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A RATE OF CONVERGENCE RESULT FOR THE SUPER-CRITICAL GALTON-WATSON PROCESS

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Let $Z_0 = 1, Z_1, Z_2, \cdots$ denote a super-critical Galton-Watson process whose non-degenerate offspring distribution has probability generating function $F(s) = \sum_{j=0}^{\infty} s^j P(Z_1 = j), 0 \leqq s \leqq 1$, where $1 < m = EZ_1 < \infty$. The Galton-Watson process evolves in such a way that the generating function $F_n(s)$ of Z_n is the nth functional iterate of $F(s)$. The convergence problem for Z_n, when appropriately normed, has been studied by quite a number of authors; for an ultimate form see Heyde [2]. However, no information has previously been obtained on the rate of such convergence. We shall here suppose that $EZ_1^2 < \infty$ in which case $W_n = m^{-n}Z_n$ converges almost surely to a non-degenerate random variable W as $n \to \infty$ (Harris [1], p. 13). It is our object to establish the following result on the rate of convergence of W_n to W.

Theorem. Let $\operatorname{var} Z_1 = \sigma^2 < \infty$. As $n \to \infty$, $m^{n/2}(W - W_n)$ converges in distribution. The limit law is given by the characteristic function relation

$$\lim_{n\to\infty} E[\exp\{it\,m^{n/2}(W - W_n)\}] = E[\exp\{-\tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}].$$

Proof. Firstly, we take $r > n$ and consider

$$
\begin{aligned}
&E[\exp\{it\,m^{n/2}(W_r - W_n)\}] \\
(1)\qquad &= \sum_{j=0}^{\infty} E[\exp\{it\,m^{n/2}(m^{-r}Z_r - m^{-n}Z_n)\} \mid Z_n = j]P(Z_n = j) \\
&= \sum_{j=0}^{\infty} \exp\{-it\,m^{-n/2}j\}\,[E[\exp\{it\,m^{n/2-r}Z_{r-n}\}]]^j P(Z_n = j) \\
&= E[\exp\{-it\,m^{-n/2}\}\,E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^{Z_n}.
\end{aligned}
$$

Next, we let $r \to \infty$ in (1) keeping n fixed. Given $\varepsilon > 0$ we can choose N so large that $\sum_{j=N+1}^{\infty} P(Z_n = j) < \varepsilon$. Also, since W_{r-n} converges in distribution to W, $E[\exp\{it\,m^{n/2-r}Z_{r-n}]]$ converges to $E[\exp\{it\,m^{-n/2}W\}]$, uniformly in any finite t interval, so we can find R so large that

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$$\left| E[\exp\{itm^{n/2-r}Z_{r-n}\}] - E[\exp\{itm^{-n/2}W\}] \right| < \varepsilon$$

for $r > R$. Consequently,

$$\sum_{j=0}^{\infty} \exp\{-itm^{-n/2}j\} [E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j P(Z_n = j)$$

$$= \sum_{j=0}^{\infty} \exp\{-itm^{-n/2}j\} [E[\exp\{itm^{-n/2}W\}]]^j P(Z_n = j)$$

$$+ \sum_{j=0}^{R} \exp\{-itm^{-n/2}j\}\{[E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j - [E[\exp\{itm^{-n/2}W\}]]^j\} P(Z_n = j)$$

$$+ \sum_{j=R+1}^{\infty} \exp\{-itm^{-n/2}j\}\{[E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j - [E[\exp\{itm^{-n/2}W\}]]^j\} P(Z_n = j),$$

and

$$\left| \sum_{j=R+1}^{\infty} \exp\{-itm^{-n/2}j\}\{[E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j - [E[\exp\{itm^{-n/2}W\}]]^j\} P(Z_n = j) \right|$$

$$\leqq 2 \sum_{j=R+1}^{\infty} P(Z_n = j) \leqq 2\varepsilon,$$

while for $r > R$,

$$\left| \sum_{j=0}^{R} \exp\{itm^{-n/2}j\} \{[E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j - [E[\exp\{itm^{-n/2}W\}]]^j\} P(Z_n = j) \right|$$

$$\leqq \sum_{j=0}^{R} \left| [E[\exp\{itm^{n/2-r}Z_{r-n}\}]]^j - [E[\exp\{itm^{-n/2}W\}]]^j \right| P(Z_n = j)$$

$$\leqq \varepsilon \sum_{j=0}^{R} jP(Z_n = j) \leqq \varepsilon m^n$$

since $EZ_n = m^n$. However, ε is arbitrary and n fixed so that

$$E[\exp\{itm^{n/2}(W - W_n)\}] = \lim_{r\to\infty} E[\exp\{itm^{n/2}(W_r - W_n)\}]$$

$$(2) \qquad = \sum_{j=0}^{\infty} \exp\{-itm^{-n/2}j\} [E[\exp\{itm^{-n/2}W\}]]^j P(Z_n = j)$$

$$= E[E[\exp\{itm^{-n/2}(W - 1)\}]]^{Z_n}.$$

Now,

$$E[E[\exp\{itm^{-n/2}(W - 1)\}]]^{Z_n}$$

$$= \int_0^{\infty} [E[\exp\{itm^{-n/2}(W - 1)\}]]^{xm^n} dP(m^{-n}Z_n \leqq x)$$

$$= P(Z_n = 0 + m^n \log E[\exp\{itm^{-n/2}(W-1)\}] \int_0^\infty P(m^{-n}Z_n > x) \tag{3}$$

$$\times [E[\exp\{itm^{-n/2}(W - 1)\}]]^{xm^n} dx,$$

using integration by parts. Furthermore, under the hypotheses of the theorem, W has mean 1 and variance $\sigma^2/(m^2 - m)$ ([1], p. 13) so we may expand $E[\exp\{itm^{-n/2}(W-1)\}]$ in the form

$$E[\exp\{itm^{-n/2}(W-1)\}] = 1 - \tfrac{1}{2}t^2\sigma^2 m^{-n}(m^2 - m)^{-1} + o(t^2 m^{-n}). \tag{4}$$

Thus,

$$m^n \log E[\exp\{itm^{-n/2}(W-1)\}] \to -\tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1} \tag{5}$$

and

$$[E[\exp\{itm^{-n/2}(W-1)\}]]^{m^n} \to \exp\{-\tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1}\} \tag{6}$$

as $n \to \infty$. Also, using (4), we have for fixed t and suitably large n,

$$\left| E[\exp\{itm^{-n/2}(W-1)\}] \right|^{m^n} < \exp\{-\tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1} + o(t^2)\}$$

$$< \exp\{-\tfrac{1}{4}t^2\sigma^2(m^2 - m)^{-1}\},$$

so that the integrand in the rightmost term of (3) is bounded in absolute value by $\exp\{-\frac{1}{4}t^2\sigma^2 x(m^2 - m)^{-1}\}$. It then follows from Fatou's lemma together with (5), (6) and since $P(m^{-n}Z_n > x) \to P(W > x)$ as $n \to \infty$, that

$$\lim_{n\to\infty} E[\exp\{itm^{n/2}(W - W_n)\}]$$

$$= P(W = 0) - \tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1} \int_0^\infty P(W > x)\exp\{-\tfrac{1}{2}t^2\sigma^2 x(m^2 - m)^{-1}\}dx$$

$$= \int_0^\infty \exp\{-\tfrac{1}{2}t^2\sigma^2 x(m^2 - m)^{-1}\}dP(W \leqq x)$$

$$= E[\exp\{-\tfrac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}],$$

again using integration by parts. This completes the proof of the theorem.

Unfortunately, the distribution function corresponding to the characteristic function $E[\exp\{-\frac{1}{2}t^2\sigma^2(m^2 - m)^{-1}W\}]$ does not seem to have a useful general representation. However, in certain particular cases it may be recognised simply.

For example, in the case of a fractional linear generating function for the offspring distribution ([1], pp. 9, 17) it is easily found to have an atom at zero and a Laplace density elsewhere.

References

[1] HARRIS, T. E. (1963) *The Theory of Branching Processes*. Springer-Verlag, Berlin.

[2] HEYDE, C. C. (1970) Extension of a result of Seneta for the super-critical Galton-Watson process. *Ann. Math. Statist.* **41** (in press).