horizon open-loop optimal input profile $\bar{\mathbf{u}}^*(\cdot; \mathbf{x}(t), t, t+T_p): [t, t+T_p] \to U$ and the corresponding finite horizon open-loop optimal state trajectory $\bar{\mathbf{x}}^*(\cdot; \mathbf{x}(t), t, t+T_p)$ on $[t, t+T_p]$ are given, the optimal value of the objective functional can be written as

$$J^*(\mathbf{x}(t), t, t+T_p) = \int_t^{t+T_p} (\|\bar{\mathbf{x}}^*(s; \mathbf{x}(t), t, t+T_p)\|_Q^2 + \|\bar{\mathbf{u}}^*(s; \mathbf{x}(t), t, t+T_p)\|_R^2)\,\mathrm{d}s + \|\bar{\mathbf{x}}^*(t+T_p; \mathbf{x}(t), t, t+T_p)\|_P^2. \quad (23)$$

For any $\tau \in (t, t+\delta]$, the closed-loop control is taken in terms of equation (7). For the nominal system without disturbances, the closed-loop state trajectory of the system (1) is then given by

$$\mathbf{x}(s) = \bar{\mathbf{x}}^*(s; \mathbf{x}(t), t, t+T_p), \quad s \in [t, \tau]. \quad (24)$$

We now calculate the value of the objective functional for any $\tau \in (t, t+\delta]$, if a feasible (suboptimal) input profile

$$\bar{\mathbf{u}}(s) = \begin{cases} \bar{\mathbf{u}}^*(s; \mathbf{x}(t), t, t+T_p) & \text{for } s \in [\tau, t+T_p] \\ K\bar{\mathbf{x}}(s; \mathbf{x}(t+\delta), t+\delta) & \text{for } s \in [t+T_p, \tau+T_p] \end{cases} \quad (25)$$

is assumed to be applied to the system. We denote that by $\bar{J}(\mathbf{x}(\tau), \tau, \tau+T_p) := J(\mathbf{x}(\tau), \bar{\mathbf{u}}(\cdot))$ with $\bar{\mathbf{u}}(\cdot)$ according to equation (25). The generated finite horizon open-loop state trajectory is the same as the open-loop state trajectory given by the optimization at time t, except for the last part on $[t+T_p, \tau+T_p]$, i.e.

$$\bar{\mathbf{x}}(s; \mathbf{x}(\tau), \tau) = \bar{\mathbf{x}}^*(s; \mathbf{x}(t), t, t+T_p), \quad s \in [\tau, t+T_p]. \quad (26)$$

In order to characterize the contribution of the state trajectory on $[t+T_p, \tau+T_p]$ to the value function, we use the results in Lemma 1: Since the feasibility of the optimization problem at time t implies that $\bar{\mathbf{x}}^*(t+T_p;\mathbf{x}(t),t,t+T_p)\in\Omega$ and on $[t+T_p,\tau+T_p]$ the system model is controlled by the linear state feedback (see equation (25)), that part of the state trajectory stays in Ω and obeys inequality (18). We want to remind that the "real" time is now $\tau\in(t,t+\delta]$ and we discuss the *predicted open-loop state trajectory in the controller*. In this situation, $\mathbf{x}(t)$ and t in inequality (18) have to be replaced by $\bar{\mathbf{x}}(s;\mathbf{x}(\tau),\tau)$ and s, respectively. Then, integrating (18) from $t+T_p$ to $\tau+T_p$ with $\bar{\mathbf{x}}(t+T_p;\mathbf{x}(\tau),\tau)=\bar{\mathbf{x}}^*(t+T_p;\mathbf{x}(t),t,t+T_p)$ yields the following relationship:

$$\|\bar{\mathbf{x}}(\tau+T_p;\ \mathbf{x}(\tau),\tau)\|_P^2\le\|\bar{\mathbf{x}}^*(t+T_p;\ \mathbf{x}(t),t,t+T_p)\|_P^2-\int_{t+T_p}^{\tau+T_p}\|\bar{\mathbf{x}}(s;\mathbf{x}(\tau),\tau)\|_{Q^*}^2\,\mathrm{d}s. \tag{27}$$

Then, the value of the objective functional for any $\tau\in(t,t+\delta]$ is

$$\begin{aligned}\bar{J}(\mathbf{x}(\tau),\tau,\tau+T_p)&=\int_\tau^{\tau+T_p}(\|\bar{\mathbf{x}}(s;\mathbf{x}(\tau),\tau)\|_Q^2+\|\bar{\mathbf{u}}(s)\|_R^2)\,\mathrm{d}s\\&\quad+\|\bar{\mathbf{x}}(\tau+T_p;\mathbf{x}(\tau),\tau)\|_P^2\\&=\int_\tau^{t+T_p}(\|\bar{\mathbf{x}}^*(s;\mathbf{x}(t),t,t+T_p)\|_Q^2\\&\quad+\|\bar{\mathbf{u}}^*(s;\mathbf{x}(t),t,t+T_p)\|_R^2)\,\mathrm{d}s\\&\quad+\int_{t+T_p}^{\tau+T_p}\|\bar{\mathbf{x}}(s;\mathbf{x}(\tau),\tau\|_{Q^*}^2\,\mathrm{d}s\\&\quad+\|\bar{\mathbf{x}}(\tau+T_p;\mathbf{x}(\tau),\tau)\|_P^2,\end{aligned}$$

where equations (25) and (26) are used. Because of inequality (27), the above equality becomes

$$\begin{aligned}\bar{J}(\mathbf{x}(\tau),\tau,\tau+T_p)\le\int_\tau^{t+T_p}&(\|\bar{\mathbf{x}}^*(s;\mathbf{x}(t),t,t+T_p)\|_Q^2\\&+\|\bar{\mathbf{u}}^*(s;\mathbf{x}(t),t,t+T_p)\|_R^2)\,\mathrm{d}s\\&+\|\bar{\mathbf{x}}^*(t+T_p;\mathbf{x}(t),t,t+T_p)\|_P^2.\end{aligned}$$

Combining it with (23) yields

$$\begin{aligned}\bar{J}(\mathbf{x}(\tau),\mathbf{x}(\tau),\tau)&\le J^*(\mathbf{x}(t),t,t+T_p)\\&\quad-\int_t^\tau(\|\bar{\mathbf{x}}^*(s;\mathbf{x}(t),t,t+T_p)\|_Q^2\\&\quad+\|\bar{\mathbf{u}}^*(s;\mathbf{x}(t),t,t+T_p)\|_R^2)\,\mathrm{d}s.\end{aligned}$$

It follows from substituting equations (7) and (24) into the above inequality that

$$\bar{J}(\mathbf{x}(\tau),\tau,\tau+T_p)\le J^*(\mathbf{x}(t),t,t+T_p)-\int_t^\tau(\|\mathbf{x}(s)\|_Q^2+\|\mathbf{u}^*(s)\|_R^2)\,\mathrm{d}s. \tag{28}$$

It is clear, by the optimality of J^*, that we have for any $\tau\in(t,t+\delta]$,

$$\begin{aligned}J^*(\mathbf{x}(\tau),\tau,\tau+T_p)&\le\bar{J}(\mathbf{x}(\tau),\tau,\tau+T_p)\\&\le J^*(\mathbf{x}(t),t,t+T_p)\\&\quad-\int_t^\tau(\|\mathbf{x}(s)\|_Q^2+\|\mathbf{u}^*(s)\|_R^2)\,\mathrm{d}s\end{aligned}$$

as required. □

Because $Q>0$ and $R>0$, Lemma 3 implies by induction that the optimal value function is non-increasing. Now we are able to state the asymptotic stability result for the closed-loop system (20).

Theorem 1. Suppose that

(a) assumptions (A1)–(A3) are satisfied,
(b) the Jacobian linearization of the nonlinear system (1) is stabilizable,
(c) the open-loop optimal control problem (3) with equation (4) subject to equation (5) is feasible at time $t=0$.

Then, for a sufficiently small sampling time $\delta>0$ and in the absence of disturbances, the closed-loop system with the model predictive control (7) is nominally asymptotically stable. Let $X\subseteq\mathbb{R}^n$ denote the set of all initial states for which assumption (c) is satisfied, then, X gives a region of attraction for the closed-loop system.

Proof. From Lemma 1, assumption (b) implies that a terminal penalty matrix P and a terminal region Ω in the form of equation (11) can be found by the procedure in Section 3. According to Lemma 2, for a sufficiently small $\delta>0$, feasibility of the open-loop optimal control problem at each time $t>0$ is guaranteed by assumption (c).

For $\mathbf{x}(t)=\mathbf{0}$, the optimal solution to the optimization problem is $\bar{\mathbf{u}}^*(\cdot;\mathbf{0},t,t+T_p):[t,t+T_p]\to\mathbf{0}$, i.e. $\mathbf{u}^*(\tau)=\mathbf{0}$ for all $\tau\in[t,t+\delta]$. Due to $\mathbf{f}(\mathbf{0},\mathbf{0})=\mathbf{0}$, then $\mathbf{x}=\mathbf{0}$ is an equilibrium of the closed-loop system (20).

Now we define a function $V(\mathbf{x})$ for the closed-loop system (20) as follows:

$$V(\mathbf{x}):=J^*(\mathbf{x},t,t+T_p). \tag{29}$$

Then, $V(\mathbf{x})$ has the following properties:

- $V(\mathbf{0})=0$ and $V(\mathbf{x})>0$ for $\mathbf{x}\neq\mathbf{0}$,
- $V(\mathbf{x})$ is continuous at $\mathbf{x}=\mathbf{0}$,
- along the trajectory of the closed-loop system starting from any $\mathbf{x}_0\in X$ there is for $0\le t_1<t_2\le\infty$

$$V(\mathbf{x}(t_2))-V(\mathbf{x}(t_1))\le-\int_{t_1}^{t_2}\|\mathbf{x}(t)\|_Q^2\,\mathrm{d}t. \tag{30}$$

The first two properties follow from Lemma A.1 in Chen (1997) and the third property is due to Lemma 3 and $R > 0$. As a central consequence, we can take the standard argument used, for example, in Hahn (1967) and Khalil (1992) to prove that the equilibrium $\mathbf{x} = \mathbf{0}$ is stable per Definition 1. That is, for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$, such that $\|\mathbf{x}(0)\| < \eta(\varepsilon)$ implies $\|\mathbf{x}(t)\| < \varepsilon$ for all $t \geq 0$. Moreover, there exists a constant $\beta \in (0, \infty)$ such that along the closed-loop trajectory one has

$$V(\mathbf{x}(t)) \leq \beta, \quad \forall t \geq 0. \tag{31}$$

In the following, we show that there exists $\eta > 0$ such that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for all $\|\mathbf{x}(0)\| < \eta$, without having to use a continuous differentiability assumption on $V(\mathbf{x})$. This implies that the equilibrium $\mathbf{x} = \mathbf{0}$ is asymptotically stable from Definition 2. Finally, it is shown that X is a region of attraction for the closed-loop system.

We start out with inequality (30) to prove the asymptotic stability. By induction, we have

$$V(\mathbf{x}(\infty)) \leq V(\mathbf{x}(0)) - \int_0^{\infty} \|\mathbf{x}(t)\|_Q^2 \, \mathrm{d}t. \tag{32}$$

Due to $V(\mathbf{x}(\infty)) \geq 0$ and $V(\mathbf{x}(0)) \leq \beta$, the integral $\int_0^{\infty} \|\mathbf{x}(t)\|_Q^2 \mathrm{d}t$ exists and is bounded. Let $\varepsilon_1 < \varepsilon$ be such that $\|\mathbf{x}(t)\| \leq \varepsilon_1$, then, $\mathbf{x}(t)$ is on the compact set $\{\|\mathbf{x}\| \leq \varepsilon_1\}$ for all $t \in [0, \infty)$. Moreover, $\mathbf{u}^*(t) \in U$ for all $t \in [0, \infty)$ with U being compact. Because $\mathbf{f}$ is continuous in $\mathbf{x}$ and $\mathbf{u}$, then, $\mathbf{f}(\mathbf{x}(t), \mathbf{u}^*(t))$ is bounded for all $t \in [0, \infty)$. Thus, $\mathbf{x}(t)$ is uniformly continuous in t on $[0, \infty)$ (Desoer and Vidyasagar, 1975). Consequently, $\|\mathbf{x}(t)\|_Q^2$ is uniformly continuous in t on $[0, \infty)$, because $\|\mathbf{x}\|_Q^2$ is uniformly continuous in $\mathbf{x}$ on the compact set $\{\|\mathbf{x}\| \leq \varepsilon_1\}$. Due to $Q > 0$, it follows from Barbalat's Lemma (Khalil, 1992) that

$$\|\mathbf{x}(t)\| \to \mathbf{0} \quad \text{as } t \to \infty. \tag{33}$$

Then, the equilibrium point $\mathbf{x} = \mathbf{0}$ of the system (20) is asymptotically stable. Clearly,

$$W_\beta := \{\mathbf{x} \in X \mid V(\mathbf{x}) \leq \beta\} \tag{34}$$

is a region of attraction.

Furthermore, for all $\mathbf{x}(0) \in X$, there exists a finite time T such that $\mathbf{x}(T) \in W_\beta$. This can be shown by contradiction: Assume that $\mathbf{x}(t) \notin W_\beta$ for all $t \geq T$. It follows from inequality (30) that for all $t \geq T$

$$\begin{aligned} V(\mathbf{x}(t+\delta)) - V(\mathbf{x}(t)) &\leq -\int_t^{t+\delta} \|\mathbf{x}(\tau)\|_Q^2 \, \mathrm{d}\tau \\ &\leq -\delta \inf\{\|\mathbf{x}\|_Q^2 \mid \mathbf{x} \notin W_\beta\} \\ &\leq -\delta \cdot \gamma, \end{aligned} \tag{35}$$

where $\gamma > 0$, because of the positive definiteness of $V(\mathbf{x})$. By induction, $V(\mathbf{x}(t)) \to -\infty$ as $t \to \infty$ that contradicts however the fact that $V(\mathbf{x}) \geq 0$. Thus, any trajectory of equation (20) starting from X enters into W_β in a finite time. Then, the asymptotic stability of equation (20) in X follows by the fact that W_β is a region of attraction.

Finally, X has also the property that any closed-loop trajectory starting from X stays in X. This can be proven again by contradiction: We assume that the closed-loop trajectory starting from any $\mathbf{x}(0) \in X$ has left X at some time $t_1 > 0$, i.e. $\mathbf{x}(t_1) \notin X$. From Lemma 2, we know that the optimization problem at time t_1 with initial condition $\bar{\mathbf{x}}(t_1; \mathbf{x}(t_1), t_1) = \mathbf{x}(t_1)$ is feasible. This contradicts that X is the set of all initial states for which assumption (c) is satisfied. Together with the achieved asymptotic stability, X gives a region of attraction for the closed-loop system. □

Remark 4.2. The given stability conditions are only sufficient and not necessary. The fact that the linearized system is not stabilizable does of course not imply that there exists no linear feedback controller being able to stabilize the nonlinear system locally.

Remark 4.3. When applying this control scheme to practical systems, the numerical optimization employed may not find the globally optimal input profile at every time step, due to real time computation time restrictions or because the optimizer is for example caught in a local optimum. Even though optimal performance might be lost this way, stability can be guaranteed nevertheless, as the stability guarantee does not depend on the optimality of the solution found but merely on its feasibility, as long as the problem is feasible and the optimizer finds any feasible solution at time $t = 0$ and as long as each optimization problem is initialized by the shifted feasible solution from the previous step.

Remark 4.4. If the nonlinear system is open-loop asymptotically stable, the nonlinear terminal inequality constraint $\mathbf{x}(t + T_p) \in \Omega$ can be removed, without loss of the achieved stability (Chen and Allgöwer, 1997b).

5. EXAMPLE

5.1. *Control problem and simulation results*

As an example for demonstrating the proposed control scheme, we consider a system described by the following ODEs:

$$\dot{x}_1 = x_2 + u(\mu + (1 - \mu)x_1), \tag{36a}$$

$$\dot{x}_2 = x_1 + u(\mu - 4(1 - \mu)x_2). \tag{36b}$$

This system is a modification of the system used in Mayne and Michalska (1990) in that it is unstable and its linearized system is stabilizable (but not controllable) for any $\mu \in (0, 1)$. In addition, it is assumed that the input u has to satisfy the following constraint:

$$U = \{u \in \mathbb{R}^1 | -2.0 \leq u \leq 2.0\}. \quad (37)$$

For this unstable constrained system, assumptions (A1)–(A3) are satisfied. The weighting matrices Q and R in the objective functional (4) are chosen as

$$Q = \begin{pmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{pmatrix}, \qquad R = 1.0. \quad (38)$$

Assume $\mu = 0.5$ for the moment. In order to find a terminal penalty matrix P and the largest possible terminal region Ω, we follow the procedure described in Section 3. First, solving the linear optimal control problem with the weighting matrices given in equation (38) for the locally linearized system, we get a linear locally stabilizing state feedback gain

$$K = [2.118 \quad 2.118]. \quad (39)$$

The largest eigenvalue of the closed-loop linearized system has real part $\lambda_{max}(A_K) = -1.0$. Then, we choose a constant $\kappa = 0.95$ which implies that the unique solution of the Lyapunov equation (9),

$$P = \begin{pmatrix} 16.5926 & 11.5926 \\ 11.5926 & 16.5926 \end{pmatrix} \quad (40)$$

is positive definite, symmetric and can be used as a terminal penalty matrix. After that, $\alpha_1 = 12.5$ is found to specify a region Ω_{α_1} in the form of equation (11), in which the linear feedback control satisfies the constraint (37). Finally, we find a region Ω_α defined by equation (11) with $\alpha = 0.025$ such that inequality (16) is satisfied. However, this region is very small, because of the small value (0.1774) of $\lambda_{min}(P)/\|P\|$. From the simple optimization (19) outlined in Remark 3.1, we can derive a less conservative terminal region Ω_α with $\alpha = 0.7$ as follows:

$$\Omega_\alpha = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x}^T P \mathbf{x} \leq 0.7\}. \quad (41)$$

The open-loop optimal control problem described by equations (3)–(5) is solved in discrete time with a sampling time of $\delta = 0.1$ time-units and a prediction horizon of $T_P = 1.5$ time-units. A few trajectories corresponding to different initial conditions of the unstable constrained system (36) with $\mu = 0.5$ controlled by the proposed quasi-infinite horizon nonlinear predictive controller with parameters (38), (40) and (41) are shown in Fig. 1. The solid lines represent closed-loop trajectories; the dashed line is the boundary of the terminal region given by equation (41); the dash-dotted lines denote the predicted open-loop trajectories that are found by solving the optimization problem at time $t = 0$ and

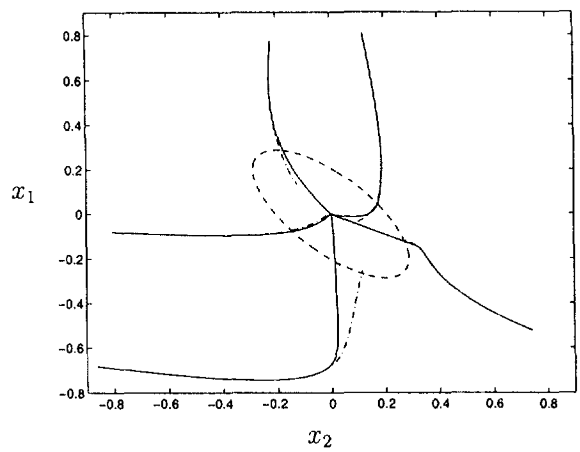

Fig. 1. Trajectories of the unstable constrained system ($\mu = 0.5$) controlled by the proposed nonlinear predictive controller for different initial conditions.

are only of finite horizon. It is clearly seen that the finite horizon open-loop trajectories end in the terminal region, as expected to be achieved by the terminal inequality constraint.

It should be emphasized that the linear state feedback with gain K as in equation (39) is *not explicitly used* to calculate the closed-loop control. Like standard model predictive controllers, the closed-loop control is determined by solving online the optimization problem given by equations (3)–(5) repeatedly. For the chosen prediction horizon T_p and sampling time δ, the optimization problem is feasible at each time. Thus, for the nominal system without disturbances, the stability conditions given in Section 4.2 are all satisfied. The closed-loop trajectories in Fig. 1 are guaranteed to converge to the origin. Figure 2 shows time profiles for the closed-loop system for two selected initial conditions (solid lines and dashed lines, respectively). It can be seen that the input constraint (37) is not violated.

5.2. *Discussion of computational burden*

The proposed nonlinear MPC scheme has significant computational advantages when compared to other existing MPC approaches. To show this, we compare the proposed controller (case A) to two other predictive controllers (cases B and C):

A P given by equation (40), terminal inequality constraint $\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) \in \Omega_\alpha$ with Ω_α given by equation (41), $T_p = 1.5$,
B $P = 0$, no terminal constraint, $T_p = 3.5$,
C $P = 0$, terminal equality constraint $\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) = \mathbf{0}$, $T_p = 3.5$.

Controller A is designed with the proposed method and has guaranteed stability. For controller B, there is no terminal constraint and the terminal

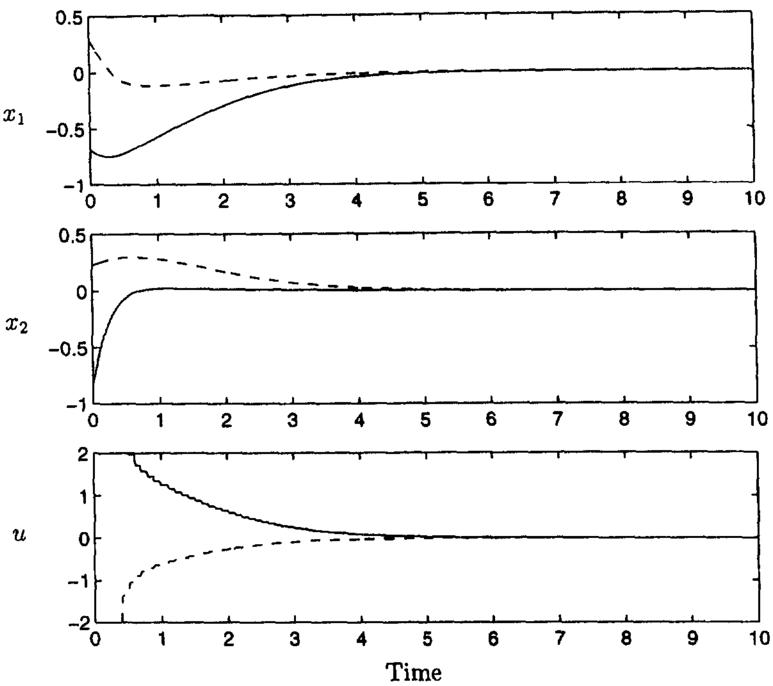

Fig. 2. Time profiles for the closed-loop system from two initial conditions.

Table 1. Comparison of elapsed CPU time.

Initial state		Elapsed CPU time (s)		
$x_1(0)$	$x_2(0)$	Controller A	Controller B	Controller C
− 0.683	− 0.864	859	1521	*
− 0.523	0.744	818	1492	*
0.808	0.121	615	1638	*
0.774	− 0.222	570	1522	5729
0.292	0.228	820	1724	*
− 0.08	− 0.804	696	1544	5093

states are not penalized. This controller corresponds to the nonlinear MPC scheme usually used in applications. Closed-loop stability can only be achieved by tuning the prediction (control) horizon T_p. Here, $T_p = 3.5$ time-units is the shortest prediction horizon determined by trial and error such that the closed-loop system is stable (for $T_p = 1.5$ time-units, the closed-loop system is unstable). For controller C, the well-known terminal equality constraint $\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) = \mathbf{0}$ is used. Hence, a terminal state penalty does not make sense. Closed-loop stability is also guaranteed for this controller, if the optimization problem at time $t = 0$ is feasible.

For a total simulation time of 10 time-units, the elapsed CPU times are shown in Table 1 for some different initial conditions, where the controllers A, B and C use the same optimization routine NAG E04UCF [Numerical Algorithms Group, 1993] and the same integration algorithm [Mitchell & Gauthier Associates, 1991] with the same numerical parameters (optimality tolerance, feasibility tolerance, integration step, etc.). The symbol "*" indicates that the optimization problem is not feasible at time $t = 0$ for the corresponding initial condition. It is clearly seen that controller A needs significantly less CPU time than controllers B and C. Here, controller B might be treated somewhat unfairly. In practice, one can use techniques such as blocking or confounding to reduce on-line computation time. However, an important drawback of controller B is that stability can only be achieved by tuning parameters such as the prediction horizon. The big difficulty for controller C is the infeasibility of the optimization problem caused by the terminal equality constraint. Despite the fact that the terminal constraint $\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) = \mathbf{0}$ needs only be satisfied with feasibility tolerance 10^{-4}, the optimization problem is not feasible at time $t = 0$ for most initial conditions in Table 1. Thus, no stability can be guaranteed. Figure 3 shows two trajectories of the constrained system controlled by controllers A, B and C. We see that there is no big difference in control performance.

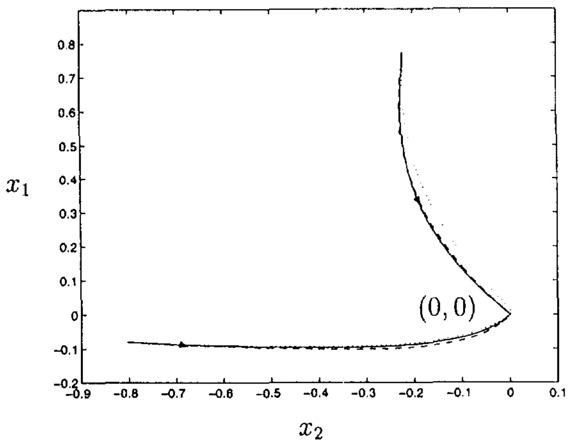

Fig. 3. Comparison of nonlinear predictive controllers: A ( ), B (– –), C (···) for two selected initial conditions.

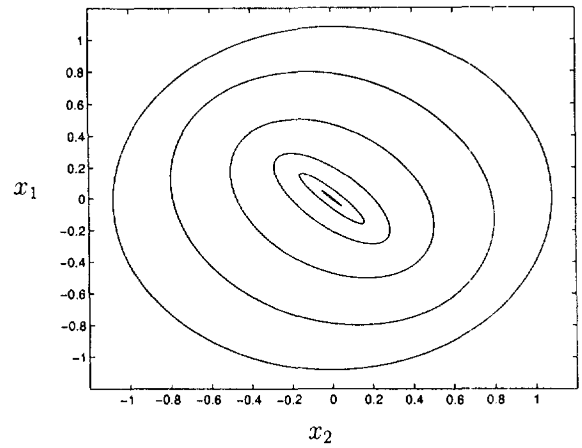

Fig. 4. Terminal region vs μ: the ellipses from the outside to the inside correspond to $\mu = 0.9, 0.8, 0.7, 0.5, 0.3, 0.1$.

5.3. *Discussion on terminal region*

The model parameter $\mu \in (0, 1)$ describes the nonlinearity of the system (36). It is immediately clear that the smaller μ, the stronger nonlinear the system behaves. For $\mu = 1$, the system is linear. For some given μ's, we follow the procedure in Section 3 to determine the terminal regions. As we do so, the linear feedback gain K is determined by solving the linear optimal control problem based on the Jacobian linearization with weighting matrices Q, R as in equation (38), and $\kappa = 0.95$ is chosen. The results are shown in Fig. 4. It can be seen that the stronger nonlinear the system is, the smaller the terminal region becomes. For the system (36) with $\mu = 0.9$, the input constraint (37) determines the size of the terminal region directly.

The stability conditions discussed in Section 4.2 are only sufficient. In particular, it is very difficult, if not impossible, to find *the* largest terminal region for a given nonlinear system. From the Lyapunov equation (9), P increases with κ, and very rapidly as κ is near to $-\lambda_{\max}(A_K)$. A large P implies strong penalty for the states at the end of the finite horizon, but does not automatically imply a large terminal region Ω_α. For a given model parameter $\mu = 0.5$, some terminal regions for different κ are shown in Fig. 5. We see that the terminal region extends first with κ, but it becomes smaller as κ approaches $-\lambda_{\max}(A_K) = 1.0$. It seems that a constant κ near to the absolute value of the largest eigenvalue of A_K corresponds to the largest possible terminal region. However, with this κ the matrix P will be also large. From the structure of the objective functional, we know that a very strong penalty of the terminal states may have a bad influence on the achievement of the control performance that is specified by the finite horizon cost. Thus, we may have to trade off between a large terminal region and good achievement of the desired control performance.

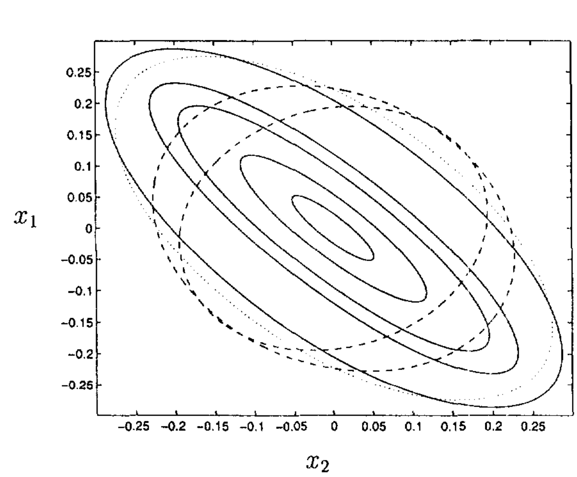

Fig. 5. Terminal region vs. κ: the solid ellipses from the inside to the outside correspond to $\kappa = 0.15, 0.35, 0.55, 0.75, 0.95$; the dotted ellipse is for $\kappa = 0.97$ and the dashed ellipses from the outside to the inside correspond to $\kappa = 0.99, 0.995$.

6. CONCLUSIONS

In this paper we proposed a quasi-infinite horizon nonlinear MPC scheme with guaranteed stability. The setup differs from the standard setup with quadratic objective functionals only in that a terminal state penalty term $(\mathbf{x}(t + T_p)^T P \mathbf{x}(t + T_p))$ is added to the finite horizon objective functional and an additional terminal inequality constraint $(\mathbf{x}(t + T_P) \in \Omega)$ has to be satisfied. These two terms do not however constitute additional tuning parameters that can be chosen freely, but have to be determined off-line such that the terminal region Ω has an invariance property. We have proven that this choice will guarantee asymptotic stability of the closed loop independent of the choice of the performance parameters Q and R in the quadratic objective functional, if a feasible solution to the optimization problem at time $t = 0$ exists. Thus, a separation between performance and stability issues is achieved in some sense. A terminal state penalty matrix P and a terminal region Ω can be determined off-line in a straightforward manner, essentially involving the solution of a linear stabilization problem and a Lyapunov equation. This is outlined in the procedure given in the paper.

The main advantage of this scheme is its guaranteed asymptotic stability. In addition, the quasi-infinite horizon nonlinear MPC scheme is computationally more attractive than other known nonlinear MPC schemes that also guarantee asymptotic stability (terminal equality constraint, infinite horizon). This was also demonstrated with the control of the unstable and constrained system in the example.

The results presented in this paper must however be viewed only as a further step towards a practical nonlinear MPC theory. As usual we have assumed that there is no model/plant mismatch, that no disturbances are acting on the system and that the whole state vector can be measured. We do however not need to assume that at every step the globally optimal input profile is found numerically. Stability does only require feasible solutions to the optimization problem. It should however be pointed out that the given conditions for nominal asymptotic stability are only sufficient.

Current investigations focus on robustness properties of this control scheme, on a further reduction of the computational burden and on a generalization of the setup to include more general objective functionals and additional state constraints.

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