A quasi-separation theorem for LQG optimal control with IQ constraints*

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Abstract

We consider the deterministic, the full observation and the partial observation LQG optimal control problems with finitely many IQ (integral quadratic!) constraints, and show that Wohnam's famous Separation Theorem for stochastic control has a generalization to this case. Although the problems of filtering and control are not independent, we show that the interdependence of these two problems is so superficial that in effect, they are problems which can be treated separately. It is in this context that the label Quasi-Separation Theorem is to be understood. We conclude with a discussion of computation issues and show how gradient-type optimization algorithms can be used to solve these problems. In this way, a systematic computation algorithm is derived.

1 Introduction

In this paper, we consider the deterministic, the full observation and the partial observation LQG optimal control problems subject to IQ (integral quadratic) constraints. In the unconstrained case, Wohnam's Separation Theorem [9] is a towering result. It states that in the unconstrained partially observed case, the optimal control is obtained by solving both a filtering problem and a control problem, and that these two problems can be solved separately. In [5, 6], it is shown that a Separation Theorem holds in the case of linear integral constraints. In this paper, we generalize these results to the case of IQ constraints.

We show that unlike the unconstrained or the linearly constrained cases, a Separation Theorem in the true sense of the word does not hold for the LQG problem subject to IQ constraints. Although the optimal control is calculated by solving a control problem and a filtering problem, the control and filtering problems can not be solved separately - the solution of the control problem is dependent on the solution of the filtering Riccati equation. However, this dependence adds no complication to the control or the filtering problems. It is in this context that the label Quasi-Separation Theorem should be understood.

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In the literature, it has been said that the use of multiple-objective LQG (and in particular, LQG control subject to IQ constraints) in practise has been limited because it is computationally extensive. We conclude this paper by examining computational issues. We show that the optimal control can be calculated by solving a certain finite dimensional optimization problem referred to in the literature as an optimal parameter selection problem [8]. Optimal parameter selection problems can be solved using standard gradient-type optimization algorithms so long as the gradient of the cost functional can be calculated. In this section, we show how this cost functional gradient can be determined. In the deterministic case, it is shown in [4] that calculating the gradient is an easy problem: it is equivalent to solving an unconstrained LQ optimal control problem. We focus on the optimal parameter selection problem for the full observation and the partial observation cases, and show that problem of calculating the gradient is equivalent to solving an unconstrained full observation or partial observation LQG problem respectively. It follows from the Separation Theorem for unconstrained LQG that there is a Separation Theorem for calculating the gradient. We also derive an alternative form of the gradient that is easily to calculate. We show that the gradient can be calculated by solving a first order matrix differential equation in addition to an unconstrained LQ problem. Thus, the gradient can be easily calculated and efficient optimization algorithms can be used to solve the optimal parameter selection problem which in turn, gives the optimal control. It is appropriate to mention here that the software package MISER 3.1 [3] is designed to solve optimization problems of this type.

2 Deterministic Case

In this section, we summarize some relevant results from [4]. Assume that $T < \infty$ and denote by $L_2^n[0,T]$ the Hilbert space of $\mathbb{R}^n$-valued, measurable square integrable functions on $[0,T]$ with inner product

$$\langle x, y \rangle \quad = \quad \int_0^T x_t' \cdot y_t \, dt \tag{1}$$

Let $\xi \in \mathbb{R}^n$ be a given vector. Suppose that $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are continuous matrix valued functions on $[0,T]$. For every $u \in L_2^m[0,T]$, define $x \in L_2^n[0,T]$ as the solution of the linear system

$$\dot{x}_t \quad = \quad A(t)x_t + B(t)u_t, \quad x_0 \; = \; \xi \tag{2}$$

Let $z = (\bar{z}, 0) \in L_2^n[0,T] \times L_2^m[0,T]$ where $\bar{z}$ is the solution of the differential equation (2) with $u_t = 0$. Define the set

$$\mathcal{Y} \quad = \quad \{(x,u) \in L_2^n[0,T] \times L_2^m[0,T] : \dot{x}_t = A(t)x_t + B(t)u_t, x_0 = 0\} \tag{3}$$

It follows then that $z + \mathcal{Y}$ is the set of solutions of the linear system (2). Moreover, $\mathcal{Y}$ is a closed subspace of $L_2^n[0,T] \times L_2^m[0,T]$ and therefore, $z + \mathcal{Y}$ is an affine subspace of $L_2^n[0,T] \times L_2^m[0,T]$.

Define the functionals $f_i : L_2^n[0,T] \times L_2^m[0,T] \to \mathbb{R}$, $i = 1, \cdots, N$ by

$$f_i(x,u) \quad = \quad \frac{1}{2}\int_0^T (x_t' Q_i(t) x_t + u_t' R_i(t) u_t)\, dt + \frac{1}{2} x_T' H_i x_T + \int_0^T (a_i'(t) x_t + b_i'(t) u_t)\, dt + h_i' x_T \tag{4}$$

where H_i, $Q_i(t) \in \mathbb{R}^{n\times n}$, $R_i(t) \in \mathbb{R}^{m\times m}$, $a_i(t) \in \mathbb{R}^n$, $b_i(t) \in \mathbb{R}^m$ are continuous functions of $t \in [0,T]$ and $R_i(t) \geq 0$ $(i = 1, \cdots, N)$, $Q_i(t) \geq 0$ and $H_i \geq 0$ $(i = 0, \cdots, N)$ and $R_0(t) > 0$ for each $t \in [0,T]$. Note that this allows for the case of linear integral constraints. Let $c_i \in \mathbb{R}$ $i = 1, \cdots N$ be given constants. The deterministic LQ optimal control problem subject to IQ constraints can be stated as follows:

$$\left\{ \begin{array}{c} f_0(x,u) \to \min \\ f_i(x,u) \le c_i, \quad i = 1, \cdots, N \\ (x,u) \in z + \mathcal{Y} \end{array} \right. \tag{5}$$

We make the following assumption:

Assumption 2.1 *There exists $(x,u) \in L_2^n[0,T] \times L_2^m[0,T]$ which is feasible for (5).*

We have the following result on the existence of an optimal solution (x^*, u^*) for (5).

Theorem 2.1 *Suppose that Assumption 2.1 is true. Then there exists a unique optimal solution (x^*, u^*) of (5).*

Proof: Let (x,u) be feasible for (5) and $\sigma = f_0(x,u)$. Define

$$H_\sigma \;=\; \{(x,u) \in L_2^n[0,T] \times L_2^m[0,T] : f_0(x,u) \le \sigma\}$$

and consider the problem

$$\left\{ \begin{array}{c} f_0(x,u) \to \min \\ f_0(x,u) \le \sigma \\ f_i(x,u) \le c_i, \quad i = 1, \cdots, N \\ (x,u) \in z + \mathcal{Y} \end{array} \right. \tag{6}$$

Since $f_0(x,u)$ is strictly convex on $L_2^n[0,T] \times L_2^m[0,T]$, the constraints of (6) define a bounded, closed, convex subset of the Hilbert space $L_2^n[0,T] \times L_2^m[0,T]$. Since every continuous, convex functional defined on a Hilbert space achieves its minimum on every bounded, closed, convex set [1, Theorem 2.6.1], it follows that there exists (x^*, u^*) which is optimal for (6). Furthermore, the uniqueness of (x^*, u^*) follows from the strict convexity of $f_0(x,u)$. Now we show that (x^*, u^*) is optimal for (5). Note first that (x^*, u^*) is feasible for (5). Suppose that (x^*, u^*) is not optimal for (5). Then there exists $(\bar{x}, \bar{u})$ which is feasible for (5) and $f_0(\bar{x}, \bar{u}) < f_0(x^*, u^*) \le \sigma$. However, this implies that $(\bar{x}, \bar{u})$ is feasible for (6) and hence, it follows from the optimality of (x^*, u^*) for (6) that $f_0(x^*, u^*) \le f_0(\bar{x}, \bar{u})$ - a contradiction. Therefore, $f_0(x^*, u^*) \le f_0(\bar{x}, \bar{u})$ for every feasible solution $(\bar{x}, \bar{u})$ of (5). The result follows. ■

We summarize the results obtained in [4]. For every $\lambda = (\lambda_1, \cdots, \lambda_N) \ge 0$ and $\zeta = (x,u) \in z + \mathcal{Y}$ let

$$g(\zeta, \lambda) \;=\; f_0(\zeta) + \sum_{i=1}^{N} \lambda_i \, f_i(\zeta, \lambda) \tag{7}$$

and define the Lagrangian

$$L(\zeta, \lambda) \;=\; g(\zeta, \lambda) - \lambda' c \tag{8}$$

Also, we shall denote

$$Q(t,\lambda) \;=\; \sum_{i=1}^{N} \lambda_i Q_i(t) \qquad H(\lambda) \;=\; \sum_{i=1}^{N} \lambda_i H_i, \qquad a(t,\lambda) \;=\; \sum_{i=1}^{N} \lambda_i a_i(t)$$

with a similar interpretation for $R(t,\lambda), b(t,\lambda)$ and $h(\lambda)$. We have the following result

Proposition 2.1 *Let $\lambda \geq 0$ be given and $\zeta(\lambda) = (x(\lambda), u(\lambda))$ denote the optimal state-control pair for the problem*

$$\left\{ \begin{array}{l} \min_\zeta g(\zeta, \lambda) \\ \\ \zeta \in z + \mathcal{Y} \end{array} \right. \tag{9}$$

Then the optimal control $u(\lambda)$ is

$$u_t(\lambda) \quad = \quad -R^{-1}(t, \lambda) \left[B'(t) P(t, \lambda) x_t + B'(t) d(t, \lambda) + b(t, \lambda) \right] \tag{10}$$

and $x(\lambda)$ is the solution of (2) with $u = u(\lambda)$. The optimal cost is

$$g(\zeta(\lambda), \lambda) \quad = \quad \frac{1}{2}\xi' P(0, \lambda)\xi + d'(0, \lambda)\xi + \frac{1}{2}p(0, \lambda) \tag{11}$$

Proof: Let $\lambda \geq 0$ be fixed. Then (9) is just an unconstrained LQ optimal control problem, and the result follows immediately. ∎

We make the following assumption:

Assumption 2.2 *For every $\lambda_i \geq 0$, $i = 1, \cdots, N$ (not all equal to zero), there exists $(x, u) \in z + \mathcal{Y}$ such that*

$$\sum_{i=1}^{N} \lambda_i (f_i(x, u) - c_i) \quad < \quad 0$$

Remark 2.1 *A sufficient condition for Assumption 2.2 to hold is the existence of $(\bar{x}, \bar{u}) \in z + \mathcal{Y}$ such that $f_i(\bar{x}, \bar{u}) < c_i$, for $i = i, \cdots, N$.*

Theorem 2.2 *Let $g(\zeta(\lambda), \lambda)$ be given by (11). Then there exists a $\lambda^* \geq 0$ which is optimal for*

$$\max_\lambda \{ g(\zeta(\lambda), \lambda) - \lambda' c \} \tag{12}$$

$$\dot{P} \quad = \quad -PA - A'P + PBR^{-1}(\lambda)B'P - Q(\lambda), \quad P(T) \ = \ H(\lambda) \tag{13}$$

$$\dot{d} \quad = \quad -\left[A - BR^{-1}(\lambda)\, B'P\right] d - a(\lambda) + PBR^{-1}(\lambda)\, b(\lambda), \quad d(T) \ = \ h(\lambda) \tag{14}$$

$$\dot{p} \quad = \quad \left[B'd + b(\lambda)\right]' R^{-1}(\lambda) \left[B'd + b(\lambda)\right], \quad p(T) \ = \ 0 \tag{15}$$

$$\lambda \quad \geq \quad 0 \tag{16}$$

Furthermore, the optimal control u^* of (5) exists and is given by

$$u_t^* \quad = \quad -R^{-1}(t, \lambda^*) \left[B'(t) P(t, \lambda^*) x_t + B'(t) d(t, \lambda^*) + b(t, \lambda^*) \right] \tag{17}$$

Proof: This is an immediate consequence of the Lagrange Duality Theorem [7, Theorem 1, pp 224] which is true under Assumption 3.2. ∎

Remark 2.2 *The notation $P(t, \lambda^*)$ is to interpreted as the solution of the Riccati equation (13) when $\lambda = \lambda^*$. A similar interpretation holds for $d(t, \lambda^*)$, $p(t, \lambda^*)$.*

From Theorem 2.2, it follows that the optimal control u^* for the LQ problem subject to IQ constraints is calculated by solving the finite dimensional optimization problem (12)-(16). In the literature, (12)-(16) is known as an optimal parameter selection problem, and can be solved using gradient-type optimization algorithms if the gradient of the cost functional (12) can be determined. We return to this issue in Section 5.

3 Full observation stochastic case

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $x = \{x_t : t \in [0,T]\}$, $u = \{u_t : t \in [0,T]\}$ be stochastic processes on $(\Omega, \mathcal{F}, P)$. Define the sets

$$\begin{aligned} X &= \left\{x : x_t \in L_2^n(\Omega, \mathcal{F}, P), \text{ and } E|x_t|^k \text{ is bounded for any } k > 0,\ t \in [0,T]\right\} \quad (18) \\ U &= \{u : u_t \in L_2^m(\Omega, \mathcal{F}, P) \text{ for all } t \in [0,T] \text{ and } E\int_0^T |u_t|^k dt < \infty \text{ for all } k > 0\} \quad (19) \end{aligned}$$

Let $\{\mathcal{F}_t\}$ be an increasing family of σ-algebras such that $\mathcal{F}_t \subset \mathcal{F}$. Let $\{W_t : t \in [0,T]\}$ be a standard Brownian motion such that W_t is an $\mathbb{R}^j$-valued random variable. Assume that W is adapted to $\{\mathcal{F}_t\}$. Let

$$\begin{aligned} \mathcal{X} &= \{x \in X : x \text{ non-anticipative with respect to } \{\mathcal{F}_t\}\} \quad (20) \\ \mathcal{U} &= \{u \in U : u \text{ non-anticipative with respect to } \{\mathcal{F}_t\}\} \quad (21) \end{aligned}$$

For every $u \in \mathcal{U}$, let $x \in \mathcal{X}$ be the solution of the stochastic differential equation

$$dx_t = A(t)x_t dt + B(t)u_t dt + C(t)dW_t, \quad x_0 \sim N(\xi, \Sigma_0) \quad (22)$$

with $A(t), B(t)$ as in (2) and $C(t)$ an $\mathbb{R}^{n\times j}$-valued continuous function. We assume that x_0 and $\{W_t\}$ are mutually independent. Let

$$\mathcal{Y} = \{(x,u) \in \mathcal{X} \times \mathcal{U} : dx_t = A(t)x_t dt + B(t)u_t dt, x_0 = 0\} \quad (23)$$

and $z = (\bar{z}, 0) \in \mathcal{X} \times \mathcal{U}$ where $\bar{z}$ is the solution of the stochastic differential equation

$$d\bar{z}_t = A(t)\bar{z}_t dt + C(t)dW_t, \quad \bar{z}_0 \sim N(\xi, \Sigma_0) \quad (24)$$

Then $z + \mathcal{Y}$ is an affine subspace of $\mathcal{X} \times \mathcal{U}$, and is the set of solutions of (22). Define the functionals $f_i : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, $i = 0, \cdots, N$ by

$$f_i(x,u) = E\left[\frac{1}{2}\int_0^T [x_t' Q_i(t) x_t + u_t' R_i(t) u_t]\, dt + \frac{1}{2}\, x_T' H_i x_T + \int_0^T [a_i'(t)x_t + b_i'(t)u_t]\, dt + h_i' x_T\right] \quad (25)$$

where $Q_i(t), R_i(t), a_i(t), b_i(t)$ are the same as in (4). The full information LQG problem subject to IQ constraints is

$$\begin{cases} f_0(x,u) \to \min \\ f_i(x,u) \le c_i, \quad i = 1, \cdots, N \\ (x,u) \in z + \mathcal{Y} \end{cases} \quad (26)$$

Analogous to Assumption 2.1, we make the following assumption:

Assumption 3.1 *There exists $(x,u) \in \mathcal{X} \times \mathcal{U}$ which is feasible for (26).*

The following result for the existance of an optimal solution (x^*, u^*) for (26) follows from Assumption (3.1). It can be proved in exactly the same manner as Theorem 2.1.

Theorem 3.1 *Suppose that Assumption 3.1 holds. Then there exists a unique optimal solution (x^*, u^*) of (26).*

Let $\zeta = (x, u)$. We define, as in the deterministic case, the Lagrangian $L(\zeta, \lambda)$ by (8) and $g(\zeta, \lambda)$ by (7) where $f_i(\zeta)$ is now given my (25). We have the following result:

Proposition 3.1 *For every $\lambda \geq 0$, there exists a unique solution $\zeta(\lambda) = (x(\lambda), u(\lambda))$ for the problem*

$$\begin{cases} \min_\zeta g(\zeta, \lambda) \\ \zeta \in z + \mathcal{Y} \end{cases}$$

The optimal control $u(\lambda)$ is

$$u_t(\lambda) \quad = \quad -R^{-1}(t, \lambda)\left[B'(t)P(t, \lambda)x_t + B'(t)d(t, \lambda) + b(t, \lambda)\right] \tag{27}$$

and $x(\lambda)$ is the solution of (22) with $u = u(\lambda)$. The optimal cost is

$$g(\zeta(\lambda), \lambda) \quad = \quad \frac{1}{2}\xi'\, P(0, \lambda)\, \xi + d'(0, \lambda)\, \xi + \frac{1}{2} p(0, \lambda) + \frac{1}{2}\int_0^T tr\{C'(t)\, P(t, \lambda)\, C(t)\}\, dt \tag{28}$$

Proof: Suppose that $\lambda \geq 0$ is fixed. Let $X_t = \sigma\{x_s : s \in [0, t]\}$ and

$$\mathcal{V} = \{u \in U : u_t \text{ is measurable with respect to } X_t\}$$

Then the optimal control for the problem

$$\begin{cases} \min_{(x,u)} g((x, u), \lambda) \\ (x, u) \text{ satisfies (22)} \\ u \in \mathcal{V} \end{cases}$$

is given by (27). In [2, Corollary 4.1, pp 163] it is also shown that (27) is also optimal over the class $u \in \mathcal{U}$ and the result follows immediately. ∎

To solve (26) we need the following assumption:

Assumption 3.2 *For every $\lambda_i \geq 0$, $i = 1, \cdots, N$ (not all equal to zero), there exists $(x, u) \in z + \mathcal{Y}$ such that*

$$\sum_{i=1}^{N} \lambda_i (f_i(x, u) - c_i) \quad < \quad 0$$

Under Assumption 3.2, we have the following result:

Theorem 3.2 *Let $g(\zeta(\lambda), \lambda)$ be given by (28). Then there exists a $\lambda^* \geq 0$ which is optimal for the problem*

$$\begin{cases} \max_\lambda \{g(\zeta(\lambda), \lambda) - \lambda' c\} \\ \text{subject to: (13) − (15)} \\ \lambda \geq 0 \end{cases} \tag{29}$$

Furthermore, the optimal control for (26) is

$$u_t^* \quad = \quad -R^{-1}(t, \lambda^*)\left[B'(t)P(t, \lambda^*)x_t + B'(t)d(t, \lambda^*) + b(t, \lambda^*)\right] \tag{30}$$

Proof: This is an immediate consequence of the Lagrange Duality Theorem [7, Theorem 1, pp 224] which is true under Assumption 3.2. ∎

Unlike the unconstrained [9] and the linearly constrained [5, 6] cases, certainty equivalence does not hold in the full-information LQG problem subject to IQ constraints because the optimization problems (12)-(16) and (29) do not generally have the same optimal solution. This is due to the dependence of λ^* (and hence the solution of the control problem) on the intensity of the channel noise, as expressed by $C(t)$. In the unconstrained case, it is shown by Wohnam [9] that dependence of the control problem on the intensity of the channel noise will generally occur when the cost functional is not quadratic. On the issue of calculating λ^*, the problem (29) (as in the deterministic case (12)-(16)) is an optimal parameter selection problem. This is a finite dimensional optimization problems over $\lambda \in \mathbb{R}^N$ and can be solved using gradient-type optimization algorithms so long as the gradient of the cost functional $g(\zeta(\lambda), \lambda) - \lambda' c$ as given in (29), with respect to the parameter λ can be calculated.

4 Partial observation stochastic case

The following assumptions are made in addition to the ones for the full observation case in Section 3. Let $F(t)$ and $G(t)$ be continuous matrix valued valued functions of $t \in [0, T]$ such that $F(t) \in \mathbb{R}^{p \times n}$ and $G(t) \in \mathbb{R}^{p \times k}$. Let $\{V_t : t \in [0, T]\}$ be a standard Brownian motion such that V_t is an $\mathbb{R}^k$-valued random variable for every $t \in [0, T]$. We assume that x_0, $\{V_t\}$ and $\{W_t\}$ are mutually independent. Consider the partially observed linear system

$$
\begin{aligned}
dx_t &= A(t)x_t dt + B(t)u_t dt + C(t)dW_t, \quad x_0 \sim N(\xi, \Sigma_0) &\quad (31)\\
dy_t &= F(t)x_t dt + G(t)\, dV_t, \quad y_0 = 0 &\quad (32)
\end{aligned}
$$

The class of feasible controls for the system (31) is defined as in [2, 9] for the unconstrained partially observed LQG problem: let $(C^p[0, T], \|\cdot\|)$ be the Banach space of continuous $\mathbb{R}^p$-valued functions on $[0, T]$ with the sup norm $\|\cdot\|$ defined by

$$
\|g\| = \sup_{t \in [0,T]} |g(t)|, \quad g \in C^p[0, T]
$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^p$. For every $t \in [0, T]$, define the operator $\pi_t : C^p[0, T] \to C^p[0, T]$ by

$$
(\pi_t\, g)(s) = \begin{cases} g(s), & s \in [0, t] \\ g(t), & s \in [t, T] \end{cases}
$$

Let Ψ be the set of functions $\psi : [0, T] \times C^p[0, T] \to \mathbb{R}^m$ which satisfy the following properties:

1. For every $\psi \in \Psi$, there exists $K_\psi \in \mathbb{R}$ such that $|\psi(t, g) - \psi(t, h)| \leq K_\psi \|g - h\|$ for every $g, h \in C^p[0, T]$ and $t \in [0, T]$. (Uniform Lipschitz condition).
2. $\psi(\cdot, \cdot)$ is Borel measurable.
3. $\psi(t, 0)$ is bounded.

The class of feasible controls is the set

$$
\bar{\mathcal{U}} = \{u \in U : \text{there exists } \psi \in \Psi \text{ with } u_t = \psi(t, \pi_t y) \text{ for every } t \in [0, T],\ y \text{ given by (32)}\} \quad (33)
$$

where U us given by (19). Let $f_i(x,u)$ be defined as in (25). The partially observed LQG optimal control problem subject to IQ constraints can be defined as follows:

$$\left\{\begin{array}{c} \min f_0(x,u) \\ f_i(x,u) \leq c_i, \quad i=1,\cdots,N \\ (x,u) \text{ satisfies } (31), u \in \bar{\mathcal{U}} \end{array}\right. \tag{34}$$

As with the full observation case, we need the following assumptions:

Assumption 4.1 *There exists (x,u) such that $u \in \bar{\mathcal{U}}$, x satisfies (31) and $f_i(x,u) \leq c_i$ for $i=1,\cdots,N$.*

Assumption 4.2 *For every $\lambda_i \geq 0$, $i=1,\cdots,N$ (not all zero), there exists (x,u) satisfying $u \in \bar{\mathcal{U}}$ and (31) such that*

$$\sum_{i=1}^{N} \lambda_i (f_i(x,u) - c_i) \quad < \quad 0$$

Using standard techniques, we can transform the partial observation problem (34) into a full observation problem of the form (26). To summarize this, we need the following basic results from Kalman filtering theory. For every $u \in \mathcal{U}$ let $\mathcal{G}_t^u = \sigma\{y_s : s \in [0,t]\}$ be the σ-field generated by the output y of (31)-(32) when the input of (31) is u. Let $\mathcal{G}_t^0$ correspond to the case when $u=0$. The following result is proven in [2].

Lemma 4.1 *If u_t is $\mathcal{G}_t^0$-measurable and $\mathcal{G}_t^0 = \mathcal{G}_t^u$, then the conditional distribution of x_t given $\mathcal{G}_t^u$ is Gaussian with mean $\hat{x}_t = E[x_t|\mathcal{G}_t^u]$ and covariance $\Sigma(t)$ where*

$$d\hat{x}_t \quad = \quad A(t)\hat{x}_t dt + B(t)u_t dt - \Sigma(t)F(t)\left(G(t)G'(t)\right)^{-1} d\nu_t, \quad \hat{x}_0 = \xi \tag{35}$$

$$\dot{\Sigma} \quad = \quad \Sigma A + A'\Sigma - \Sigma F' \left(GG'\right)^{-1} F\Sigma + CC', \quad \Sigma(0) \ = \ \Sigma_0 \tag{36}$$

and the innovations process ν is given by $d\nu_t = dy_t - F(t)\hat{x}_t dt = G(t)d\hat{w}_t$ where $\hat{w}$ is a Brownian motion adapted to $\{\mathcal{G}_t^0\}$ and satisfies

$$E\left[\nu_t\right] = 0, \qquad E[\nu_t\, \nu_t'] = \int_0^t G(s)\, G'(s)\, ds, \qquad E\left[\nu_t \hat{x}_t'\right] = 0.$$

Furthermore the optimal state estimate, and the optimal state estimate error are orthogonal; that is

$$E[(x_t - \hat{x}_t)\,\hat{x}_t'] \quad = \quad 0$$

In particular, when $u \in \bar{\mathcal{U}}$, the conditions of Lemma 4.1 are satisfied. Using the results in Lemma 4.1, it is easy to show that

$$f_i(x,u) = f_i(\hat{x},u) + \frac{1}{2}\int_0^T \operatorname{tr}\{Q_i(t)\Sigma(t)\}dt + \frac{1}{2}\operatorname{tr}\left\{H_i\Sigma(T)\right\} \tag{37}$$

Denote

$$\hat{c}_i \quad = \quad c_i - \frac{1}{2}\int_0^T \operatorname{tr}\{Q_i(t)\,\Sigma(t)\}dt - \frac{1}{2}\operatorname{tr}\left\{H_i\Sigma(T)\right\} \tag{38}$$

It follows that the partially observed LQG optimal control problem subject to IQ constraints is equivalent to the following full observation problem:

$$\left\{\begin{array}{c} f_0(\hat{x},u) \to \min \\ f_i(\hat{x},u) \leq \hat{c}_i, \quad i=1,\cdots,N \\ (\hat{x},u) \text{ satisfies } (35), \ u \in \bar{\mathcal{U}} \end{array}\right. \tag{39}$$

We are now in the position to prove the following generalization of the Separation theorem.

Theorem 4.1 (Quasi-Separation Theorem) *Let $g(\hat{\zeta}(\lambda), \lambda)$ be given by (41). Then there exists a $\lambda^* \geq 0$ which is optimal for the problem:*

$$\left\{ \begin{array}{c} \max_\lambda \{g(\hat{\zeta}(\lambda), \lambda) - \lambda' \hat{c}\} \\ \text{Subject to: } (13) - (15) \\ \lambda \geq 0 \end{array} \right. \tag{40}$$

where $g(\hat{\zeta}(\lambda), \lambda)$ is given by

$$g(\hat{\zeta}(\lambda), \lambda) = \frac{1}{2}\xi' P(0,\lambda)\xi + d'(0,\lambda)\xi + \frac{1}{2}p(0,\lambda) + \frac{1}{2}\int_0^T tr\{, '(t)\, P(t,\lambda), , (t)\}\, dt \tag{41}$$

with

$$, (t) = \Sigma(t)F(t)(G(t)G'(t))^{-1}G(t)$$

Furthermore the optimal control for (34) exists and is given by

$$u_t^* = -R^{-1}(t,\lambda^*)\,[B'(t)P(t,\lambda^*)\hat{x}_t + B'(t)d(t,\lambda^*) + b(t,\lambda^*)] \tag{42}$$

where $\hat{x}_t$ is the solution of (35) with u_t given by (42).

Proof: It is shown in [2, Lemma 11.3, pp 191] that if $u \in \bar{\mathcal{U}}$, then the conditions of Lemma 4.1 are satisfied; that is u is non-anticipative with respect to $\{\mathcal{G}_t^0\}$ and $\mathcal{G}_t^0 = \mathcal{G}_t^u$. Consider the full observation problem

$$\left\{ \begin{array}{c} f_0(\hat{x}, u) \to \min \\ f_i(\hat{x}, u) \leq \hat{c}_i, \quad i = 1, \cdots, N \\ (\hat{x}, u) \text{ satisfies } (35),\ u_t\ \mathcal{G}_t^0\text{–measurable} \end{array} \right. \tag{43}$$

Then (43) is exactly of the form (26). Moreover, the class of feasible controls for (43) contains the set $\bar{\mathcal{U}}$. By Assumption 4.1, the problem (43) satisfies the conditions of Theorem 3.2, so there exists a unique optimal state-control pair (x^*, u^*) for (43).

For every $\lambda \geq 0$, we can define the functional $g(\hat{x}, u, \lambda)$ as in (7) with $f_i(\hat{x}, u)$ given by (25). Under Assumption 4.2, the conditions for Theorem 3.2 are satisfied for (43) and the optimal parameter selection problem associated with the problem (43) is given by (40). Moreover, the optimal control u^* for (43) is given by (42) where λ^* is the optimal solution of (40). Since $u^* \in \bar{\mathcal{U}}$, it follows from $\bar{\mathcal{U}} \subset \{u \in \mathcal{U} :$ u non-anticipative with respect to $\mathcal{G}_t^0\}$ that u^* is optimal for (39) and hence, optimal for (34). ■

The reader should note the following. First, certainty equivalence does not hold. This is no surprise since certainty equivalence does not hold in the full observation case because the optimal control is dependent on the intensity of the channel noise. As can be seen in (41), channel noise intensity as given by $C(t)$ and $G(t)$ effect the solution of the control problem in the partial observation case. As stated earlier, it is shown in [9] that dependence of the control problem on the channel noise intensity in the unconstrained case generally occurs when the cost functional is not quadratic.

A second, more important observation is that the Separation Theorem does not hold in the sense of [9]. To see this, the reader should observe (38), (41) and (40). From these equations, it is clear that the solution λ^* of (40) is dependent on the error covariance $\Sigma(t)$ associated with the filtering problem (35)-(36). Thus the problems of filtering and control are not separate. However, when solving (40) the dependence of λ^*

(and hence the solution of the control problem) on $\Sigma(t)$ adds no complication. Since $\Sigma(t)$ is independent of λ, it needs to be calculated only once, and the optimization problem (40) may be solved with no further re-calculation of $\Sigma(t)$ (and hence, with no further reference to the filtering problem). It is in this sense that the control and filtering problems are separate, and hence our naming Theorem 4.1 a Quasi-Separation Theorem.

5 Optimal parameter selection problems and a separation theorem for gradient calculations

In view of Theorems 2.2, 3.2 and 4.1, the optimal control for the deterministic, the full observation and the partial observation LQG problems with IQ constraints is obtained by solving the finite dimensional optimization problems (12)-(16), (29) and (40) respectively. In the literature, these finite dimensional optimization problems are known as optimal parameter selection problems, and the interested reader may refer to [8] for more details. Optimal parameter selection problems can be solved as mathematical programming problems (using efficient gradient-type optimization algorithms) so long as the value of the cost functional and gradient of the cost functional can be calculated for any given λ. In this section, we derive the gradient of the cost functional for the problems (12)-(16), (29) and (40).

In the optimal parameter selection problem for the deterministic LQ case, a key part of calculating the gradient of the cost functional is solving an certain unconstrained LQ optimal control problem - this is proven in [4]. In the optimal parameter selection problems for the full observation and partial observation problems, we show that a similar result holds; that is, a key step in calculating the gradient of the cost functional is solving an unconstrained full observation and partial observation LQG problem respectively. By virtue of the Separation theorem for unconstrained LQG, it follows that there is a Separation theorem associated with calculating the gradient.

We shall calculate the gradient of the optimal parameter selection problems (12)-(16), (29) and (40) by working with a general optimization problem with quadratic cost and quadratic constraints. The LQG problems as stated in (5) and (26) are special cases of problems of this form and the partially observed LQG problem is solved by transforming it into a problem of this type (namely, a full observation problem). The general problem which we shall work with can be stated as follows: Let X be a Hilbert space with inner product$\langle \cdot, \cdot \rangle$ and Ω a closed subspace of X. Let $Q_i : X \to X$, $i = 0, \cdots, N$ be symmetric operators such that $Q_0 > 0$ and $Q_i \geq 0$ for $i = 1, \cdots, N$. Let $a_i \in X$ for $i = 1, \cdots, N$. The general quadratic optimization problem is

$$\left\{ \begin{array}{c} f_0(x) \to \min \\ f_i(x) \leq c_i, \quad i = 1, \cdots, N \\ x \in \Omega \end{array} \right. \tag{44}$$

where $f_i : X \to \mathbb{R}$ are the linear-quadratic cost functionals

$$f_i(x) \quad = \quad \frac{1}{2} \langle Q_i x, x \rangle + \langle a_i, x \rangle \tag{45}$$

The optimal parameter selection problems (12)-(16), (29) and (40) correspond to the dual problem of (44) with $f_i(x)$ given by (45). We shall derive the gradient of the cost functional of the optimal parameter selection problems by examining the dual problem associated with (44). The dual problem associated with (44) is an optimization problem over $\lambda \in \mathbb{R}^N$ and may be stated as follows:

$$\left\{ \begin{array}{c} J(\lambda) = \frac{1}{2} \langle Q(\lambda) \cdot x, x \rangle + \langle a(\lambda), x \rangle \to \max \\ Q(\lambda) \cdot x + a(\lambda) \in X^{\perp} \\ x \in X, \ \lambda \geq 0 \end{array} \right. \tag{46}$$

It should be noted that for every $\lambda \in \mathbb{R}^N$, $\lambda \geq 0$, there is a unique $x(\lambda) \in X$ satisfying the constraint

$$Q(\lambda) \cdot x + a(\lambda) \quad \in \quad X^{\perp}$$

and that $x(\lambda)$ is the optimal solution of the following optimization problem over $x \in \Omega$

$$\left\{ \begin{array}{c} \frac{1}{2} \langle Q(\lambda) \cdot x, x \rangle + \langle a(\lambda), x \rangle \to \min \\ x \in \Omega \end{array} \right. \tag{47}$$

Moreover, $x(\lambda)$ is a smooth function of $\lambda \geq 0$. The optimal parameter selection problems (12)-(16), (29) and (40) correspond to the dual problem (46). The following theorem gives the gradient $\frac{dJ(\lambda)}{d\lambda}$ where $J(\lambda)$ is as stated in (46).

Theorem 5.1 *For every $\lambda \in \mathbb{R}^N$, $\lambda \geq 0$ the gradient of $J(\lambda)$ with respect to λ is*

$$\frac{dJ(\lambda)}{d\lambda} \quad = \quad \left[\frac{dJ(\lambda)}{d\lambda_1}, \cdots, \frac{dJ(\lambda)}{d\lambda_N} \right]'$$

$$\frac{dJ(\lambda)}{d\lambda_i} \quad = \quad f_i(x(\lambda)) - c_i \tag{48}$$

where $x(\lambda)$ is the unique $x \in X$ which satisfies the constraint $Q(\lambda) \cdot x + a(\lambda) \in X^{\perp}$ and $f_i(\lambda)$ is given by (45).

Proof: By the chain rule

$$\frac{dJ(\lambda)}{d\lambda_i} \quad = \quad \frac{\partial J(\lambda)}{\partial \lambda_i} + \frac{\partial J(\lambda)}{\partial x(\lambda)} \cdot \frac{\partial x(\lambda)}{\partial \lambda_i}$$

From (46), we obtain

$$\frac{\partial J(\lambda)}{\partial x(\lambda)} \quad = \quad Q(\lambda) \cdot x(\lambda) + a(\lambda) \ \in \ X^{\perp}$$

On the other hand, $x(\lambda) \in X$ for every $\lambda \geq 0$ and hence

$$\frac{\partial x(\lambda)}{\partial \lambda_i} \quad \in \quad X$$

It follows that for every $\lambda \geq 0$

$$\frac{\partial J(\lambda)}{\partial x(\lambda)} \cdot \frac{\partial x(\lambda)}{\partial \lambda_i} \quad = \quad 0$$

The result follows from the fact that

$$\frac{\partial J(\lambda)}{\partial \lambda_i} \quad = \quad \frac{1}{2} \langle Q_i \cdot x(\lambda), x(\lambda) \rangle + \langle a_i, x(\lambda) \rangle - c_i \ = \ f_i(x(\lambda)) - c_i$$

$\blacksquare$

Thus, for every $\lambda \geq 0$, the gradient of $J(\lambda)$ is obtained in the following way. First, we solve the unconstrained optimization problem (47) over $x \in \Omega$ and obtain the optimal solution $x(\lambda)$. Once $x(\lambda)$ has been obtained, the components of $\frac{dJ(\lambda)}{d\lambda}$ are obtained by evaluating the constraint functionals $f_i(x) - c_i$ at $x(\lambda)$. Since the cost functionals of the optimal parameter selection problems (12)-(16), (29) and (40) correspond to the cost functional $J(\lambda)$ of a dual problem of the form (46), we obtain the following result from Theorem 5.1.

Theorem 5.2 *Let $\lambda \in \mathbb{R}^N$, $\lambda \geq 0$ and $J(\lambda) = g(\zeta(\lambda), \lambda) - \lambda' c$ where $g(\zeta(\lambda), \lambda)$ is given by (11). The gradient of the cost functional $J(\lambda)$ evaluated at λ is*

$$\frac{dJ(\lambda)}{d\lambda} = \left[\frac{\partial J(\lambda)}{\partial \lambda_1}, \cdots, \frac{\partial J(\lambda)}{\partial \lambda_N}\right]$$

$$\frac{\partial J(\lambda)}{\partial \lambda_j} = \frac{1}{2}\int_0^T [\beta' Q_j \beta + \eta' R_j \eta] \, dt + \frac{1}{2}\beta'(T) H_j \beta(T) + \int_0^T [a_j' \beta + b_j' \eta] \, dt + h_p'(T) \beta(T) - c_j \quad (49)$$

where $\beta(t)$ is the solution of the equation

$$\dot{\beta}(t) = A(t) \beta(t) + B(t) \eta(t), \quad \beta(0) = \xi \quad (50)$$

with $\eta(t)$ given by

$$\eta(t) = -R^{-1}(\lambda, t) [B'(t) P(t) \beta(t) + B'(t) d(t) + b(\lambda, t)] \quad (51)$$

As stated before, the problem of calculating the gradient of $J(\lambda)$ is equivalent to solving an unconstrained LQ optiimal control problem, and evaluating the value of the constraints with this optimal control. This can be clearly seen from Theorem 5.1.

The optimal parameter selection problem (29) is the dual problem of the full observation problems (26). By Theorem 5.1, the gradient of the cost functionals of (29) and (40) is given as follows.

Theorem 5.3 *Let $\lambda \in \mathbb{R}^N$, $\lambda \geq 0$ be given. Then the cost functional of the problem (29) is of the form*

$$J(\lambda) = \frac{1}{2}\xi' P(0, \lambda)\xi + d'(0, \lambda)\xi + \frac{1}{2}p(0, \lambda) + \frac{1}{2}\int_0^T tr\{C'(t) P(t, \lambda), C(t)\} \, dt - \lambda' c \quad (52)$$

The gradient of $J(\lambda)$ evaluated at λ is

$$\frac{dJ(\lambda)}{d\lambda} = \left[\frac{\partial J(\lambda)}{\partial \lambda_1}, \cdots, \frac{\partial J(\lambda)}{\partial \lambda_N}\right]$$

$$\frac{\partial J(\lambda)}{\partial \lambda_j} = E\left[\frac{1}{2}\int_0^T [\beta' Q_j \beta + \eta' R_j \eta] \, dt + \frac{1}{2}\beta'(T) H_j \beta(T) + \int_0^T [a_j' \beta + b_j' \eta] \, dt + h_p'(T) \beta(T)\right] - c_j \quad (53)$$

where

$$d\beta_t = A(t)\beta_t dt + B(t)\eta_t dt + C_t dW_t, \quad x_0 \sim N(\xi, \Sigma_0) \quad (54)$$

with η_t given by

$$\eta_t = -R^{-1}(\lambda, t) [B'(t) P(t) \beta_t + B'(t) d(t) + b(\lambda, t)] \quad (55)$$

As in the deterministic case, the problem of calculating the gradient is equivalent to solving an unconstrained full observation LQG control problem, and evaluating the constraints with this optimal control.

The partially observed problem (34) is solved by transforming it into the full observation problem (43). By Theorem 5.1, the gradient of the cost functional of the optimal parameter selection problem (40) is obtained by solving an unconstrained, partially observed LQG problem. For this reason, there is a Separation theorem for calculating the gradient.

Theorem 5.4 (Gradient Separation Theorem) *Let $\lambda \in \mathbb{R}^N$, $\lambda \geq 0$ be given. Then the cost functional of the problem (40) is*

$$J(\lambda) \quad = \quad \frac{1}{2}\xi' P(0,\lambda)\xi + d'(0,\lambda)\xi + \frac{1}{2}p(0,\lambda) + \frac{1}{2}\int_0^T tr\{,'(t)\, P(t,\lambda), ,(t)\}\, dt - \lambda' c \tag{56}$$

where $,(t) = \Sigma(t)F(t)(G(t)G'(t))^{-1}G(t)$. *The gradient of* $J(\lambda)$ *evaluated at* λ *is*

$$\frac{dJ(\lambda)}{d\lambda} \quad = \quad \left[\frac{\partial J(\lambda)}{\partial \lambda_1}, \cdots, \frac{\partial J(\lambda)}{\partial \lambda_N}\right]$$

$$\frac{\partial J(\lambda)}{\partial \lambda_j} = E\left[\frac{1}{2}\int_0^T \left[\hat{\beta}'\, Q_j\, \hat{\beta} + \eta'\, R_j\, \eta\right]\, dt + \frac{1}{2}\,\hat{\beta}_T'\, H_j\, \hat{\beta}_T + \int_0^T \left[a_j'\, \hat{\beta} + b_j'\, \eta\right]\, dt + h_p'(T)\, \hat{\beta}_T\right] - c_j \tag{57}$$

where

$$d\hat{\beta}_t \quad = \quad A(t)\hat{\beta}_t dt + B(t)\eta_t dt - \Sigma(t)\, F(t)\, (G(t)\, G'(t))^{-1}\, d\nu_t, \qquad x_0 \;=\; \xi \tag{58}$$

$$\eta_t \quad = \quad -R^{-1}(\lambda,t)\left[B'(t)\, P(t)\, \hat{\beta}_t + B'(t)\, d(t) + b(\lambda,t)\right] \tag{59}$$

ν is the innovations process given by $d\nu_t = dy_t - F(t)\hat{\beta}_t dt$ and y is the solution of the stochastic differential equations

$$d\beta_t \quad = \quad A(t)\beta_t dt + B(t)\eta_t dt + C(t)dW_t, \qquad x_0 \;\sim\; N(\xi, \Sigma_0) \tag{60}$$

$$dy_t \quad = \quad F(t)\beta_t dt + G(t)\, dV_t, \qquad y_0 \;=\; 0 \tag{61}$$

where W and V are standard Brownian motions which satisfy the conditions stated in Section 4.

For computational purposes, the following expression for the gradient is the most useful. As observed in Theorem 4.1, when calculating the optimal control for the partially observed case, the problems of 'control' and 'filtering' are not independent but rather, the 'control' problem (namely, the optimal parameter selection problem (40)) depends on the solution $\Sigma(t)$ of the filtering Riccati equation. This dependence on $\Sigma(t)$ can be seen in the expression for the gradient of the cost functional of (40) which is stated in Theorem 5.5. Once again however, once $\Sigma(t)$ has been determined, the problem of calculating the gradient can be carried out independently of the 'filtering' problem and hence, the control problem can be solved independently of the filtering problem.

Theorem 5.5 *Let $K(t) = 0$ for the optimal parameter selection problem (12)-(16), $K(t) = C(t)$ for (29) and $K(t) = \Sigma(t)\, H(t)(G(t)\, G'(t))^{-1}F(t)$ for (40). Let $\lambda > 0$ be given. Then the gradient of the cost functional $J(\lambda)$ evaluated at λ for (12)-(16), (29) and (40) is*

$$\frac{dJ(\lambda)}{d\lambda} \quad = \quad \left[\frac{\partial J(\lambda)}{\partial \lambda_1}, \cdots, \frac{\partial J(\lambda)}{\partial \lambda_N}\right]$$

$$\frac{\partial J(\lambda)}{\partial \lambda_j} = \frac{1}{2}\int_0^T [\beta' Q_j \beta + \eta' R_j \eta]\, dt + \frac{1}{2}\beta'(T) H_j \beta(T) + \int_0^T [a_j' \beta + b_j' \eta]\, dt + h_p'(T)\beta(T) - c_j$$
$$+\frac{1}{2}\int_0^T tr\left\{\left(PBR^{-1}(\lambda) R_j R^{-1}(\lambda) B' P + Q_j\right)\Lambda_K\right\} dt + \frac{1}{2} tr\left\{\Lambda_K(T)\cdot H_j\right\} \quad (62)$$

where $\beta(t)$ is the solution of the costate equation

$$\dot{\beta}(t) = A(t)\beta(t) + B(t)\eta(t), \quad \beta(0) = \xi \quad (63)$$

with $\eta(t)$ given by

$$\eta(t) = -R^{-1}(\lambda, t)\,[B'(t) P(t) \beta(t) + B'(t) d(t) + b(\lambda, t)] \quad (64)$$

$\Lambda_K(t)$ *is the solution of*

$$\dot{\Lambda}_K = \left(A - BR^{-1}(\lambda)B'P\right)\Lambda_K + \Lambda_K\left(A - BR^{-1}(\lambda)B'P\right)' + KK', \quad \Lambda_K(0) = 0 \quad (65)$$

and $P(t)$, $d(t)$ are the solutions of the differential equations (13)-(14).

6 Conclusion

We have studied the LQ and LQG optimal control problems subject to IQ constraints. We have shown that the classic Separation Theorem result of Wohnam does not hold, but a generalization of this result which we call a Quasi-Separation Theorem is true. We show that the optimal control is determined by solving a finite dimensional optimization problem, and derive the gradient of its cost functional so that efficient algorithms for finite dimensional optimization problems can be used to calculate the optimal solution. We show that the problem of calculating the gradient is equivalent to solving an unconstrained LQG problem, and a Separation theorem for this gradient calculation (which we call a Gradient Separation Theorem) is proven. Since repeated calculations of the gradient are needed when implementing these optimization algorithms, the optimal control is calculated by solving a sequence of unconstrained LQG problems.

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