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A QUESTION OF C. C. YANG ON THE UNIQUENESS OF ENTIRE FUNCTIONS

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1. Introduction and Main Results

Let f and g be two nonconstant entire functions. If f and g have the same a-points with the same multiplicities, we denote this by $f=a\rightleftarrows g=a$ for simplicity's sake (see, [1]). It is assumed that the reader is familiar with the notations of the Nevanlinna Theory (see, for example, [2]). We denote by $S(r, f)$ any quantity satisfying $S(r, f)=o(T(r, f))$ as $r\rightarrow\infty$ except possibly for a set of r of finite linear measure.

M. Ozawa has proved the following theorem:

THEOREM A (see [1]). *Let f and g be entire functions of finite order. Assume that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $\delta(0, f)>1/2$. Then $f\cdot g\equiv 1$ unless $f\equiv g$.*

In [3] H. Ueda has shown that in Theorem A the order restriction of f and g can be removed. He proved the following theorem:

THEOREM B. *Let f and g be entire functions. Assume that $f=0\rightleftarrows g=0$, $f=1\rightleftarrows g=1$ and $\delta(0, f)>1/2$. Then $f\cdot g\equiv 1$ unless $f\equiv g$.*

In [4] C. C. Yang has asked: what can be said about the relationship between two entire functions f and g if $f=0\rightleftarrows g=0$ and $f'=1\rightleftarrows g'=1$?

In this paper we answer the question posed by C. C. Yang. In fact, we prove the following theorem:

THEOREM 1. *Let f and g be two nonconstant entire functions. Assume that $f=0\rightleftarrows g=0$, $f'=1\rightleftarrows g'=1$ and $\delta(0, f)>1/2$. Then $f'g'\equiv 1$ unless $f\equiv g$.*

The assumption "$\delta(0, f)>1/2$" in Theorem 1 is best possible. Indeed, consider

$$f(z)=-\frac{1}{2}e^{2z}-\frac{1}{2}e^{z}, \qquad g(z)=\frac{1}{2}e^{-2z}+\frac{1}{2}e^{-z}.$$

Then $f=0\rightleftarrows g=0$, $f'=1\rightleftarrows g'=1$ and $\delta(0, f)=1/2$. $f\not\equiv g$ and $f'\cdot g'\not\equiv 1$ are evident.

In place of Theorem 1, we prove more generally the following theorem

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which includes Theorem B and Theorem 1.

THEOREM 2. *Let f and g be two nonconstant entire functions. Assume that $f=0 \rightleftarrows g=0$, $f^{(n)}=1 \rightleftarrows g^{(n)}=1$ and $\delta(0, f)>1/2$, where n is a nonnegative integer. Then $f^{(n)} \cdot g^{(n)} \equiv 1$ unless $f \equiv g$.*

Theorem 2 is the best possible. Indeed, let

$$f(z)=-\frac{1}{2^n}e^{2z}+\frac{(-1)^{n+1}}{2^n}e^z,$$

$$g(z)=\frac{(-1)^{n+1}}{2^n}e^{-2z}-\frac{1}{2^n}e^{-z},$$

where n is a non-negative integer. It is easy to see that $f=0 \rightleftarrows g=0$, $f^{(n)}=1 \rightleftarrows g^{(n)}=1$ and $\delta(0, f)=1/2$, but $f \not\equiv g$ and $f^{(n)} \cdot g^{(n)} \not\equiv 1$. This shows that $\delta(0, f)>1/2$ is needed.

2. Some Lemmas

The following Lemmas will be needed in the proof of our theorems.

LEMMA 1 (see [2]). *Let f be a nonconstant entire function, n be a nonnegative integer. Then*

$$T(r, f^{(n)}) \leqq T(r, f)+S(r, f).$$

LEMMA 2. *Under the same conditions of Lemma 1, we have*

$$N\left(r, \frac{1}{f^{(n)}}\right) \leqq T(r, f^{(n)})-T(r, f)+N\left(r, \frac{1}{f}\right)+S(r, f).$$

Proof. We note that

$$m\left(r, \frac{1}{f}\right) \leqq m\left(r, \frac{1}{f^{(n)}}\right)+m\left(r, \frac{f^{(n)}}{f}\right)$$

$$=m\left(r, \frac{1}{f^{(n)}}\right)+S(r, f). \tag{1}$$

By the first fundamental theorem (see [2]). we have from (1),

$$T(r, f)-N\left(r, \frac{1}{f}\right) \leqq T(r, f^{(n)})-N\left(r, \frac{1}{f^{(n)}}\right)+S(r, f). \tag{2}$$

Thus

$$N\left(r, \frac{1}{f^{(n)}}\right) \leqq T(r, f^{(n)})-T(r, f)+N\left(r, \frac{1}{f}\right)+S(r, f), \tag{3}$$

which proves Lemma 2.

LEMMA 3. *Let g be a nonconstant entire function, n be a nonnegative integer. Then*

$$N\left(r, \frac{1}{g^{(n)}}\right) \leqq N\left(r, \frac{1}{g}\right) + S(r, g).$$

Proof. By Lemma 2 we have

$$N\left(r, \frac{1}{g^{(n)}}\right) \leqq T(r, g^{(n)}) - T(r, g) + N\left(r, \frac{1}{g}\right) + S(r, g).$$

From Lemma 1 we have

$$T(r, g^{(n)}) \leqq T(r, g) + S(r, g).$$

Hence

$$N\left(r, \frac{1}{g^{(n)}}\right) \leqq N\left(r, \frac{1}{g}\right) + S(r, g), \tag{4}$$

which proves Lemma 3.

LEMMA 4. *Assume that the conditions of Theorem 2 are satisfied. Then*

$$T(r, f) = O(T(r, f^{(n)})) \qquad r(\notin E),$$

$$T(r, g) = O(T(r, f^{(n)}) \qquad (r \notin E),$$

where E is a set of finite linear measure.

Proof. From (1) we get

$$(\delta(0, f) + o(1))T(r, f) \leqq T(r, f^{(n)}) + S(r, f).$$

Hence we have

$$T(r, f) \leqq \left(\frac{1}{\delta(0, f)} + o(1)\right) T(r, f^{(n)}) \qquad (r \notin E), \tag{5}$$

that is

$$T(r, f) = O(T(r, f^{(n)})) \qquad (r \notin E).$$

By Milloux's basic result (see, for example, [2, Theorem 3.2]), we have

$$T(r, g) < N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(n)} - 1}\right) + S(r, g). \tag{6}$$

We note that

$$\begin{aligned} N\left(r, \frac{1}{g}\right) = N\left(r, \frac{1}{f}\right) &\leqq (1 - \delta(0, f) + o(1)) T(r, f) \\ &\leqq (1 - \delta(0, f) + o(1)) \left(\frac{1}{\delta(0, f)} + o(1)\right) T(r, f^{(n)}) \\ &= \left(\frac{1}{\delta(0, f)} - 1 + o(1)\right) T(r, f^{(n)}) \qquad (r \notin E) \end{aligned} \tag{7}$$

and

$$N\Big(r, \frac{1}{g^{(n)}-1}\Big)=N\Big(r, \frac{1}{f^{(n)}-1}\Big)\leqq T(r, f^{(n)})+O(1). \tag{8}$$

From (6), (7), (8) we obtain

$$T(r, g)\leqq\Big(\frac{1}{\delta(0, f)}+o(1)\Big)T(r, f^{(n)})+S(r, g),$$

that is

$$T(r, g)=O(T(r, f^{(n)})) \qquad (r\notin E).$$

This completes the proof of Lemma 4.

LEMMA 5. *Let f_1 and f_2 be two nonconstant entire functions, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1f_1+c_2f_2\equiv c_3$, then*

$$T(r, f_1)<N\Big(r, \frac{1}{f_1}\Big)+N\Big(r, \frac{1}{f_2}\Big)+S(r, f_1).$$

Proof. By the second fundamental theorem (see [2]), we have

$$T(r, f_1)<N\Big(r, \frac{1}{f_1}\Big)+N\Big(r, \frac{1}{f_1-\frac{c_3}{c_1}}\Big)+S(r, f_1)$$

$$=N\Big(r, \frac{1}{f_1}\Big)+N\Big(r, \frac{1}{f_2}\Big)+S(r, f_1),$$

which proves Lemma 5.

LEMMA 6 (see [5], [6]). *Let $f_1, f_2, \cdots, f_n$ be linearly independent entire functions satisfying $\sum_{i=1}^{n} f_i\equiv 1$. Then for $j=1, 2, \cdots, n$ we have*

$$T(r, f_j)<\sum_{i=1}^{n} N\Big(r, \frac{1}{f_i}\Big)+O(\log r+\log T(r)) \qquad (r\notin E),$$

where $T(r)$ denotes the maximum of $T(r, f_i)$, $i=1, 2, \cdots, n$.

This is a special case of a result of R. Nevanlinna (see, [5, P_{116}]).

To prove our theorems, we also need the following result, which is interesting by itself.

LEMMA 7. *Let f_1, f_2 and f_3 be three entire functions satisfying*

$$\sum_{i=1}^{3} f_i\equiv 1. \tag{9}$$

If $f_1\not\equiv$constant, and

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) \leqq (\lambda + o(1))T(r) \qquad (r \notin E) \tag{10}$$

where $T(r) = \max_{i=1,2,3}\{T(r, f_i)\}$, *and* $\lambda < 1$, *then* $f_2 \equiv 1$ *or* $f_3 \equiv 1$.

Proof. Suppose neither f_2 nor f_3 are constants. If f_1, f_2 and f_3 are linearly independent, by Lemma 6 and (10) we have

$$T(r, f_j) < \sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) + o(T(r))$$

$$\leqq (\lambda + o(1))T(r) \qquad (r \notin E,\ j = 1, 2, 3)$$

and hence

$$T(r) \leqq (\lambda + o(1))T(r) \qquad (r \notin E) \tag{11}$$

which is impossible. If f_1, f_2 and f_3 are linearly dependent, there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$\sum_{i=1}^{3} c_i f_i \equiv 0 \tag{12}$$

Assume $c_1 \neq 0$, from (9), (12) we have

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 \equiv 1, \tag{13}$$

and

$$T(r, f_i) = (1 + o(1))T(r) \qquad (i = 1, 2, 3). \tag{14}$$

By Lemma 5 and (10), (13), (14) we also obtain (11), which is impossible. Assume $c_1 = 0$, from (9), (12) we have

$$f_1 + \left(1 - \frac{c_2}{c_3}\right) f_2 \equiv 1$$

and

$$T(r, f_i) = (1 + o(1))T(r) \qquad (i = 1, 2, 3),$$

giving a contradiction as before.

Suppose that $f_2 \equiv c (\neq 0)$. If $c \neq 1$, from (9) we have

$$f_1 + f_3 = 1 - c \tag{15}$$

and

$$T(r, f_i) = (1 + o(1))T(r) \qquad (i = 1, 2, 3).$$

By Lemma 5 and (10), (14), (15) we obtain (11), which is impossible. Therefore $c = 1$, that is, $f_2 \equiv 1$.

Suppose that $f_3 \equiv c\ (\neq 0)$. In a similar manner we get $f_3 \equiv 1$.

This completes the proof of Lemma 7.

LEMMA 8. *If, in addition to the assumptions of Theorem* 2, $f^{(n)}\equiv g^{(n)}$, *then* $f\equiv g$.

Proof. Suppose that $f\not\equiv g$. From $f^{(n)}\equiv g^{(n)}$, we have

$$f(z)=g(z)+p(z),$$

where $p(z)$ ($\not\equiv 0$) is a polynomial of degree at most $n-1$.

From $\delta(0, f)>0$ we know that f is a transcendental entire function. Thus we get

$$T(r, p)=o(T(r, f))$$

and

$$T(r, g)=(1+o(1))T(r, f).$$

By the second fundamental theorem (see, [2, Theorem 2.5]), we have

$$\begin{aligned} T(r, f) &< N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f-p}\right)+S(r, f) \\ &= N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{g}\right)+S(r, f) \\ &= 2N\left(r, \frac{1}{f}\right)+S(r, f) \\ &\leqq 2(1-\delta(0, f))T(r, f)+S(r, f). \end{aligned} \tag{16}$$

Since

$$2(1-\delta(0, f))<1,$$

so (16) is a contradiction. Hence $f\equiv g$.

3. Proof of Theorem 2

From $f^{(n)}=1 \rightleftarrows g^{(n)}=1$, we have

$$f^{(n)}-1=e^{\alpha}(g^{(n)}-1), \tag{17}$$

where α is a entire function.

Let $f_1=f^{(n)}$, $f_2=e^{\alpha}$, $f_3=-e^{\alpha}g^{(n)}$. From (17) we have

$$\sum_{i=1}^{3} f_i\equiv 1$$

and

$$\sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right)=N\left(r, \frac{1}{f^{(n)}}\right)+N\left(r, \frac{1}{g^{(n)}}\right). \tag{18}$$

By Lemma 2 and Lemma 4 we have

$$N\left(r, \frac{1}{f^{(n)}}\right) \leqq T(r, f^{(n)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f^{(n)}). \tag{19}$$

By Lemma 3 and Lemma 4 we have

$$\begin{aligned} N\left(r, \frac{1}{g^{(n)}}\right) &\leqq N\left(r, \frac{1}{g}\right) + S(r, g) \\ &= N\left(r, \frac{1}{f}\right) + S(r, f^{(n)}). \end{aligned} \tag{20}$$

From (18), (19), (20) we obtain

$$\begin{aligned} \sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) &\leqq T(r, f^{(n)}) - T(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f^{(n)}) \\ &\leqq T(r, f^{(n)}) - T(r, f) + 2(1-\delta(0, f))T(r, f) + S(r, f^{(n)}) \\ &= T(r, f^{(n)}) - (2\delta(0, f) - 1)T(r, f) + S(r, f^{(n)}) \end{aligned} \tag{21}$$

By Lemma 1 and Lemma 4 we have

$$T(r, f^{(n)}) \leqq T(r, f) + S(r, f^{(n)}). \tag{22}$$

Noting $2\delta(0, f) - 1 > 0$, from (21), (22), we get

$$\begin{aligned} \sum_{i=1}^{3} N\left(r, \frac{1}{f_i}\right) &\leqq T(r, f^{(n)}) - (2\delta(0, f) - 1)T(r, f^{(n)}) + S(r, f^{(n)}) \\ &= 2(1-\delta(0, f) + o(1))T(r, f^{(n)}) \\ &\leqq (\lambda + o(1))T(r) \qquad (r \notin E), \end{aligned}$$

where $\lambda = 2(1-\delta(0, f)) < 1$. By Lemma 7, we have $f_2 \equiv 1$ or $f_3 \equiv 1$.

If $f_2 \equiv 1$, from (17) we have $f^{(n)} \equiv g^{(n)}$. By Lemma 8, we get $f \equiv g$. If $f_3 \equiv 1$, from (17) we have $g^{(n)} = -e^{-\alpha}$, $f^{(n)} = -e^{\alpha}$, and hence $f^{(n)} \cdot g^{(n)} \equiv 1$. This completes the proof of Theorem 2.

REFERENCES

[1] M. OZAWA, Unicity theorems for entire functions, J. d'Analyse Math. **30** (1976), 411-420.

[2] W.K. HAYMAN, Meromorphic functions, Oxford. 1964.

[3] H. UEDA, Unicity theorems for meromorphic or entire functions II, Kodai Math. J. **6** (1983), 26-36.

[4] C.C. YANG, On two entire functions which together with their first derivatives have the same zeros, J. Math. Anal. Appl., **56** (1976), 1-6.

[5] R. NEVANLINNA, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Paris, Gauthier-Villars, 1929.
[6] F. GROSS, Factorization of meromorphic functions, U.S. Govt. Printing Office *Publication, Washington, D.C., 1972.*

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