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A RADON TRANSFORM ON SPHERES THROUGH THE ORIGIN IN R^n AND APPLICATIONS TO THE DARBOUX EQUATION

BY

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ABSTRACT. On domain $C^\infty(R^n)$ we invert the Radon transform that maps a function to its mean values on spheres containing the origin. Our inversion formula implies that if $f \in C^\infty(R^n)$ and its transform is zero on spheres inside a disc centered at 0, then f is zero inside that disc. We give functions $f \notin C^\infty(R^n)$ whose transforms are identically zero and we give a necessary condition for a function to be the transform of a rapidly decreasing function. We show that every entire function is the transform of a real analytic function. These results imply that smooth solutions to the classical Darboux equation are determined by the data on any characteristic cone with vertex on the initial surface; if the data is zero near the vertex then so is the solution. If the data is entire then a real analytic solution with that data exists.

In 1917 Radon inverted the first "Radon transform" **[18]**. This transform, R, maps a function on R^n to a function on the set of hyperplanes in R^n. If f is a continuous function of compact support on R^n then Rf evaluated on a hyperplane is the integral of f over that hyperplane in its natural measure. The case $n = 2$ has many applications in science, engineering, and medicine **[2]**, **[3]**, **[15]**, **[21]** and the transform on R^n (n arbitrary) has many applications to partial differential equations **[13]**, **[14]**. Generalizations of this Radon transform to integrations over certain spheres and ellipsoids have been studied by John and others **[13]**, **[19]** again in connection with partial differential equations. Moreover these examples are all special cases of the generalized Radon transform: given smooth manifolds X, Y, and a class of submanifolds of X, $\{H_y | y \in Y\}$, one specifies smooth measures μ_y on each H_y. The generalized Radon transform R from $C_0^\infty(X)$ to functions on Y takes $f \in C_0^\infty(X)$ to the integrals of f over the manifolds H_y in the measures μ_y **[7]**. In many cases restrictions on the support of Rf imply restrictions on the support of f **[10]**; this fact is useful in applications to partial differential equations **[11]**, **[14]**.

In this article we define a Radon transform over spheres passing through the origin in R^n. If $f \in C(R^n)$, the transform $\hat{f}$ evaluated on a sphere containing 0 is the mean value of f over that sphere in its natural measure. Our main result, Theorem 1, is an inversion formula for this transform: if $f \in C^\infty(R^n)$ then $f(x)$ is determined by the values of $\hat{f}$ on spheres that lie inside the disc of radius $|x|$

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centered at the origin. The theorem is proven by using facts about spherical harmonics, Gegenbauer polynomials and the classical Radon transform on R^n. Our theorem implies the following support restriction (Corollary 2). If $f \in C^\infty(R^n)$ and $\hat{f}$ is zero when evaluated on spheres inside a disc centered at 0 then f is zero in that disc. To give perspective to this result, for each $q \in N$, we define $f_q \in C^q(R^n)$ such that $\hat{f}_q$ is identically zero (Example 1). Then we observe other interesting properties of this transform that help characterize its range (Propositions 4 and 5) and finally we apply our transform to C^∞ solutions of the classical D. rboux equation ($n \geqslant 2$)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial}{\partial r} - \Delta_x\right)u = 0 \tag{1}$$

where Δ_x is the Laplacian in R^n ($u(x, r) \in C^\infty(R^n \times R)$). If the initial value for (1) is $u(x, 0) = f(x)$ then it is well known that $u(x, r)$ is the spherical mean of f over the sphere of radius $|r|$ centered at x **[13]**. From our results on spherical means we conclude that a C^∞ solution to (1) is determined on each compact subset of $R^n \times R$ by data on a corresponding compact subset of the characteristic cone $|x - x_0| = |r|$ for fixed $x_0 \in R^n$; if the data is zero near x_0 then so is the solution. Finally we show that real analytic data on the cone determines a real analytic solution for (1).

Rhee and Chen discovered a different inversion formula for this transform that is valid in odd dimensions on a different class of functions than $C^\infty(R^n)$. Rhee also obtained a representation of certain solutions to the Darboux equation **[1]**, **[19]**.

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We now define our Radon transform. Let $y = 2p\omega$ where $\omega \in S^{n-1}$, $p \in R$, let $d\Omega$ be the standard measure on S^{n-1} and let ω_n be the volume of S^{n-1} in this measure. If $f \in C(R^n)$ then

$$\hat{f}(y) = \frac{1}{\omega_n}\int_{\xi \in S^{n-1}} f(p(\omega + \xi))\, d\Omega(\xi). \tag{2}$$

This is just the mean value of f over the sphere with center $y/2$ and radius $|y|/2$ in its standard measure. The map (2) from $C^\infty(R^n)$ to $C^\infty(R^n)$ is a generalized Radon transform in the sense above; away from the origin it is an elliptic Fourier integral operator (see (11) and **[9]**, **[17]**).

Our main theorem will be proved using facts about Gegenbauer polynomials and spherical harmonics. Let $n > 2$ and let $\lambda = (n-2)/2$. The Gegenbauer polynomial $C_l^\lambda(t)$ of degree $l = 0, 1, 2, \ldots$ is orthogonal to all polynomials of degree less than l on $[-1, 1]$ with weight function $(1 - t^2)^{\lambda - 1/2}$. A function on S^{n-1}, $Y_l(\omega)$, is a spherical harmonic of degree l if Y_l is the restriction of a homogeneous harmonic polynomial of degree l in R^n. If $\xi \in S^{n-1}$ and $\cdot$ denotes the standard inner product on R^n, then $Y_l(\omega) = C_l^\lambda(\omega \cdot \xi)$ is a spherical harmonic of degree l. If Y_l is a spherical harmonic, $f \in C(R^n)$, and $p \in R$ define

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$$f_l(p) = \frac{\int_{\omega \in S^{n-1}} f(p\omega)\overline{Y_l(\omega)}\, d\Omega(\omega)}{\int_{\omega \in S^{n-1}} |Y_l(\omega)|^2\, d\Omega(\omega)}.$$

Then f_l depends on the choice of Y_l and $f_l(-p) = (-1)^l f_l(p)$; for $f \in C^\infty(R^n)$, $f_l(p) = p^l g(p)$ where $g \in C^\infty(R)$ is even. In the Hilbert space $L^2(S^{n-1}, d\Omega)$ a spherical harmonic of degree l is orthogonal to any polynomial of lower degree. In fact one can choose a complete orthonormal system in $L^2(S^{n-1}, d\Omega)$ consisting of spherical harmonics and having $O(l^{n-2})$ elements of degree l. If Y_l is a member of such a system, $f \in C(R^n)$ and $p \in R$ then the corresponding coefficient of the series in this system for $f(p\omega)$ is $f_l(p)$ and if $f \in C^\infty(R^n)$ this series converges uniformly absolutely on compact subsets of R^n to $f(p\omega)$ **[6]**, **[20]**.

We can now state our main theorem.

THEOREM 1. *Let $n > 2$, $\lambda = (n-2)/2$ and let Y_l be a spherical harmonic of degree l on S^{n-1}.*

(3) *If $f \in C(R^n)$ and $s > 0$ then*

$$\hat{f}_l(s) = (2/s)^{n-1}\frac{\omega_{n-1}}{\omega_n C_l^\lambda(1)}\int_0^s C_l^\lambda(r/s) f_l(r) r^{2\lambda}\left(1 - (r/s)^2\right)^{\lambda - 1/2} dr.$$

(4) *If $f \in C^\infty(R^n)$, $p > 0$ and $K = \Gamma(l+1)/(2^{n-1}\Gamma(l+2\lambda)\lambda)$ then*

$$\begin{aligned} f_l(p) &= Kp^{-2\lambda}(d/dp)^{n-1}\int_0^p \hat{f}_l(s) C_l^\lambda(p/s)\left((p/s)^2 - 1\right)^{\lambda - 1/2} s^{4\lambda}\, ds \\ &= Kp^{-4\lambda - 1}\int_0^p (d/ds)^{n-1}\left[s^{4\lambda+1}\hat{f}_l(s)\right] C_l^\lambda(p/s)\left((p/s)^2 - 1\right)^{\lambda - 1/2} s^{2\lambda}\, ds. \end{aligned}$$

Note that if $f \in C^\infty(R^n)$ then $\hat{f}$ is smooth and so $\hat{f}_l(p)p^{-l}$ is in $C^\infty([0, \infty))$; this implies that the integrals in (4) are in $C^\infty([0, \infty))$.

For the case $n = 2$ Cormack **[2]** proved that

$$\hat{f}_l(s) = \frac{2}{(\pi s)}\int_0^s f_l(r) T_{|l|}(r/s)\left(1 - (r/s)^2\right)^{-1/2} dr, \tag{5}$$

$$f_l(p) = \frac{d}{dp}\int_0^p \hat{f}_l(s) T_{|l|}(p/s)\left((p/s)^2 - 1\right)^{-1/2} ds \tag{6}$$

where $T_{|l|}$ is the Chebychev polynomial of the first kind of degree $|l|$ **[6]**, $f_l(r)$ is the lth Fourier coefficient of $f(r\omega)$ and $\omega \in S^1$. The other results of this paper are true in this case as well. Their proofs are as given here for $n > 2$ except the formulas (5) and (6) replace (3) and (4).

PROOF OF THEOREM 1. The proof follows from the relation between our transform (2) and the classical Radon dual transform on R^n. For $g(\xi, p) \in C(S^{n-1} \times R)$ and $y \in R^n$, the classical Radon dual transform is defined by

$$R^*g(y) = \int_{\xi \in S^{n-1}} g(\xi, y \cdot \xi)\, d\Omega(\xi). \tag{7}$$

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Let $\omega \in S^{n-1}$, let S be the sphere of radius 1 centered at ω and let $d\Omega_1$ be its standard measure. Let $H = \{\xi \in S^{n-1} | \xi \cdot \omega = 0\}$ and define $\chi\colon S^{n-1} \to S$, $\xi \to 2(\xi \cdot \omega)\xi$; then χ is a $2-1$ cover of $S - \{0\}$ by $S^{n-1} - H$.

Let $d\tau$ be the standard measure on $H \cong S^{n-2}$. If $\tau \in H$ and $\theta, \phi \in [0, \pi]$ the maps

$$(\tau, \theta) \to (\sin\theta)\tau + (\cos\theta)\omega = \xi \tag{8}$$

and

$$(\tau, \phi) \to (\sin\phi)\tau + (1 + \cos\phi)\omega = \xi' \tag{9}$$

give coordinates on S^{n-1} and S respectively in which their measures become $d\Omega = (\sin\theta)^{n-2}\, d\theta\, d\tau$ and $d\Omega_1 = (\sin\phi)^{n-2}\, d\phi\, d\tau$ respectively. In coordinates (8) and (9) $\chi(\tau, \theta) = (\tau, 2\theta)$; using these expresssions for $d\Omega$, $d\Omega_1$, and χ it is straightforward to check for $f \in C(R^n)$, $y = 2p\omega \in R^n$ that

$$\begin{aligned} \hat{f}(y) &= \frac{1}{\omega_n} \int_{\xi' \in S} f(p\xi')\, d\Omega_1 \\ &= \frac{1}{\omega_n} (2/|y|)^{n-2} \int_{\xi \in S^{n-1}} f((y \cdot \xi)\xi) |y \cdot \xi|^{n-2}\, d\Omega. \end{aligned} \tag{10}$$

If we use (7) and (10) we get the following lemma.

LEMMA. *If* $\tilde{f}(\xi, p) = f(p\xi)|p|^{n-2}$ *then*

$$\hat{f}(y) = (1/\omega_n)(2/|y|)^{n-2} R^*\tilde{f}(y). \tag{11}$$

To prove (3) we assume $f(y) = f_l(s) Y_l(y')$ where $s = |y|$, $y' = y/|y|$ and Y_l is a spherical harmonic of degree l. Then $\tilde{f}(\xi, p) = \tilde{f}_l(p) Y_l(\xi)$ where $\tilde{f}_l(p) = |p|^{n-2} f_l(p)$ and $\tilde{f}_l(p) = (-1)^l \tilde{f}_l(-p)$. Since $\tilde{f}$ is continuous, a result of Ludwig **[14**, Lemma 5.1] establishes that $R^*\tilde{f}(y) = W(s) Y_l(y')$ where

$$W(s) = \frac{\omega_{n-1}}{C_l^\lambda(1)} \frac{2}{s} \int_0^s C_l^\lambda(r/s) f_l(r) r^{2\lambda} \left(1 - (r/s)^2\right)^{\lambda - 1/2} dr. \tag{12}$$

Equation (3) follows by using (12) in (11).

We recall formula (30) of **[4]** that states for $0 < r \leqslant p$

$$\begin{aligned} &\int_r^p s^{2\lambda-1} C_l^\lambda(r/s) C_l^\lambda(p/s) \left(1 - (r/s)^2\right)^{\lambda-1/2} \left((p/s)^2 - 1\right)^{\lambda-1/2} ds \\ &\qquad = \frac{\pi}{2^{n-3}} \left(\frac{\Gamma(l + 2\lambda)}{\Gamma(l+1)\Gamma(\lambda)} \right)^2 \frac{(p-r)^{n-2}}{\Gamma(n-1)}. \end{aligned} \tag{13}$$

To prove (4) one multiplies (3) by $C_l^\lambda(p/s)((p/s)^2 - 1)^{\lambda-1/2} s^{4\lambda}$ and integrates from zero to p. Then one simplifies the integral using (13) and finally differentiates $n-1$ times with respect to p. These manipulations are all legitimate because both $f_l(p)p^{-l}$ and $\hat{f}_l(p)p^{-l}$ are in $C^\infty([0, \infty))$. The result is the first equality in (4). To prove the second equality, one changes the first integral in (4) to an integral from zero to one, brings $(d/dp)^{n-1}$ inside the integral and uses Leibnitz's rule. This finishes the proof of Theorem 1.

Our first two corollaries follow from Theorem 1 and properties of spherical harmonics.

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COROLLARY 2. *If $f \in C^\infty(R^n)$ and $A \geqslant 0$ then the values of $\hat{f}(y)$ for $|y| \leqslant A$ determine $f(x)$ for $|x| \leqslant A$. If $\hat{f}(y) = 0$ for $|y| \leqslant A$ then $f(x) = 0$ for $|x| \leqslant A$.*

COROLLARY 3. *The Radon transform* (2) *with domain $C^\infty(R^n)$ is invertible.*

It is interesting to compare Corollary 2 to the "hole theorem" for the classical Radon transform, R, on R^n, if $f \in C^\infty(R^n)$ is *rapidly decreasing* and $Rf = 0$ when evaluated on hyperplanes *not intersecting* a disc in R^n then $f = 0$ *outside* of that disc [**10**, Theorem 2.1].

To give perspective to our results we now define functions f such that $\hat{f} \equiv 0$.

EXAMPLE 1. Given natural numbers q and n let integers k and l satisfy: $k > q$; if n is even (respectively odd) then k is even (respectively odd); l is even and $l > k + n - 2$. Let Y_l be a spherical harmonic of degree l on S^{n-1} and define $f \in C^q(R^n)$ by $f(x) = |x|^k Y_l(x/|x|)$. Using (3) and orthogonality properties of the Gegenbauer polynomials one sees that $\hat{f} \equiv 0$.

Our next proposition will help determine the range of (2) on rapidly decreasing functions. Condition (14) is quite similar to the integrals that characterize the range of the classical Radon transform on these functions [**10**, Theorem 4.1] (see also [**14**, Theorem 2.1(d)]).

PROPOSITION 4. *Let f be a rapidly decreasing function on R^n.*

(i) *If $1 < k + n - 1 \leqslant l$ and $k + n - 1$ and l are both even or both odd then*

$$\int_0^\infty \hat{f}_l(s)/s^k \, ds = 0. \tag{14}$$

(ii) *If $\hat{f}_l$ is a real function not identically zero then it changes sign at least l times in R.*

PROOF. Part (i) is proved by first using (3) to express $\hat{f}_l$ in terms of f_l. One can then use Fubini's theorem to switch the integrals because of the restrictions on k and since f is rapidly decreasing. Finally by changing variables in the inner integral one sees that

$$\int_0^\infty \hat{f}_l(s)/s^k \, ds = K' \int_0^\infty f_l(r) r^{-k} \int_0^1 C_l^\lambda(t)(1 - t^2)^{\lambda - 1/2} t^{k+n-3} \, dt \, dr$$

for a suitable constant K'. Using the hypotheses on k and orthogonality relations of the Gegenbauer polynomials one sees that the inner integral is zero. This proves (i).

To prove (ii) we recall that $\hat{f}_l$ is odd (respectively even) if l is odd (respectively even) and show that there are at least $[l/2]$ connected components of the zero set of $\hat{f}_l$ at which $\hat{f}_l$ changes sign in $[0, \infty)$ (not including, if l is odd, the component containing zero). Here $[l/2]$ is the largest integer not greater than $l/2$. Choose an element s_j from each such component, $j = 1, \ldots, m$, where m is the number of components. If $m < [l/2]$, then

$$\int_0^\infty \hat{f}_l(s) s^{-l+n-1} \prod_{j=1}^m \left(s^2 - s_j^2\right) ds \tag{15}$$

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is zero by (i). However, the integrand of (15) is either nonpositive or nonnegative and not identically zero. This implies that the integral in (15) is nonzero. This contradiction shows $m \geqslant [l/2]$ and proves the proposition.

We define a real analytic function on R^n to be entire if its power series at zero converges everywhere.

PROPOSITION 5. *If g is an entire function on R^n then there is a unique real analytic function f such that $\hat{f} = g$.*

ADDENDUM. The authors have recently shown that f is entire. The proof uses bounds on spherical harmonics on R^n when extended to C^n.

PROOF. If f exists, it is unique by Corollary 3. If $h(x) = |x|^{l+2k} Y_l(x/|x|)$ then (3) and integral 7.311 #2 of [**8**, p. 826] show that

$$\hat{h}_l(s) = \left(\frac{s}{2}\right)^{l+2k} \frac{\Gamma(n/2)\Gamma(2k+l+n-1)}{\Gamma(k+n/2)\Gamma(k+l+n-1)}. \tag{16}$$

Choose a complete orthonormal system of spherical harmonics for $L^2(S^{n-1}, d\Omega)$. Here the "spherical harmonic series" for a function $q \in C^\infty(R^n)$ will be the series in this system for $q(s\omega)$.

Let g be entire. If g_l is a coefficient of a spherical harmonic of degree l in the spherical harmonic series for g then $g_l(s) = \sum_{k=0}^\infty a_k s^{l+2k}$ is an entire function. Standard convergence results for analytic functions [**12**, pp. 27, 34] and properties of spherical harmonics [**20**] show that the spherical harmonic series for g converges uniformly, absolutely on compact subsets of R^n even when each coefficient a_k in each g_l is replaced by its absolute value. Using (16) one can formally invert the equation $g_l = \hat{f}_l$ term by term getting a formal power series for f_l. Because g_l is entire and the "inversion factors"

$$\frac{2^{l+2k}\Gamma(k+n/2)\Gamma(k+l+n-1)}{\Gamma(n/2)\Gamma(2k+l+n-1)}$$

are bounded by 2^{l+2k} the power series for $f_l(p)$ is term by term majorized by the series for $g_l(2p)$. Therefore, if f is the function that has spherical harmonic series with coefficients f_l given above then f is real analytic and $\hat{f} = g$. This proves the proposition.

Our final theorem applies the theory developed above for our Radon transform (2) to the classical Darboux equation (1). An equation related to (1) was used by Poisson to solve the equation of propagation of sound in R^3 [**5**], [**16**], [**22**]. It is well known [**13**] that if $u(x, r)$ solves (1) and $u(x, 0) = f(x)$ then $u(x, r)$ is the mean value of f over the sphere of radius $|r|$ centered at x. Therefore

$$u(x, |x|) = \hat{f}(2x). \tag{17}$$

Equation (1) has been studied in the case when the constant $(n-1)$ is replaced by an arbitrary complex number [**5**], [**22**] and in this case, too, the solution $u(x, r)$ is related to the spherical means of f.

Our final theorem proves that solutions to (1) are determined by data on a characteristic cone with vertex on the initial surface.

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THEOREM 6. *Let* $x_0 \in R^n$ *and* $A > 0$.

(i) *If* u *is a* C^∞ *solution of* (1) *then its values on the set* $B = \{(x, r) \in R^n \times R \mid |x - x_0| + |r| \leqslant 2A\}$ *are determined by its values on the truncated characteristic cone* $|x - x_0| = |r| \leqslant A$. *In particular if* $u = 0$ *on the truncated cone then* $u = 0$ *on* B.

(ii) *If* $g(x)$ *is an entire function on* R^n *then there is a unique real analytic solution to* (1) *with data* $u(x, r) = g(x)$ *on the characteristic cone* $|x - x_0| = |r|$.

The proof of Theorem 6 follows from Corollaries 2 and 3 as well as Proposition 5 and the relation (17).

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