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A random walk through models of nonlinear clustering

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Abstract. I present a few simple models of the mass function of collapsed objects. The emphasis is on apparently unrelated models which end up giving the same answer for the number density and merger histories of virialized clumps. I also comment briefly on models of the spatial distribution of the clumps, and how they can be used to model the spatial distribution of the mass.

1. Introduction

In what follows I will describe a few simple, toy models of the growth of clustering. The emphasis will be on the interrelations between these models, more than on the exact agreement with the results of simulations. All the models I will discuss are hierarchical in the sense that small things form first, and big things form later by mergers of the small things; there is no fragmentation.

2. The mass function and merger histories of collapsed objects

In the following discussion I will focus on how one might estimate the probability that a randomly chosen particle belongs to a clump which contains m particles. All the models use the initial spatial distribution of the mass to estimate how clusters grow. This means that the models are most likely to be accurate if the mass was cold initially.

2.1. Gravity as an effective length-scale

To begin, consider a distribution of particles arranged at random along a line—we will consider three-dimensional distributions shortly. This random distribution is supposed to represent the initial spatial distribution. We wish to model what gravitational evolution does to this distribution. Presumably, gravity being a force of attraction, near neighbours will begin to move towards each other. Suppose that if two particles collide, they merge. We would like to estimate the number of clumps containing m particles at a time t after the initial distribution of single particles started to cluster. There are two natural choices for the order in which things happened.

The first is to assume that clusters grow by a process similar to percolation. The intuitive idea is that gravity can rearrange things only on small scales at early times, but on increasingly larger scales at later times—the length scale associated with gravity increases with time. To model this, draw a line outwards from each particle; assume that the length of this line increases monotonically with time. Two particles are said to be friends, and to have merged with each other, if the line from one of them touches the other. If we define clumps by requiring that friend-of-friends are in the same clump, then this defines a set of clusters as a function of link-length. The size of a typical cluster must grow as the friend-of-friend length increases; large link-lengths mean late times, reflecting the fact that massive clusters are not present initially, but are more common later.

If we keep track of the clumps at time t_1 which are part of a larger clump at $t_2 > t_1$, then we have stored some information about the merger history of an object. This means that we can start to address questions such as: How does the distribution of clump sizes evolve? Are the most massive objects at t_2 made of the most massive objects which were present at t_1? How different are the merger histories of objects which contain the same number of particles?

The link-length model above provides analytic answers to all these questions if the initial distribution of points was Poisson. In the one-dimensional case, the probability that a clump contains m particles when the critical link-length is l equals

$$\eta(m,l) = \left[\prod_{i=1}^{m-1} \int_0^l \exp(-\bar{n}x_i)\bar{n}\, dx_i\right] \mathrm{e}^{-\bar{n}l} = \mathrm{e}^{-\bar{n}l}\left[1-\mathrm{e}^{-\bar{n}l}\right]^{m-1} \quad \text{where } m \geq 1, \quad (1)$$

and $\bar{n}$ denotes the average density of particles. The term in square brackets can be understood as follows. The probability that the particles are at $(dl_1, \cdots, dl_n)$ equals $\prod \bar{n}\, dl_i$ times the probability that there are no particles in between: $\prod \exp(-\bar{n}l_i)$, where l_i denotes the distance between particles i and $i+1$. If we change one of the l_is to any value between zero and l, but keep the other l_js fixed, then we will still produce a valid configuration; hence the integration over the allowed range for each l_i. The endpoint of the clump is determined by requiring that there be no particle closer than l to it, hence the final exponential on the right hand side of the square brackets.

In what follows, it will prove convenient to define

$$b \equiv 1 - \exp(-\bar{n}l). \quad (2)$$

Initially $l = 0$, so equation (1) says that all clumps have size $m = 1$ initially. At later times, $l \to \infty$, and the clump distribution tends to an exponential. In the process, b changes from zero to unity.

The fraction of mass which is in m clumps is

$$f(m,b) = \frac{m\eta(m,b)}{\sum m\eta(m,b)} = (1-b)\, m\eta(m,b). \quad (3)$$

The number density of m-clumps is the average density $\bar{n}$ times $f(m,b)/m$, and is often called the universal mass function.

Consider an m-clump at t_2 when the link-length was $l_2 \geq l_1$. At $t_1 \leq t_2$ the link-length was shorter, and so not all m particles would have been 'friends-of-friends'. In other words, at $t_1 \leq t_2$, the m-clump was partitioned into smaller subclumps. A modification of the argument above allows us to write down the distribution of subclump sizes. Namely,

$$p(n_1, \cdots, n_m; l_1|m; l_2) = n! \,\frac{\eta(n, l_{21})}{\eta(m, l_2)} \prod_{i=1}^{m} \frac{\eta(i, l_1)^{n_i}}{n_i!}, \tag{4}$$

where n_i denotes the number of subclumps which contain i particles; the total number of particles is $\sum i n_i = m$ and they are partitioned into $\sum n_i = n$ clumps. The $n!$ factor comes from not caring about the order of the subclumps, the factor of $\eta(m, l_2)$ in the denominator comes from the fact that we know we have an m-clump at t_2, and $\eta(n, l_{21} \equiv l_2 - l_1)$ comes from noting that the distance between the right-end of one subclump and the left-end of the next subclump must be greater than l_1 but less than l_2. This gives a factor of $\exp(-\bar{n}l_1) - \exp(-\bar{n}l_2)$, which we write as $\exp(-\bar{n}l_1)[1 - \mathrm{e}^{-\bar{n}(l_2-l_1)}]$, for each subclump except the rightmost, for which the factor is simply $\exp(-\bar{n}l_2)$. The term in the product sign is just the probability of having the correct set of subclumps, and assuming that if subclumps which contain the same number of particles are exchanged, the partition of m is unchanged.

Because $\eta(i, l_1)$ depends on i only in the exponent, equation (4) can be simplified considerably:

$$p(n_1, \cdots, n_m|m) = \left(\frac{n!}{\prod_{i=1}^{m} n_i!}\right) \left(\frac{b_2 - b_1}{b_2}\right)^{n-1} \left(\frac{b_1}{b_2}\right)^{m-n}, \tag{5}$$

where the b_is are defined by equation (2). This partition function contains all the information required for quantifying how different the various trees in the forest of possible merger histories of an m-clump are. For example, the probability that an m-clump was in n pieces at $t_1 < t_2$ is given by summing over the set $\pi(n|m)$ of partitions of m which have n parts:

$$p(n; b_1|m; b_2) = \sum_{\pi(n|m)} p(n_1, \cdots, n_m|m) = \binom{m-1}{n-1} \left(\frac{b_1}{b_2}\right)^{m-n} \left(\frac{b_2 - b_1}{b_2}\right)^{n-1}; \tag{6}$$

the number of subclumps follows a Binomial distribution.

The average fraction of M which is in m-subclumps at b_1 is

$$f(m; b_1|M; b_2) = \sum_{\pi(M)} \frac{m n_m}{M}\, p(n_1, \cdots, n_M|M). \tag{7}$$

The conditional mass function is related to this fraction similarly to how the unconditional mass function is related to $f(m, b)$: namely, $\mathcal{N}(m, b_1|M, b_2) = (M/m) f(m, b_1|M, b_2)$.

It is useful to rewrite the partition function (equation 4) above as follows. Let $p(S_n = m)$ denote the probability that the sum of n independent variables each distributed according to equation (1) equals m. Then

$$p(S_n = m; b) = \binom{m-1}{n-1} (1 - b)^n \, b^{m-n}. \tag{8}$$

This, with equation (6) for the probability of having m subclumps, allows one to verify that equation (4) also equals

$$p(n_1,\cdots,n_m|m) = \frac{p(n;b_1|m;b_2)}{p(S_n=m;b_1)}\,n!\prod_{i=1}^{m}\frac{\eta(i,l_1)^{n_i}}{n_i!}. \tag{9}$$

Thus, the partition function expresses the fact that, other than the requirement that the sum of the masses of the subclumps should equal the mass of the parent halo, there are no additional correlations between subclumps.

All the above was worked out for the special case of a Poisson distribution of points on a line. To generalize these results to two (or d) dimensions we must be able to compute the area (volume) of interestection of n circles (d-dimensional spheres). Expressions for these quantites are in Kratky (1978, 1981). Rather than showing the results of doing this here, I will now turn to another model of hierarchical clustering.

2.2. Gravity and a critical density for collapse

The link-length model above was useful for illustrating how one might write down expressions for the merger histories of clumps. It is a bad model for gravity for the following reason. The $m-1$ separations between the m particles of an m-clump may all be a small fraction of the critical link-length l, but it is also possible that they are all a substantial fraction of l. As a result, the model allows m-clumps to have range of sizes. The model says that the gravitational influence of a clump extends over the region it occupies plus $2l$ (from the two end points). Because m-clumps come in a range of sizes, this would say that the gravitational influence of some m-clumps extends over a greater range than others. However, one might have thought that it is the mass of a clump which determines the range of its influence, and not the space which it occupies.

To allow for this, we must modify the criterion used for identifying clumps. Rather than using only the list of initial separations, use the initial densities. Assume that an initial region has collapsed to form a clump if the density within it exceeds a certain threshold value. In addition, assume that when a region collapses, all the particles within the region remain within it. This means that one is interested in finding regions which are isolated in the following sense: the points within the region must be sufficiently close to each other that density within the region exceeds the critical value, but this set of points must be sufficiently isolated from any other set so that the density within any larger region containing the points is less than the critical value.

In this case, the probability a clump contains m particles is

$$\eta(m,b) = \frac{(mb)^{m-1}\mathrm{e}^{-mb}}{m!} \qquad \text{where } b=\frac{1}{1+\delta_{\mathrm{c}}} \qquad \text{and } m\geq 1 \tag{10}$$

where $1+\delta_{\mathrm{c}}$ denotes the ratio of the critical density required for collapse to the average background density (Sheth 1995). In this model, we assume that $\delta_{\mathrm{c}}(t)\gg 1$ initially, and that it decreases with time. Notice that this distribution differs from equation (1) at low masses. At late times, this distribution tends to one which astronomers associate with Press & Schechter (1974), rather than the simple exponential distribution of the previous subsection. The forest of merger trees associated with this model is given by inserting equation (10) in (9), using the distribution $p(S_n = m, b)$ associated with equation (10) rather than equation (8), and using the Binomial distribution of subclumps (equation 6) (Sheth 1996).

In both this and the previous model, cosmology only enters when translating the critical link-length or density to cosmological time. The order in which mergers happen is the same for all cosmologies; it depends only on the initial distribution of density fluctuations. This is a powerful and simplifying idealization which is in good qualitative agreement with numerical simulations of clustering from cold, initially Gaussian density fluctuation fields.

The mass function and merger histories associated with equation (10) are in quantitative agreement with simulations, whereas the model associated with equation (1) is not. The quantitative agreement between the model description of the forest of merger histories and the simulations shows that any additional correlations between subclumps, other than those required by mass conservation, must be small.

2.3. Random walks and the excursion set approach

The mass function (equation 10) above can be derived by rephrasing the critical density requirement slightly. If one imagines computing the density ρ in concentric spheres centred on a randomly chosen particle in a Poisson distribution, then $\rho(v)$ will execute a random walk as v increases. The requirement that the density exceed a certain value at v but be less than this value for all $V > v$ means that the problem of computing the mass function can be cast in terms of a barrier crossing problem associated with random walks (Epstein 1983). The continuum limit of this process has been studied by Bond et al. (1991), and has come to be called the excursion set model of the clump mass function. This approach allows one to compute the conditional mass function as well—it is the continuum limit of the distribution one gets by inserting equation (10) in equation (7)—but the full partition function of merger histories associated with this model has not yet been worked out (Sheth & Pitman 1997 discuss a special case in which the merger history forest can be solved for). Progress can be made, however, if one uses the idea that, other than mass conservation, there are no additional correlations to account for. The resulting model is in good agreement with simulations (Sheth & Lemson 1999).

I have not seen a random walk derivation of the exponential distribution associated with the link-length model I presented earlier.

2.4. Binary merger models

The models described above are also solutions of the Smoluchowski binary merger model:

$$\frac{dn(m,t)}{dt} = \sum_{i=1}^{m-1} \frac{K(i,m-i)}{2} n(i,t)n(m-i,t) - n(m,t)\sum_{i>0} K(m,i)n(i,t). \quad (11)$$

The expression above expresses the fact that the number of clumps of mass m increases if smaller clumps merge with each other to form an m-clump, and it decreases because m-clumps are themselves merging with other clumps. Note that there is no fragmentation in this model—the destruction rate which is the second term on the right hand side is a consequence of mergers, not fragmentation.

If we set $n(m,t) = (1-b)\eta(m,b)$, where $b \equiv 1-\exp(-t)$, then equation (1) solves the case in which $K(i,j)$ is a constant, independent of both i and j. It has been used to describe the growth of linear polymers. Our second model, equation (10), is the solution to $K = i+j$; it is associated with the growth of branched polymers (e.g. Sheth & Pitman 1997).

One of the virtues of writing these models using Smoluchowski's equation is that it shows clearly how the formation rate of m-clumps evolves with time. It is given by the first term on the right hand side of equation (11). It is a simple matter to verify that the continuum limit of this expression equals that which was recently found by Percival & Miller (1999) from the random walk approach.

2.5. Peaks in Gaussian random fields

Another model for the clump distribution is to suppose that clumps are associated with peaks in the initial density field. Following Bardeen et al. (1986), one often smoothes the initial density fluctuation field with a filter of scale R, and then identifies peaks in the smoothed field. In this case, the density of peaks of height ν is

$$n(\nu)\,d\nu = \frac{\exp(-\nu/2)}{(2\pi)^2 R_*^3}\,G(\gamma,\gamma\nu^{1/2})\,d\nu \tag{12}$$

where $\nu = \delta_c^2/\sigma_0^2(R)$, $R_* = \sqrt{3}\sigma_1/\sigma_0$, $\gamma(R) = \sigma_1^2/\sigma_0\sigma_2$ and σ_0, σ_1 and σ_2 depend on the shape of the power spectrum of the initial density fluctuation field (Section IV in Bardeen et al. 1986) and

$$G(\gamma,y) = \int_0^\infty \mathrm{d}x\, f(x)\,\frac{\exp[-(x-y)^2/2(1-\gamma^2)]}{\sqrt{2\pi(1-\gamma^2)}}$$

with $f(x)$ given by equation (A19) in Bardeen et al. If we define

$$f(\nu)\,d\nu \equiv (m/\bar\rho)\,n(\nu)\,d\nu, \tag{13}$$

and use the fact that the mass under a Gaussian filter is $m = \bar\rho\,(2\pi)^{3/2}R^3$, then we have a quantity which one might interpret as the fraction of mass which is in peaks of height ν.

Unfortunately, this is not really the sort of quantity we can compare with a mass function of collapsed clumps. In simulations clumps have a range of masses, whereas in this picture all peaks have the same mass m whatever their height ν. Although it is tempting to identify the higher peaks with the more massive objects, the expression above does not show how to do this self-consistently.

If, instead, we smoooth the density field with a range of filter sizes R, and identify collapsed objects with peaks of height $\delta_c/\sigma_0(R)$, then, because $\sigma_0(R)$ decreases as $R \propto m^{1/3}$ increases, we have a model in which massive objects are associated with higher peaks. In this case, the associated mass function of collapsed peaks is given by

$$\nu f(\nu) = \frac{\exp(-\nu/2)}{\sqrt{2\pi}}\left(\frac{R}{R_*}\right)^3 \frac{H(\gamma,\gamma\nu^{1/2})}{3}\,\frac{R^2\sigma_2(R)}{\sigma_0(R)}\,\frac{dm/m}{d\nu/\nu}, \tag{14}$$

where

$$H(\gamma, y) = \int_0^\infty \mathrm{d}x\, x f(x)\, \frac{\exp[-(x-y)^2/2(1-\gamma^2)]}{\sqrt{2\pi(1-\gamma^2)}},$$

and we have again set $m/\bar{\rho} = (2\pi)^{3/2} R^3$. At large ν, $H \approx \gamma\nu^{1/2}\, G$, and this expression is the same as the previous one. At smaller ν, however, this expression differs from equation (12).

If the initial spectrum of density fluctuations was a power law, $P(k) \propto k^n$, then equation (14) for the mass function associated with peaks becomes

$$\nu f(\nu) = \sqrt{\frac{\nu}{2\pi}} \exp\left(-\frac{\nu}{2}\right) \frac{H(\gamma, \gamma\nu^{1/2})}{\nu^{1/2}} \frac{(5+n)^2}{12\sqrt{6(3+n)}}. \tag{15}$$

Note that this expression explicitly depends on the shape of the power spectrum.

This should be compared with the continuum limit of equation (10),

$$\nu f(\nu) = \sqrt{\frac{\nu}{2\pi}} \exp\left(-\frac{\nu}{2}\right), \tag{16}$$

which is often called the Press–Schechter formula. Note that, when expressed as a function of ν rather than m, this mass function is the same for all $P(k)$.

The mass function which actually fits numerical simulations is better approximated by

$$\nu f(\nu) = A(p)\Big(1 + (a\nu)^{-p}\Big)\sqrt{\frac{a\nu}{2\pi}} \exp\left(-\frac{a\nu}{2}\right), \tag{17}$$

where $a \approx 0.7$, $p = 0.3$ and $A(p)$ is determined by requiring that $\int d\nu f(\nu) = 1$; $A \approx 0.322$ (Sheth & Tormen 1999). Like the Press–Schechter function, this mass function also has no dependence on the shape of $P(k)$.

These various mass functions are shown in Fig. 1. Whereas the Press-Schechter, excursion set, critical collapse density, binary merger model, is not in quantitative agreement with the mass function of collapsed objects in the simulations, it is at least qualitatively consistent. The peaks model fits the simulations rather well at high masses (large ν) but is not so accurate at smaller masses. (I set $n = -1.5$ in the peaks formula which is about the right value for ΛCDM models.)

Equation (16) is associated with the assumption that clumps form from a spherical collapse (e.g. Press & Schechter 1974; Bond et al. 1991). Modifying the excursion set argument to allow for ellipsoidal collapse is relatively straightforward (Sheth, Mo & Tormen 2000). In essence, the spherical model has $\delta_{sc} \approx 1.686$ with no m dependence for the critical collapse density, whereas ellipsoidal collapse has $\delta_{ec}(m)$. At large m, $\delta_{ec}(m) \to \delta_{sc}$; this simply reflects the fact that only the most massive clusters are spherical. Accounting for the difference between ellipsoidal and spherical collapse appears to increase agreement between the excursion set predictions and the simulations.

Modifying the peaks mass function to include the effects of ellipsoidal collapse can be done by using the mass dependent $\delta_{ec}(m)$ when identifying peaks; this is relatively straightforward and I have not bothered to show the results

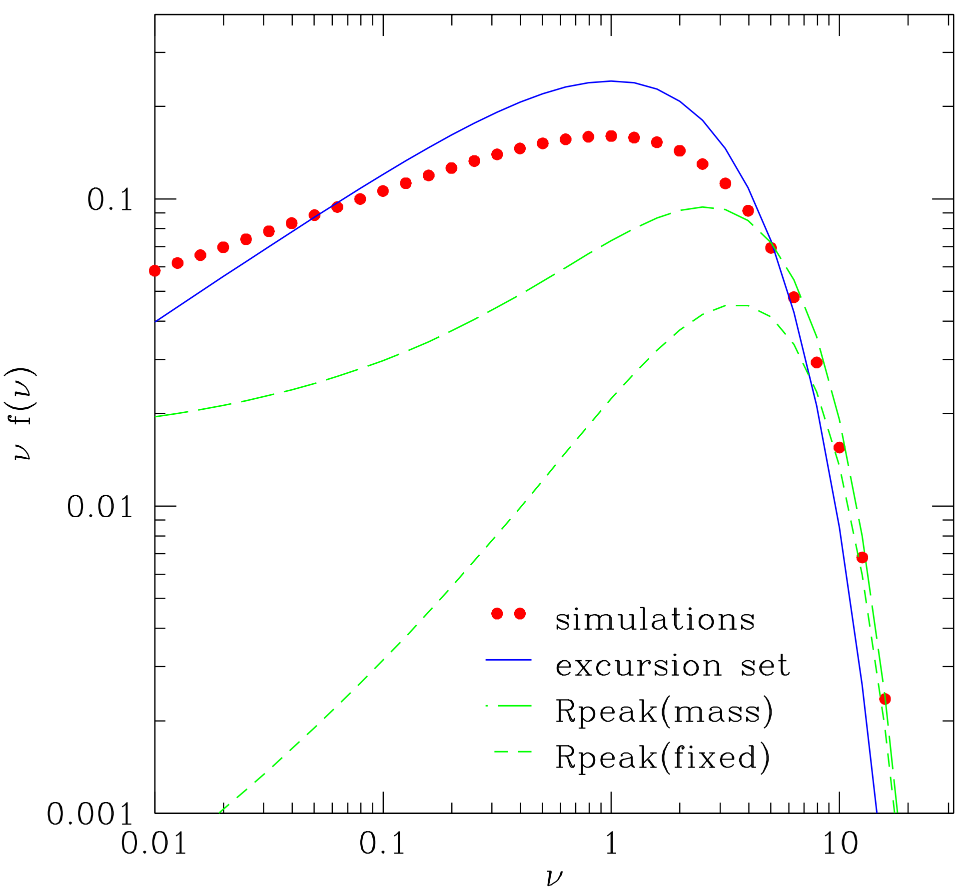

Figure 1. Various formulae for the mass function of collapsed objects. Filled circles show the mass function measured in numerical simulations of clustering in a cold dark matter dominated universe. Solid curve shows the excursion set spherical collapse formula, short dashed line shows the result of using a fixed smoothing scale to define peaks, and then simply counting peaks as a function of height, and long dashed line shows the result of assuming that more massive objects are associated with peaks on large smoothing scales, whereas less massive objects are peaks on smaller scales.

here. Extending the peaks model to provide a description of the forest of merger histories remains an open problem (but see Manrique et al. 1998 for an initial attempt to do this). Also, I have not seen any discussion of how one might include a mass dependent $\delta_{ec}(m)$ into the Smoluchowski binary merger model of the mass function.

3. The spatial distribution of mass

In this section I describe two approaches to modelling how the distribution of the density field evolves in the nonlinear regime. Both approaches are quite different from the linear and higher order perturbation theory approaches (e.g. Scoccimarro, these proceedings) which solve the equations of motion to make their estimates.

The essential idea is that if all the mass is in collapsed objects, then one can describe the distribution of the mass by making models of the number and spatial distribution of clumps, and of the internal distribution of mass within clumps (Neyman & Scott 1959; Scherrer & Bertschinger 1991). Numerical simulations show that the density run within a clump depends on clump mass; Navarro, Frenk & White (1997) have provided a simple fitting formula for this mass dependence. The previous section described models of the mass function of collapsed objects. All that remains is to build a model for the spatial distribution of clumps.

Mo & White (1996) described how knowledge of the merger history tree of clumps allows one to describe the spatial distribution of clumps. They argued that the two point correlation function of clumps can be computed if the second moment of the distribution of clump merger histories is known. At large separations, they argued that knowledge of only the conditional and unconditional clump mass functions was necessary for modelling the clump correlation functions. Sheth & Tormen (1999) showed that, in fact, in this limit, knowledge of only the unconditional mass function is sufficient.

Mo, Jing & White (1997) used equation (16) to write down estimates of the variance and higher order moments of the large scale clump distribution. They also did this for the 'fixed smoothing scale' peaks mass function of equation (12). The corresponding expressions for the peak mass function in equation (14) are got by replacing their equation (25) with

$$h_k = \frac{(-1)^k}{k!}\frac{(\gamma\nu^{1/2})^k}{H(\gamma,\gamma\nu^{1/2})}\frac{\partial^k H(\gamma,\gamma\nu^{1/2})}{\partial y^k}\Bigg|_{y=\gamma\nu^{1/2}}.$$

In the approach above, one solves for the distribution of the mass by writing down estimates of the two-point and higher order correlation functions. If one is interested in writing down the probability that a randomly placed cell contains mass M, then one must compute the appropriate sum over all these correlation functions.

It is possible to build model for this probability distribution function directly (Sheth 1998). The idea is to modify slightly the random walk, excursion set model of the clump mass function. In this modified excursion set approach, the same model which yields equations (10) and (16) for the mass function says that the probability that a random cell contains mass m is Generalized Poisson and Inverse Gaussian, respectively. Remarkably, this Inverse Gaussian distribution is also predicted by the perturbation theory based model of Scoccimarro & Frieman (1999).

The fundamental quantity in the excursion set model for the clump mass function is the distribution of first crossings, by Brownian motion random walks, of a barrier of fixed height $B = \delta_{sc}$. If one thinks of clumps as being objects which have collapsed to a very small size, then the clump mass function is like the distribution of mass in cells of vanishingly small size which are not empty. This allows one to generalize the model to the case in which the cells have some non-zero size. To do so, one must study the first crossing distribution of a barrier whose shape $B(M/V)$ depends on cell size V. If $f(M)\,dM$ denotes the probability that the first crossing of $B(M/V)$ happens at M, then

$$f(M)\,dM = \frac{M}{\bar{\rho}V}\,p(M|V)\,dM, \tag{18}$$

where $p(M)\,dM$ denotes the probability that a cell of size V contains M (Sheth 1998). Of course, the first crossing distribution, and so the associated probability that a cell contains mass M, both depend on the functional form of B. In 1998, I used the spherical collapse model to specify the shape of this function.

In the limit of large V, the associated distribution of first crossings of this barrier is well approximated by

$$f(M)\,dM \approx p(B)\,dB, \tag{19}$$

where B depends on M and V. If the initial distribution of fluctuations was Gaussian then we should set $p(x)$ to be Gaussian. If we then insert this in equation (18), we have a model for the large scale probability distribution of the mass. The result of doing this is very similar to the model of the nonlinear probability distribution derived by Fosalba & Gaztañaga (1998).

Fosalba & Gaztañaga argued that their analysis could not be applied on small scales. Our excursion set model, however, can be used even on smaller scales. The only caveat is that the first crossing distribution is more complicated than the simple transformation of the Gaussian given above. In this sense, the excursion set model can be thought of as providing a simple way to extend the results presented in Fosalba & Gaztañaga to smaller, more nonlinear scales.

4. Discussion

It has become common practice to announce that cosmology, as a subject, has matured. While this is welcome news, I fear that this maturity also signifies a sort of loss of innocence. Interest has shifted from simple models which capture the essence of the nonlinear physics of gravitational instability, to detailed numerical simulations of the growth of clustering, in which the problem is solved by brute force, sometimes with no net increase in physical insight—the exponential growth in computing power in recent years has not been accompanied by a corresponding increase in our understanding of how clustering evolves.

Some aspects of the first two simple models of the growth of hierarchical clustering described above are quite general; they are not restricted to the non-linearities associated with gravitational instability in an expanding Universe.

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