Appl Math Optim 12:97–114 (1984)

Applied Mathematics and Optimization

A Random Wave Process

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Communicated by H. H. Kuo

Abstract. The parabolic or forward scattering approximation to the equation describing wave propagation in a random medium leads to a stochastic partial differential equation which has the form of a random Schrödinger equation. Existence, uniqueness and continuity of solutions to this equation are established. The resulting process is a Markov diffusion process on the unit sphere in complex Hilbert space. Using Markov methods a limiting Markov process is identified in the case of a narrow beam limit; this limiting process corresponds to a simple random translation of the beam known as "spot-dancing."

1. Introduction

Consider the reduced, scalar wave equation

$$\nabla^2 u(\mathbf{x}) + k^2 n^2(\mathbf{x}) u = 0, \qquad \mathbf{x} \in R^3, \tag{1.1}$$

when the index of refraction $n(\mathbf{x})$ is a random function. Here k denotes the free space wave number. In many physical problems one is interested in solutions of (1.1) that propagate mainly in one direction, say the x_3 direction, with negligible backscattering. Such solutions can be obtained by solving a simpler equation, a

[1]Research supported by the Natural Sciences and Engineering Research Council of Canada.

[2]Research supported by the Air Force Office of Scientific Research under Grant number AFOSR-80-0228.

parabolic or Schrödinger equation.

If we write

$$u(x_1, x_2, x_3) = v(x_1, x_2, x_3)e^{ikx_3} \tag{1.2}$$

and assume that v varies slowly in the x_3 direction so that the $\partial^2 v/\partial x_3^2$ can be neglected, then v satisfies the equation:

$$2ik\, \partial v/\partial x_3 + \Delta v + k^2(n^2(\mathbf{x}) - 1)v = 0 \tag{1.3}$$

in which Δ is the Laplace operator in the transverse coordinates

$$\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2. \tag{1.4}$$

Equation (1.3) is the parabolic or forward scattering wave equation (Klyatskin and Tatarskii (1970)). It is to be solved as an initial value problem for $x_3 > 0$ with $v(x_1, x_2, 0)$ given. The range of validity of the parabolic approximation is discussed in the literature but there seems to be no general mathematical analysis of the passage from (1.1) to (1.3). In any case the approximation will be valid when k^{-1} is small compared to the correlation lengths of $n^2(\mathbf{x}) - 1$ in the directions x_1 and x_2 which are transverse to the propagation direction x_3.

If we divide (1.3) by $2k$, rescale x_2 and x_3, let $t = x_3$ and put

$$\mu = \tfrac{1}{2}k(n^2 - 1) \tag{1.5}$$

we obtain the scaled form of (1.3)

$$\begin{aligned} &i\, \partial V/\partial t + \Delta V + \mu V = 0, \qquad t > 0 \\ &\mathbf{x} = (x_1, x_2) \in R^2, \\ &V(0, x_1, x_2) = V_0(x_1, x_2). \end{aligned} \tag{1.6}$$

Here $\mu(t, x_1, x_2)$ is a given real-valued stochastic process and $V_0(x_1, x_2)$ is a given complex-valued function that may or may not be random. The analysis of the stochastic partial differential equation (1.6) can be carried out under different hypotheses regarding the random coefficient μ. For example the method of smooth perturbations can be used when μ is small (Keller (1964), Tatarskii (1971)). The Rytov method can also be used (Tatarskii (1971)). However, in many physical problems the predictions of low order perturbation theory (usually second order) are at variance with empirical observations. This has motivated the study of (1.6) without perturbation methods in the region of "saturated" fluctuations.

In particular saturated fluctuations arise for large distances along the axis of propagation. In such cases the correlation length of μ in the direction of propagation is small compared to the range along the axis of propagation over which the solution is sought. Therefore with an appropriate rescaling of the t-axis it is convenient to assume that μ is Gaussian white noise in the t variable and then view (1.1) (with the appropriate Stratonovich correction) as an Itô stochastic partial differential equation. The parabolic and Gaussian white noise approximations have been used extensively in the study of wave propagation in a random medium (cf. Tatarskii (1971), Klyatskin (1975) and Strohbehn (1978)). A more detailed discussion of the mathematical conditions under which these approximations are valid and their physical significance are given in Dawson and Papanicolaou (1984).

It is the objective of this paper to study the process V defined by (1.6) with μ as above using the methods of both stochastic partial differential equations and infinite dimensional Markov processes. In particular the latter are required in the study of certain limiting regimes and are illustrated in this paper by applying them to the large noise narrow beam limit in section 4.

2. The Random Schrödinger Equation

This section is devoted to the definition and analysis of the stochastic process obtained from the stochastic partial differential equation (1.6) in the Gaussian white noise limit.

We first describe the underlying noise process μ which we will denote by W', the formal white noise process. Let $\mathcal{S}(R^2)$ denote the space of C^∞-functions which together with their derivatives of all orders decrease rapidly at infinity. $\mathcal{S}'(R^2)$ is the dual space of tempered distributions. Let $\{W(t): t \geqq 0\}$ be a $\mathcal{S}'(R^2)$-valued Wiener process (cf. Itô (1983)) whose law is a probability measure on $\Omega := C([0,\infty), \mathcal{S}'(R^2))$ which is characterized by the following properties:

for each $\phi \in \mathcal{S}(R^2)$, $\langle \phi, W(t)\rangle$ is a one dimensional Wiener process where $\langle \cdot,\cdot\rangle$ denotes the natural bilinear functional on $\mathcal{S}(R^2)\times \mathcal{S}'(R^2)$, and (2.1)

for each $\phi, \psi \in \mathcal{S}(R^2)$,

$$\Gamma(\phi,\psi) := E(\langle \phi, W(t)\rangle\langle \psi, W(s)\rangle)$$

$$= \min(s,t)\int_{R^2}\int_{R^2}\phi(\mathbf{y}_1)\psi(\mathbf{y}_2)Q(\mathbf{y}_1-\mathbf{y}_2)\,d\mathbf{y}_1\,d\mathbf{y}_2 \tag{2.2}$$

where $Q(\cdot)$ is a continuous symmetric positive definite function, since $\Gamma(\phi,\psi)$ is a covariance functional.

Property (2.2) implies that $W(\cdot)$ is *spatially homogeneous*, that is, $\langle \phi, W(t)\rangle$ has the same probability law as $\langle \theta_{\mathbf{y}}\phi, W(t)\rangle$ where $\theta_{\mathbf{y}}\phi(\mathbf{y}_1) = \phi(\mathbf{y}+\mathbf{y}_1)$. From the continuity of $Q(\cdot)$ it also follows that for fixed t, $W(t,\mathbf{y})$ is an ordinary random function of $\mathbf{y}$ so that $\langle \phi, W(t)\rangle$ can be written as an integral

$$\langle \phi, W(t)\rangle = \int_{R^2} W(t,\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y},$$

in fact, the mapping $\mathbf{y} \to W(t,\mathbf{y})$ is continuous from R^2 into $L^2(\Omega)$ (cf. Meiden (1980)).

In the spectral representation of Q

$$Q(\mathbf{y}) = \sigma^2 \int_{R^2} e^{-i(\mathbf{y},\mathbf{x})} \rho(d\mathbf{x}), \tag{2.3}$$

Bochner's theorem implies that $\rho(\cdot)$ is a probability measure with σ^2 a constant. Then $W(\cdot)$ has the spectral representation

$$W(t,\mathbf{y}) = \int_{R^2} e^{-i(\mathbf{x},\mathbf{y})} M(t, d\mathbf{x}) \tag{2.4}$$

where for each $t \geqq 0$, $M(t,\cdot)$ is a complex Gaussian finitely additive random measure with orthogonal increments, that is, if A and B are Borel subsets of R^2,

$$E(M(t,A)) = 0, \quad \text{and} \tag{2.5a}$$

$$E\left(M(t,A)M(t,B)^*\right) = \sigma^2 \rho(A \cap B) \cdot t, \tag{2.5b}$$

(where * denotes complex conjugate). In addition $M(t,\cdot)$ inherits from $W(t,\cdot)$ the property of independent increments in time. Finally since $M(\cdot)$ is a random spectral measure,

$$M(t,A) = M(t,-A)^* \tag{2.6}$$

which follows from the fact that $W(\cdot)$ is real-valued.

Taking the limit of equation (1.6) when $\mu\, dt$ approaches the Gaussian white noise dW in the t-direction, leads to the *Itô-Schrödinger equation*:

$$dV(t,\mathbf{y}) = \left[-i\Delta V(t,\mathbf{y}) - \tfrac{1}{2}\sigma^2 V(t,\mathbf{y})\right] dt + iV(t,\mathbf{y})\, dW(t, d\mathbf{y}) \tag{2.7}$$

for $t \geqq 0$, $\mathbf{y} = (y_1, y_2)$ and $\Delta = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$.

It turns out that it is more convenient to study the Fourier transformed version of (2.7)

$$dX(t,\mathbf{x}) = \left[i|\mathbf{x}|^2 - \tfrac{1}{2}\sigma^2\right] X(t,\mathbf{x})\, dt + i\int_{r^2} X(t,\mathbf{x}-\boldsymbol{\xi}) M(dt, d\boldsymbol{\xi}), \tag{2.8}$$

where

$$X(t,\mathbf{x}) := (1/2\pi)^2 \int_{R^2} e^{i(\mathbf{x},\mathbf{y})} V(t,\mathbf{y})\, d\mathbf{y},$$

and where for $C := [t_1, t_2]$, A Borel in R^2, $M(C,B) := M(t_2,B) - M(t_1,B)$. The terms $-\tfrac{1}{2}\sigma^2 V$ and $-\tfrac{1}{2}\sigma^2 X$ in Equations (2.7), (2.8) are the Stratonovich correction terms and arise since these equations were derived as the white noise limits of equations having smooth random coefficients (cf. Stratonovich (1965)).

There are several possible definitions of what is meant by a solution to an equation of the form (2.7) or (2.8) (refer to Miyahara (1982) for these definitions and a discussion of the relation between them). In this paper we consider an *evolution* (or "mild") *solution*. $X(\cdot,\cdot)$ is said to be an evolution solution of Equation (2.8) if:

$$X(t,\mathbf{x}) = T_t X(0,\mathbf{x}) + i\int_0^t \int_{R^2} T_{t-s}(X(s,\mathbf{x}-\xi)) \cdot M(ds, d\xi), \tag{2.9}$$

where

$$T_t f(\mathbf{x}) := \exp\left((i|\mathbf{x}|^2 - \tfrac{1}{2}\sigma^2)t\right) \cdot f(\mathbf{x}).$$

The right hand side of (2.9) is interpreted as an Itô stochastic integral (cf. Dawson and Salehi (1980, Section 2)).

An evolution solution to Equation (2.8) can be obtained by the method of Wiener-Itô expansions (cf. Dawson and Salehi (1980), Miyahara (1982)). To implement this we define recursively:

$$X_0(t,\mathbf{x}) = T_t X(0,\mathbf{x})$$

$$X_{n+1}(t,\mathbf{x}) = i\int_0^t \int_{R^2} T_{t-s} X_n(s,\mathbf{x}-\xi) \cdot M(ds, d\xi). \tag{2.10}$$

Assume that $\|X(0,\cdot)\|_{L^2(R^2)} = 1$, $\sup_{\mathbf{x}} E\{|X(0,\mathbf{x})|^2\} < \infty$, and that $X(0,\cdot)$ is independent of $M(\cdot,\cdot)$.

Theorem 2.1. *The series*

$$X(t,\cdot) = \sum_{n=0}^{\infty} X_n(t,\cdot) \tag{2.11}$$

converges strongly in $L^2(R^2)$ *with probability one and converges in the* $L^2(\Omega \times R^2, P \times \lambda)$*-norm where* λ *denotes Lebesgue measure. The resulting process* $X(\cdot,\cdot)$ *is an* $L^2(R^2)$*-valued evolution solution to Equation* (2.8), *that is*

$$\left\| X(t,\mathbf{x}) - T_t X(0,\mathbf{x}) - i\int_0^t \int_{R^2} T_{t-S} X(s,\mathbf{x}-\xi) \cdot M(ds, D\xi) \right\|_{L^2(R^2)} = 0 \tag{2.12}$$

with probability one.

Proof. The processes $\{X_n(t,\cdot)\}$ can be represented as multiple Wiener integrals of degree n and consequently are orthogonal in $L^2(P)$ (cf. Dawson and Salehi

(1980, Theorem 2.1)). Then

$$E|X_{n+1}(t,\mathbf{x})|^2 = \sigma^2\int_0^t\int_{R^2}\exp(-\sigma^2(t-s))\cdot E|X_n(s,\mathbf{x}-\xi)|^2\rho(d\xi)\,ds.$$

Therefore by Fubini's theorem

$$E\left(\int|X_{n+1}(t,\mathbf{x})|^2\,d\mathbf{x}\right) = \sigma^2\int_0^t\left[\int E|X_n(s,\mathbf{x})|^2\,d\mathbf{x}\right]\cdot\exp(-\sigma^2(t-s))\,ds.$$

This yields:

$$E\left(\int|X_0(t,\mathbf{x})|^2\,d\mathbf{x}\right) = \exp(-\sigma^2 t),$$

$$E\left(\int|X_1(t,\mathbf{x})|^2\,d\mathbf{x} = \sigma^2\exp(-\sigma^2 t)\cdot t,\right.$$

$$E\left(\int_{R^2}|X_n(t,\mathbf{x})|^2\,d\mathbf{x}\right) = \left((\sigma^2 t)^n/n!\right)\cdot\exp(-\sigma^2 t). \tag{2.13}$$

Similarly,

$$\sup_{\mathbf{x}} E|X_n(t,\mathbf{x})|^2 \leqq \left((\sigma^2 t)^n/n!\right)\cdot\exp(-\sigma^2 t)\cdot\sup_{\mathbf{x}} E\left(|X(0,\mathbf{x})|^2\right). \tag{2.14}$$

By the orthogonality in $L^2(P)$ of the X_n it can be shown that

$$E\left(\left\|\sum_{k=n}^{n+m} X_k(t,\cdot)\right\|^2\right) = \sum_{k=n}^{n+m}\left((\sigma^2 t)^k/k!\right)\cdot\exp(-\sigma^2 t). \tag{2.15}$$

From (2.15) and the Borel-Cantelli lemma we can verify that (2.11) converges in the $L^2(R^2)$-norm with probability one and also in the $L^2(\Omega\times R^2, P\times\lambda)$-norm. Furthermore, summing (2.13) and (2.14) yields,

$$E\left(\|X(t)\|^2_{L^2(R^2)}\right) = 1, \qquad \sup_{\mathbf{x}} E|X(t,\mathbf{x})|^2 \leqq \sup E\left(|X(0,\mathbf{x})|^2\right). \tag{2.16}$$

To verify (2.12) first note that the stochastic integral on the right hand side exists since

$$\int_0^t\int_{R^2}\exp(-\sigma^2(t-s))\cdot E\left(|X(s,\mathbf{x}-\xi)|^2\right)\rho(d\xi)\,ds$$
$$\leqq \left(1-\exp(-\sigma^2 t)\right)\cdot\sup_{\mathbf{x}} E\left(|X(0,\mathbf{x})|^2\right) < \infty.$$

Furthermore,

$$\int_{R^2} E\left(\left|\int_0^t \int_{R^2} T_{t-s}\left(\sum_{n=N+1}^{\infty} X_n(s,\mathbf{x}-\xi)\right)\cdot M(ds,d\xi)\right|^2\right) d\mathbf{x}$$

$$\leqq \sigma^2 \int_0^t \int_{R^2}\int_{R^2} \exp(-\sigma^2(t-s))\cdot\left(\sum_{n=N+1}^{\infty} E|X_n(s,\mathbf{x}-\xi)|^2\right)\rho(d\xi)\,d\mathbf{x}\,ds$$

$$= \sigma^2 \int_0^t \exp(-\sigma^2(t-s))\cdot\left(\sum_{n=N+1}^{\infty} E\|X_n(s)\|_{L^2(R^2)}\right) ds$$

$$\to 0 \quad \text{as } N \to \infty.$$

Therefore

$$i\int_0^t \int_{R^2} T_{t-s}\left(\sum_{n=0}^{\infty} X_n(s,\mathbf{x}-\xi)\right)\cdot M(ds,d\xi)$$

$$= \sum_{n=0}^{\infty} i\int_0^t \int_{R^2} T_{t-s} X_n(s,\mathbf{x}-\xi)\cdot M(ds,d\xi)$$

$$= \sum_{n=1}^{\infty} X_n(t,\mathbf{x}) = X(t,\mathbf{x}) - X_0(t,\mathbf{x}) \quad \text{a.s.}$$

and the proof of (2.12) is complete. □

Remark 2.1. The pathwise uniqueness can also be obtained as in Dawson and Salehi (1980). However we omit this but prove the uniqueness in law by the method of duality in the next section.

Remark 2.2. We define the moment functions:

$$m_2(t;\mathbf{x}_1,\mathbf{x}_2) := E\big(X(t,\mathbf{x}_1)X^*(t,\mathbf{x}_2)\big)$$

$$m_4(t;\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_3,\mathbf{x}_4) := E\big(X(t,\mathbf{x}_1)X(t,\mathbf{x}_2)X^*(t,\mathbf{x}_3)X^*(t,\mathbf{x}_4)\big).$$

Using Itô's lemma for Hilbert space valued processes (cf. Miyahara (1982, Theorem 2.4)), we obtain the following moment equation

$$\partial m_2(t;\mathbf{x}_1,\mathbf{x}_2)/\partial t = -\sigma^2 m_2(t;\mathbf{x}_1,\mathbf{x}_2) - i\big(|\mathbf{x}_1|^2 - |\mathbf{x}_2|^2\big)m_2(t;\mathbf{x}_1,\mathbf{x}_2) + \sigma^2\int m_2(t;\mathbf{x}_1-\mathbf{r},\mathbf{x}_2-\mathbf{r})\rho(d\mathbf{r}) \tag{2.17}$$

and a similar equation for $m_4(t;.,.,.,.)$. Using these equations it is easy to verify that the moment functions are well defined and integrable for all times t.

It is easy to verify that the stochastic process given by the evolution solution of Equation (2.8) has a weakly right continuous version. We now proceed to prove that it is actually a weakly continuous process.

Theorem 2.2. $\|X(t)\|_{L^2(R^2)} \leqq \|X(0)\|_{L^2(R^2)}$ *for all* $t \geqq 0$ *with probability one.*

Proof. Let $\{e_n\}$ be an orthonormal basis in $L^2(R^2)$. Using Itô's Lemma for Hilbert space valued processes (cf. Miyahara (1982, Theorem 2.4))

$$
\begin{aligned}
&\langle e_n, X(t)\rangle\langle e_n, X^*(t)\rangle - \langle e_n, X(0)\rangle\langle e_n, X^*(0)\rangle \\
&\quad = -\frac{1}{2}\sigma^2\int_0^t \langle e_n, X(s)\rangle\langle e_n, X^*(s)\rangle\, ds \\
&\qquad + \frac{1}{2}\sigma^2\int_0^t\int_{R^2}\int_{R^2}\int_{R^2} e_n(\mathbf{x}_1)e_n(\mathbf{x}_2)X(s,\mathbf{x}_1-\boldsymbol{\xi}) \\
&\qquad \times X^*(s,\mathbf{x}_2-\boldsymbol{\xi})\rho(d\boldsymbol{\xi})\, d\mathbf{x}_1, d\mathbf{x}_2 \\
&\qquad + i\int_0^t\int_{R^2}\left[\int_{R^2} X(s,\mathbf{x}-\boldsymbol{\xi})e_n(\mathbf{x})\, d\mathbf{x}\cdot\langle e_n, X^*(s)\rangle M(ds, d\boldsymbol{\xi})\right. \\
&\qquad \left. - \int_{R^2} X^*(s,\mathbf{x}-\boldsymbol{\xi})e_n(\mathbf{x})\, d\mathbf{x}\cdot\langle e_n, X(s)\rangle M^*(ds, d\boldsymbol{\xi})\right].
\end{aligned}
$$

Summing on n,

$$
\begin{aligned}
&\|X(t)\|^2_{L^2(R^2)} - \|X(0)\|^2_{L^2(R^2)} \\
&\quad = -\frac{1}{2}\sigma^2\int_0^t \|X(s)\|^2_{L^2(R^2)}\, ds + \frac{1}{2}\sigma^2\int_0^t \|X(s)\|^2_{L^2(R^2)}\, ds \\
&\qquad + i\int_0^t\int_{R^2}\int_{R^2}\left[X^*(\mathbf{x})X(\mathbf{x}-\boldsymbol{\xi})\, d\mathbf{x}\, M(ds, d\boldsymbol{\xi})\right. \\
&\qquad \left. - X(\mathbf{x})X^*(\mathbf{x}-\boldsymbol{\xi})\, d\mathbf{x}\, M^*(ds, d\boldsymbol{\xi})\right].
\end{aligned} \tag{2.18}
$$

The stochastic integral on the right hand side of (2.18) is well-defined since as noted above the fourth moment function is integrable. Property (2.6) of the spectral measure then implies that the last term is zero and therefore

$$\|X(t)\|_{L^2(R^2)} = \|X(0)\|_{L^2(R^2)} \tag{2.19}$$

with probability one. The result then follows by the weak continuity. □

Theorem 2.3. *The process* $X(t)$ *is almost surely a weakly continuous function from* $[0,\infty)$ *to* $L^2(R^2)$.

Proof. Let $\{e_n : n = 0,1,2,\ldots\}$ denote a complete orthonormal system for $L^2(R^2)$. Then

$$M_n(t) := \langle e_n, X(t)\rangle - \int_0^t \left\langle e_n, (i|\mathbf{x}|^2 - 1/2\sigma^2)\cdot X(s)\right\rangle ds \tag{2.20}$$

is a P-complex martingale. Moreover $M_n(t)\cdot M_n^*(t) - \langle M_n(t)\rangle$ is a P-martingale where the increasing process $\langle M_n(t)\rangle$ is given by:

$$\begin{aligned}\langle M_n(t)\rangle = \sigma^2 \int_0^t \int_{R^2}\int_{R^2} X(s, \mathbf{x}_1 - \boldsymbol{\xi}) X^*(s, \mathbf{x}_2 - \boldsymbol{\xi}) e_n(\mathbf{x}_1) \\ \times e_n(\mathbf{x}_2)\rho(d\boldsymbol{\xi})\, d\mathbf{x}_1\, d\mathbf{x}_2\, ds.\end{aligned} \tag{2.21}$$

Since $\|X(s)\|_{L^2(R^2)} \leqq \|X(0)\|_{L^2(R^2)}$, Schwarz's inequality implies that for $t > s$,

$$\langle M_n(t)\rangle - \langle M_n(s)\rangle \leqq \sigma^2(t-s) \quad \text{a.s.} \tag{2.22}$$

Hence for $k \geqq 1$, it follows from the Burkholder-Gundy inequalities that for $\delta > 0$,

$$E\left(\sup_{t \leqq s \leqq t+\delta} |M_n(s) - M_n(t)|^k\right) \leqq c_k \sigma^2 \delta^{1/2k} \tag{2.23}$$

where c_k is a constant. Inequality (2.23) together with the Kolmogorov criterion implies the a.s. continuity of $\langle e_n, X(t)\rangle$ and hence the a.s. weak continuity. □

3. The Markov Diffusion Process

Let H denote the unit ball in the complex Hilbert space $L^2(R^2)$ endowed with the weak topology so that it is compact. In this section we identify the random wave process as a Markov diffusion process on H using the viewpoint of an appropriate martingale problem.

A martingale problem is described by a pair $(\mathscr{D}, \mathscr{L})$ where $\mathscr{D} \subset C(H)$ and $\mathscr{L}$ is a linear operator defined on $\mathscr{D}$. Let $\Omega := C([0,\infty), H)$. Then a solution to the martingale problem is a mapping $h \to P_h$ from H to $M_1(\Omega)$, the space of probability measures on Ω, such that

$$P_h(X(0) = h) = 1, \quad \text{and} \tag{3.1}$$

for every $F \in \mathscr{D}$, $F(X(t)) - \int_0^t \mathscr{L}F(X(s))\, ds$ is a P_h martingale for each $h \in H$. (3.2)

The operator $\mathscr{L}$ associated with the stochastic evolution equation (2.8) involves first and second variational derivatives and is given by:

$$\begin{aligned}
\mathscr{L}F(h,h^*) = & -\frac{1}{2}\sigma^2\left[\int h(\mathbf{x})(\delta F/\delta h(\mathbf{x}))\,d\mathbf{x}+\int h^*(\mathbf{x})(\delta F/\delta h^*(\mathbf{x}))\,d\mathbf{x}\right] \\
& - i\int|\mathbf{x}|^2 h(\mathbf{x})(\delta F/\delta h(\mathbf{x}))\,d\mathbf{x} + i\int|\mathbf{x}|^2 h^*(\mathbf{x})(\delta F/\delta h^*(\mathbf{x}))\,d\mathbf{x} \\
& -\frac{1}{2}\sigma^2\int_{R^2}\int_{R^2}\int_{R^2} h(\mathbf{x}-\mathbf{r})h(\mathbf{y}+\mathbf{r}) \\
& \times(\delta^2 F/\delta h(\mathbf{x})\,\delta h(\mathbf{y}))\rho(d\mathbf{r})\,d\mathbf{x}\,d\mathbf{y} \\
& -\frac{1}{2}\sigma^2\int_{R^2}\int_{R^2}\int_{R^2} h^*(\mathbf{x}-\mathbf{r})h^*(\mathbf{y}+\mathbf{r}) \\
& \times(\delta^2 F/\delta h^*(\mathbf{x})\,\delta h^*(\mathbf{y}))\rho(d\mathbf{r})\,d\mathbf{x}\,d\mathbf{y} \\
& +\sigma^2\int_{R^2}\int_{R^2}\int_{R^2} h(\mathbf{x}-\mathbf{r})h^*(\mathbf{y}-\mathbf{r}) \\
& \times(\delta^2 F/\delta h(\mathbf{x})\,\delta h^*(\mathbf{y}))\rho(d\mathbf{r})\,d\mathbf{x}\,d\mathbf{y}
\end{aligned} \tag{3.3}$$

where $\delta F/\delta h(\mathbf{x})$ and $\delta^2 F(\delta h(\mathbf{x})\,\delta h^*(\mathbf{y})$ denote the first and second variational derivatives, respectively.

A polynomial function on H with coefficients in $\mathscr{S}$ is of the form:

$$F(h,h^*) = \sum_{m,n=1}^{k} F_{f_{n,m}}(h,h^*) \tag{3.4}$$

where $f_{n,m}\in\mathscr{S}(R^{2(m+n)})$ and

$$\begin{aligned}
F_{f_{n,m}}(h,h^*) = F_h(f_{n,m}) := & \int_{R^2}\cdots\int_{R^2} f_{n,m}(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{x}_{n+1},\ldots,\mathbf{x}_{n+m}) \\
& \cdot h(\mathbf{x}_1)\ldots h(\mathbf{x}_n)h^*(\mathbf{x}_{n+1})\ldots h^*(\mathbf{x}_{n+m})\,d\mathbf{x}_1\ldots d\mathbf{x}_{n+m}.
\end{aligned} \tag{3.5}$$

We denote by $\mathscr{D}_0$ the algebra of such polynomial functions on H. Since it separates points, it is dense in $C(H)$ and is convergence determining.

For $F\in\mathscr{D}_0$, $\mathscr{L}$ is given by:

$$\begin{aligned}
\mathscr{L}F_{f_{n,m}}(h,h^*) = & \mathscr{K}F_{f_{n,m}}(h,h^*) \\
& +\left[1+\tfrac{1}{2}\sigma^2\big((n+m)^2-2(n+m)\big)\right]F_{f_{n,m}}(h,h^*)
\end{aligned} \tag{3.6}$$

where

$$\mathscr{K}F_{f_{n,m}}(h,h^*) = \mathscr{K}^{\#}F_h(f_{n,m})$$

$$\mathscr{K}^{\#}F_h(f_{n,m}) = \frac{1}{2}\sigma^2 \sum_{\substack{j,k=1\\ j\neq k}}^{n+m} \left[F_h(K_{jk}f_{n,m}) - F_h(f_{n,m})\right] + \left(F_h(Vf_{n,m}) - F_h(f_{n,m})\right)$$

and

$$V := -i\sum_{j=1}^{n}|\mathbf{x}_j|^2 + i\sum_{j=n+1}^{n+m}|\mathbf{x}_j|^2,$$

$$K_{jk}f_{n,m} = -\int_{R^2} f_{n,m}\big(\mathbf{x}_1,\ldots,(\mathbf{x}_j-\mathbf{r}),\ldots,(\mathbf{x}_k+\mathbf{r}),\ldots,\mathbf{x}_n;\mathbf{x}_{n+1},\ldots,\mathbf{x}_{n+m}\big)\rho(d\mathbf{r}) \quad \text{for } 1 \leqq j,k \leqq n,$$

$$= -\int_{R^2} f_{n,m}\big(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{x}_{n+1},\ldots,(\mathbf{x}_j-\mathbf{r}),\ldots,(\mathbf{x}_k+\mathbf{r}),\ldots,\mathbf{x}_{n+m}\big)\rho(d\mathbf{r})$$

for $n+1 \leqq j,k \leqq n+m$, and

$$= \int_{R^2} f_{n,m}\big(\mathbf{x}_1,\ldots,(\mathbf{x}_j-\mathbf{r}),\ldots,\mathbf{x}_n;\mathbf{x}_{n+1},\ldots,(\mathbf{x}_k-\mathbf{r}),\ldots,\mathbf{x}_{n+m}\big)\rho(d\mathbf{r})$$

$$\text{for} \quad 1 \leqq j \leqq n,\ (n+1) \leqq k \leqq (n+m).$$

Theorem 3.1. (a) *The probability law $P_{X(0)}$ of the evolution solution of the Equation* (2.8) *is a solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem.* (b) *The $(\mathscr{D}_0, \mathscr{L})$-martingale problem has a unique solution and is the law of a strong Markov process with state space H.* (c) *For $h \in H_1 := \{h : h \in H, \|h\| = 1\}$, P_h is the probability law of a strongly continuous Markov diffusion process on H_1.*

Proof. (a) Let $\{X(t) : t \geqq 0\}$ denote the evolution solution of Equation (2.8) constructed in Section 2 and let $F \in \mathscr{D}_0$. Using Itô's lemma for Hilbert-space-valued processes (cf. Miyahara (1982, Theorem 2.4)), it can be shown that

$$F_{f_{n,m}}(X(t), X^*(t)) - F_{f_{n,m}}(X(0), X^*(0)) - \int_0^t \mathscr{L}F_{f_{n,m}}(X(s), X^*(s))\,ds$$

is given by the stochastic integral

$$i\int_0^t\int_{R^2}\cdots\int_{R^2} f_{n,m}(\mathbf{x}_1,\ldots,\mathbf{x}_{n+m})$$
$$\times\Bigg[\sum_{j=1}^{n} X(s,\mathbf{x}_1)\ldots X(s,\mathbf{x}_j-\xi)\ldots X^*(s,\mathbf{x}_{n+m})M(ds,d\xi)$$
$$-\sum_{j=n+1}^{n+m} X(s,\mathbf{x}_1)\ldots X^*(s,\mathbf{x}_j-\xi)\ldots X^*(s,\mathbf{x}_{n+m})M^*(ds,d\xi)\Bigg]$$
$$\times\, d\mathbf{x}_1\cdot d\mathbf{x}_{n+m} \tag{3.7}$$

which is a $P_{X(0)}$-martingale. From this it follows that the probability law of $X(\cdot)$ is a solution to the $(\mathscr{D}_0, \mathscr{L})$-martingale problem.

(b) The uniqueness is proved by the method of duality. Observe that $\mathscr{K}^{\#}$ is the restriction of a $\mathscr{D}_0$-valued Markov jump processes $\{\xi(t): t \geqq 0\}$ with jumps:

$$\text{Rate } 1: f_{n,m} \to Vf_{n,m}$$
$$\text{Rate } \tfrac{1}{2}\sigma^2: f_{n,m} \to K_{jk}f_{n,m} \quad \text{for } 1 \leqq j,k \leqq n \text{ or } (n+1) \leqq j,k \leqq (n+m),$$
$$\text{Rate } \sigma^2: f_{n,m} \to K_{jk}f_{n,m} \quad \text{for } 1 \leqq j \leqq n,\ (n+1) \leqq k \leqq (n+m). \tag{3.8}$$

It then follows from the duality relationship between $X(\cdot)$ and $\xi(\cdot)$ (cf. Dawson and Kurtz (1982, Theorem 3.1)) that any solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem must satisfy

$$E_n\left(F_{f_{n,m}}(X(t), X^*(t)\right) = E_{f_{n,m}}\left(F_h(\xi(t))\cdot\exp\left(\tfrac{1}{2}\sigma^2\left((n+m)^2-2(m+n)\right)t\right)\right). \tag{3.9}$$

Then using the results of Stroock and Varadhan (1979, Section 6.2) it follows that there is at most one solution to the martingale problem and that it is the law of a strong Markov process on H.

(c) Let P_h be the solution of the $(\mathscr{D}_0, \mathscr{L})$-martingale problem when $h \in H_1$. Let τ denote the stopping time

$$\tau := \inf\{t: \|X(t)\| \leqq 1-a\} \quad \text{for some } a > 0.$$

Assume that $P(\tau < b) > 0$ for some $b < \infty$. Then by theorem 2.2 and the strong Markov property,

$$P(\|X(b)\| < 1) > 0$$

thus yielding a contradiction of (2.19). Thus $P(\tau < \infty) = 0$ and it follows that $X(t) \in H_1$ for all t with probability one.

It remains to show that a weakly continuous function of t lying entirely in H_1 is actually strongly continuous. Assume that

$$\sum_{n=0}^{\infty} a_n(t) e_n \to \sum_{n=0}^{\infty} a_n(t_0) e_n \text{ weakly and that } \sum_{n=0}^{\infty} |a_n(t)|^2 = 1 \quad \text{for all } t$$

and

$$\sum_{n=0}^{\infty} |a_n(t_0)|^2 = 1.$$

It suffices to show that for $\varepsilon > 0$,

$$\lim_{t \to t_0} \left\| \sum_{n=0}^{\infty} a_n(t) e_n - \sum_{n=0}^{\infty} a_n(t_0) e_n \right\| < \varepsilon. \tag{3.10}$$

Choose N such that $\sum_{n=N}^{\infty} |a_n(t_0)|^2 < \frac{1}{4}\varepsilon$. Then choose $\eta > 0$ such that for $|t - t_0| \leqq \eta$, $\sum_{n=1}^{N} |a_n(t) - a_n(t_0)|^2 < \varepsilon/8$. Then (3.10) can be verified. Thus the proof of the strong continuity is complete. □

Remark. The norm $\|X(t)\|_{L^2(R^2)}$ denotes the total energy in the wave as a function of t. Thus the physical significance of part (c) of Theorem 3.1 is the conservation of energy.

Theorem 3.1 completes the characterization of the random wave process $\{X(t): t \geq 0\}$. In applications the main interest is in the identification of the probability distribution of the random variable $|X(t, \mathbf{x})|$ for $\mathbf{x} \in R^2$. Although Theorem 2.1 provides an explicit solution for the stochastic evolution equation (2.8), it provides no solution to this problem since there is no procedure for finding the probability distribution of a random variable described by a Wiener-Itô series. For this reason various limiting regimes in which these distributions can be evaluated have been studied in the literature. In the next section we consider one such limiting regime, namely, the narrow beam spot-dancing limit.

4. The Narrow Beam Spot-Dancing Limit

The random wave process is completely determined by the spectral measure $\rho(\cdot)$. In order that the stochastic evolution equation be a good approximation to the original random wave equation certain conditions on the parameters must be satisfied. However, consistent with these constraints there are a number of limiting regimes which are of physical interest (cf. Dawson and Papanicolaou (1984)). In this section we consider one of these. It is given by the family $\{\sigma_\varepsilon^2, \rho_\varepsilon\}$

such that:

$$\sigma_\varepsilon^2 = \sigma^2/\varepsilon, \tag{4.1a}$$

$$\int |\mathbf{r}|^2 \rho_\varepsilon(d\mathbf{r}) = 1/\sigma_\varepsilon^2, \quad \text{and} \tag{4.1b}$$

$$\sigma_\varepsilon^2 \int |\mathbf{r}|^4 \rho_\varepsilon(d\mathbf{r}) = O(\varepsilon). \tag{4.1c}$$

For simplicity of exposition we also assume that $\rho_\varepsilon(d\mathbf{r})$ is isotropic, that is, invariant under rotations in R^2.

It is therefore appropriate to investigate the possible existence of a limiting process as $\varepsilon \downarrow 0$. The method developed in Section 2 to study the stochastic evolution equation (2.8) encounters serious difficulties in this case. To demonstrate this recall from (2.13) that for each n,

$$E\int |X_n(t,\mathbf{x})|^2 \, d\mathbf{x} = \left((\sigma^2 t)^n / n!\right) \cdot \exp(-\sigma^2 t) \to 0 \quad \text{as } \sigma^2 \to \infty.$$

Next we consider the behaviour of the dual process $\xi(t)$ for small ε. From (3.7) and (3.8) we conclude that

$$\begin{aligned} d/dt\left(E_{f_{n,m}}(\xi(t))\right) &= \tfrac{1}{2}\sigma_\varepsilon^2\left[(n+m)^2 - 2(m+n) - 2n(n-1) - 2m(m-1)\right] f_{n,m} \\ &+ O(\varepsilon)\sigma_\varepsilon^2/2 = -\tfrac{1}{2}\sigma_\varepsilon^2 (n-m)^2 f_{n,m} + O(1) \end{aligned} \tag{4.2}$$

Therefore if $m \neq n$, then $\lim_{\varepsilon \downarrow 0} E_h(F_{f_{n,m}}(X_\varepsilon(t), X_\varepsilon^*(t))) = 0$ for $t > 0$. In view of this we cannot hope to prove the weak convergence of $X_\varepsilon(\cdot)$ as $\varepsilon \downarrow 0$ to a limiting Markov diffusion process on H_1. Nevertheless a well-defined limit theorem can be established in a slightly different context.

Let $H^s := \{h \otimes h^* : h \in H\}$ with the topology induced by the weak topology and $H_1^s := \{h \otimes h^* : h \in H_1\}$ with the topology induced by the strong topology. Consider the H^s-valued stochastic process defined by:

$$Z_\varepsilon(t;\mathbf{x},\mathbf{y}) := X_\varepsilon(t,\mathbf{x}) X_\varepsilon^*(t,\mathbf{y}). \tag{4.3}$$

Let $\mathscr{D}_0^s(\subset C(H^s)) := \{F_{f_{n,m}} \in \mathscr{D}_0 \text{ with } n = m\}$ and $F_{f_{n,n}}^s(h \otimes h^*) := F_{f_{n,n}}(h) = F_{h\otimes h^*}^s(f_{n,n})$. Then Z_ε is a solution of the $(\mathscr{D}_0^s, \mathscr{L}_\varepsilon^s)$-martingale problem where for $F \in \mathscr{D}_0^s$,

$$\mathscr{L}_\varepsilon^s F_{f_{n,n}}(h \otimes h^*) = \mathscr{K}^{\#} F_{h\otimes h^*}(f_{n,n}) + \left[1 + 2\sigma^2 n(n-1)\right] F_{h \otimes h^*}(f_{n,n}). \tag{4.4}$$

Note that $\mathscr{L}_\varepsilon^s : \mathscr{D}_0^s \to \mathscr{D}_0^s$ and that the $(\mathscr{D}_0^s, \mathscr{L}_\varepsilon^s)$-martingale problem has a unique solution by duality. Therefore Z_ε is a H^s-valued Markov diffusion process. We next establish the uniqueness for the limiting martingale problem.

Lemma 4.1. *The H^s-valued martingale problem associated with the pair $(\mathscr{D}_0^s, \mathscr{L}_0^s)$ where*

$$\mathscr{L}_0^s F_{f_{n,n}}(h \otimes h^*) = \sigma^2 F_{h\otimes h^*}\left(\sum_{j=1}^{2n}\sum_{k=1}^{2n}(\nabla_j \cdot \nabla_k) f_{n,n}\right) + F_{h\otimes h^*}(V \cdot f_{n,n}) \tag{4.5}$$

has at most one solution, $P^0_{h\otimes h^*}$ on $C([0,\infty), H^s)$.

Proof. Let $Z_0(\cdot)$ denote a solution. Then uniqueness in law follows from the duality relation:

$$E^0_{h\otimes h^*}\left(F_{f_{n,n}}(Z_0(t))\right) = E_{f_{n,n}}(F_{h\otimes h^*}(\psi(t))) \tag{4.6}$$

where $\psi(\cdot)$ satisfies the deterministic evolution

$$\partial\psi/\partial t = \sigma^2 \sum_{j=1}^{2n}\sum_{k=1}^{2n}(\nabla_j \cdot \nabla_k)\psi + V\cdot\psi, \qquad \psi(0) = f_{n,n}. \tag{4.7}$$

Let $\{\mathbf{b}(t): t \geq 0\}$ denote a Brownian motion in R^2 with generator $\sigma^2 \nabla \cdot \nabla$. From (4.1) and the Feynman-Kac formula the solution of (4.7) is given by

$$\psi(t; \mathbf{x}_1, \ldots, \mathbf{x}_{2n}) = E\Bigg[f_{n,n}(\mathbf{x}_1 + \mathbf{b}(t), \ldots, \mathbf{x}_n + \mathbf{b}(t)) \cdot \exp\left(-i\sum_{j=1}^{n}\int_0^t |\mathbf{x}_j + \mathbf{b}(s)|^2\, ds + i\sum_{j=n+1}^{2n} \times \int_0^t |\mathbf{x}_j + \mathbf{b}(s)|^2\, ds\right)\Bigg]. \qquad \square \tag{4.8}$$

Remark 4.1. The class of functions $\mathscr{D}_0^s$ can be extended to include generalized functions of the form:

$$f_{n,n}(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{x}_{n+1}, \ldots, \mathbf{x}_{2n}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \cdot \delta(\mathbf{x}_{n+1} - \mathbf{x}_1) \ldots \delta(\mathbf{x}_{2n} - \mathbf{x}_n).$$

In this case $\psi(t) = E(f(\mathbf{x}_1 + \mathbf{b}(t), \ldots, \mathbf{x}_n + \mathbf{b}(t))$. Therefore in this case, the probability-measure-valued process $|X(t,\mathbf{x})|^2 d\mathbf{x}$ is Markov and has the representation:

$$|X(t,\mathbf{x})|^2 = |X(0, \mathbf{x} + \mathbf{b}(t))|^2. \tag{4.9}$$

This random displacement of the energy distribution has given rise to the descriptive phrase "spot-dancing". Representation (4.9) also implies that the limit process Z_0 is a strongly continuous H_1^s-valued Markov diffusion process.

Theorem 4.1. *For $h \in H$, the probability laws $P^{\varepsilon}_{h\otimes h^*}$ of Z_ε converge weakly as probability measures on $C([0,\infty), H^s)$ to the law $P^{0}_{h\otimes h^*}$ of the process Z_0 associated with the $(\mathscr{D}^s_0, \mathscr{L}^s_0)$-martingale problem.*

Proof. As a first step we verify that the probability measures $P^{\varepsilon}_{h\otimes h^*}$ are uniformly tight. Since H^s is endowed with the weak topology and thus is compact, it suffices to show that for fixed $f(\mathbf{x},\mathbf{y})$ that the process

$$\iint Z_\varepsilon(t;\mathbf{x},\mathbf{y}) f(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}$$

are weakly compact in $D([0,\infty), R)$.

Using the assumed properties of $\rho(\cdot)$, Taylor's formula with remainder and (4.4), it follows that for $F_{f_{n,n}} \in \mathscr{D}^s_0$

$$F_{f_{n,n}}(Z_\varepsilon(t)) - \int_0^t \mathscr{L}^s_0 F_f(Z_\varepsilon(s))\, ds + \|f_{n,n}\|_4 \cdot O(\varepsilon) t := M_t \tag{4.10}$$

is a bounded martingale where $\|f_{n,n}\|_4 := \max\limits_{i,j,k,l} \sup\limits_{\mathbf{x}_1\ldots\mathbf{x}_{2n}} |f^{(ijkl)}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})|$ where the latter refers to the mixed fourth partial derivative.

In the special case $n=1$, the martingale M_t associated with

$$\iint Z_\varepsilon(t;\mathbf{x},\mathbf{y}) f(\mathbf{x};\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} - \int_0^t \iint Z_\varepsilon(s;\mathbf{x},\mathbf{y}) V(\mathbf{x},\mathbf{y}) f(\mathbf{x};\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}\, ds$$
$$- \int_0^t \iint Z_\varepsilon(s;\mathbf{x},\mathbf{y}) \big[\Delta_x f(\mathbf{x};\mathbf{y}) + \Delta_y f(\mathbf{x};\mathbf{y}) + 2(\nabla_1\cdot\nabla_2) f(\mathbf{x};\mathbf{y})\big]\, d\mathbf{x}\, d\mathbf{y}\, ds \tag{4.11}$$

has a representation of the form:

$$M_t = i\int_0^t \iiint f(\mathbf{x},\mathbf{y}) X_\varepsilon(s,\mathbf{x}-\boldsymbol{\xi}) X^*_\varepsilon(s,\mathbf{y}) M(ds, d\boldsymbol{\xi})\, d\mathbf{x}\, d\mathbf{y}$$
$$- i\int_0^t \iiint f(\mathbf{x},\mathbf{y}) X_\varepsilon(s,\mathbf{x}) X^*_\varepsilon(s,\mathbf{y}-\boldsymbol{\xi}) M^*(ds, d\boldsymbol{\xi})\, d\mathbf{x}\, d\mathbf{y}. \tag{4.12}$$

Then $M_t M^*_t - \langle M\rangle_t$ is a $P^{\varepsilon}_{h\otimes h^*}$-martingale where the increasing process

$$\langle M\rangle_t = \int_0^t \sum_{j=1}^{2} \bigg[\iint f_1^j(\mathbf{x};\mathbf{y}) X_\varepsilon(s,\mathbf{x}) X^*_\varepsilon(s,\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} + \iint f_2^j(\mathbf{x};\mathbf{y}) X_\varepsilon(s,\mathbf{x})$$
$$\times X^*_\varepsilon(s,\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}\bigg]^2 ds$$
$$+ O(\varepsilon) t; \quad \text{where } f_1^j(\mathbf{x},\mathbf{y}) = \partial f/\partial x_j,\ f_2^j(\mathbf{x},\mathbf{y}) = \partial f/\partial y_j. \tag{4.13}$$

The uniform tightness of the measures $P^{\varepsilon}_{h\otimes h^*}$ follows from the criterion of Holley and Stroock (1981, Theorem 1.2).

Now let $P^{0}_{h\otimes h^*}$ be any limit of $P^{\varepsilon}_{h\otimes h^*}$. From (4.10) it follows that for $F_{f_{n,n}} \in \mathscr{D}^s_0$,

$$F_{f_{n,n}}(Z_0(t)) - \int_0^t \mathscr{L}^s_0 F_f(Z_0(s))\, ds \tag{4.14}$$

is a $P^{0}_{h\otimes h^*}$-martingale thus yielding existence for the $(\mathscr{D}^s_0, \mathscr{L}^s_0)$-martingale problem. Since the uniqueness was proved in Lemma 4.1, this proves that $P^{0}_{h\otimes h^*}$ is a H^s-valued Markov diffusion process and completes the proof of the weak convergence. □

Remark 4.2. Consider the Gaussian beam:

$$Y(0,\mathbf{x}) = \exp(-|\mathbf{x}|^2/2c^2). \tag{4.15}$$

Then (4.9) implies that

$$(1/c^2)\log|Y(t,\mathbf{x})|^2 = |\mathbf{x} + \mathbf{b}(t)|^2 \tag{4.16}$$

and therefore has a non-central chi-square distribution with two degrees of freedom. This distribution which is known as the Rice-Nakagami distribution in the wave propagation literature has probability density function of the form:

$$f(u) = \exp\left(-(u + c_1^2)/2c_2^2\right)\cdot I_0\left(c_1 u^{1/2}/c_2^2\right), \qquad u > 0, \tag{4.17}$$

where $I_0(\cdot)$ is the zeroth order modified Bessel function.

The spot-dancing phenomenon and its relation to the Rice-Nakagami distribution were first discovered by Furutsu (1972) and Furutsu and Furuhama (1973). The method of Furutsu was based on an explicit calculation of the moments of all orders and the observation that the results obtained at a fixed t agreed with that given by a Gaussian random displacement of the beam.

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Accepted 13 June 1984