

# A RATIONAL TERMINAL VELOCITY EQUATION FOR BUBBLES AND DROPS AT INTERMEDIATE AND HIGH REYNOLDS NUMBERS

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## Introduction

In the process industries, many operations involve direct contact between fluid phases by dispersing one phase as drops or bubbles in another, continuous phase. Such operations include distillation in tray columns, —gas absorption using tray columns, bubble columns, and impeller-agitated vessels, — liquid/liquid extraction in spray columns and impeller-agitated vessels, and quenching of hot gas by bubbling through liquid. Chemical reaction may be present also in any of these operations. Usually, the quantitative description of mass and heat transfer in such dispersions requires knowledge of interfacial area and transfer coefficients that are given by equations incorporating the terminal velocity of a single particle as a calculation quantity, either directly, or indirectly, say via a “slip velocity”. In the motion of single bubbles and drops at terminal velocity, one may distinguish between two principal regimes in any given fluid/fluid system (Fig. 1). These regimes of motion are:

Regime 1. Viscosity of the continuous phase or of both phases has a significant effect on terminal velocity, Reynolds number is low, drag coefficient decreases as Reynolds number increases.

Regime 2. Apparently negligible effect of viscosities on terminal velocity, intermediate to high Reynolds number, drag coefficient increases as Reynolds number increases.

The numerical value of the Reynolds number at transition from regime 1. to regime 2. in any given system depends on its components. Here, we are concerned with regime 2., where motion is affected by interfacial tension, gravity, buoyancy, and form drag. With impure fluids, the contaminants affect motion. Several equations have been proposed for calculation of the terminal velocity of bubbles or of drops in regime 2. Some are correlations of experimental results<sup>7,14)</sup>, while the substitution by

Mendelson (1967)<sup>12)</sup> of particle perimeter for wavelength in a surface wave equation, and the same substitution by Marrucci *et al.* (1970)<sup>13)</sup> in a general interface wave equation (Lamb, 1932)<sup>9)</sup> provide an easily memorised equation coupled with reasonable prediction of experimental data. However, so far there is no rational explanation that justifies the use of the wave equations that are based on assumptions of two-dimensional motion, negligible fluid velocity, and interface amplitude which is much smaller than wavelength.

Here it is proposed to formulate a rationally justifiable equation for terminal velocity of both single bubbles and single drops in regime 2. (Fig. 1). The predictions made by this equation are then to be compared with those made by other equations, and with reported experimental data.

## Discussion

The basic equation for terminal velocity  $U$  of a single particle is derived from the steady-state force balance

$$(\rho_a - \rho_c)g\pi D^3/6 = (C_D\pi D^2/4)(\rho_c U^2/2) \quad (1)$$

i.e. (gravity force) — (buoyancy force) = (drag force)

$$\text{With } |\rho_a - \rho_c| = \Delta\rho \quad (2)$$

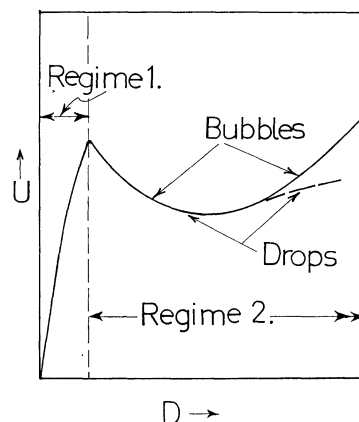


Fig. 1 Terminal velocity  $U$  vs. particle diameter  $D=[6/\pi(\text{particle volume})]^{1/3}$ , typical relation for clean fluids

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$$U = \sqrt{4gD\Delta\rho/3\rho_c C_D} \quad (3)$$

The Chemical Engineers' Handbook (1973)<sup>2)</sup> mentions several reports on bubble and drop dynamics. Among these, one finds the following equations for terminal velocity in regime 2. (Fig. 1) (with nomenclature in conformity with that used here):

Klee and Treybal (1956)<sup>7)</sup>:

$$U = 17.6\rho_c^{-0.55}\Delta\rho^{0.28}\mu_c^{0.10}\sigma^{0.18} \quad (4)$$

Thorsen, Stordalen, and Terjesen (1968)<sup>14)</sup>

$$U = 6.8/(1.65 - \Delta\rho/\rho_a) \sqrt{\sigma/D(3\rho_a + 2\rho_c)} \quad (5)$$

Marrucci, Apuzzo, and Astarita (1970)<sup>13)</sup>

$$U = \sqrt{\Delta\rho g D / 2\rho_c + 2\sigma/\rho_c D} \quad (6)$$

At the pressures at which experiments to determine gas bubble terminal velocities in liquids have usually taken place,  $\Delta\rho \approx \rho_c$ , so that Eq. (6) is virtually the same as the Mendelson (1967)<sup>12)</sup> equation.

To derive an equation for terminal velocity the following argument is put forward:

Introduction into a continuous phase fluid of a spherical fluid particle of diameter  $D$  requires work, and this increases the potential energy  $PE$  of the continuous phase, i.e.

$$PE = \pi D^3 \Delta\rho g D / 6 + \pi D^2 \sigma \quad (7)$$

potential energy being due to gravity-, buoyancy-, and interfacial tension effects.

This potential energy exists in the continuous phase ahead of the moving particle, and disappears as displaced fluid moves around-, and inward behind the particle.

Photographs of bubbles in regime 2. as defined here (Fig. 1) show wakes (Batchelor, 1967<sup>1)</sup>, Lindt, 1971<sup>10,11)</sup>, and flow separation seems generally accepted for fluid particles in this regime (Heertjes and DeNie, 1971<sup>5)</sup>, Haas *et al.*, 1972<sup>3)</sup>).

Therefore it is postulated here that potential energy of displaced continuous phase is converted to kinetic energy as it flows from front to rear of the moving particle through a vertical distance  $D$ . The resultant kinetic energy is then dissipated in the wake.

Relevant equations are:

$$\rho_c \pi D^3 v^2 / (6)(2) = \pi D^3 \Delta\rho g D / 6 + \pi D^2 \sigma \quad (8)$$

$$v^2 / 2 = \Delta\rho g D / \rho_c + 6\sigma / \rho_c D \quad (9)$$

From Eq. (8),

Force =  $\pi D^3 \Delta\rho g / 6 + \pi D \sigma = \text{constant}$  (for a given fluids combination and particle size  $D$ )

and therefore acceleration  $a = \text{constant}$  here. Then

$$v^2 / 2 = (1/2) a^2 t^2 \quad (10)$$

$$D = (1/2) a t^2 \quad (11)$$

$$a^2 = 4D^2 / t^4 \quad (12)$$

From Eqs. (9), (10), (12),

$$\frac{v^2}{2} = \frac{1}{2} \frac{4D^2}{t^4} t^2 = \frac{2D^2}{t^2} = \frac{\Delta\rho g D}{\rho_c} + \frac{6\sigma}{\rho_c D} \quad (13)$$

The time  $t$  required for conversion of potential energy to kinetic energy is the time in which the particle moves through distance  $D$  at terminal velocity  $U$ , so that

$$t = D/U \quad (14)$$

Substituting Eq. (14) into Eq. (13),

$$U^2 = D^2/t^2 = \Delta\rho g D / 2\rho_c + 3\sigma/\rho_c D \quad (15)$$

From relevant cited literature, it seems that in systems of relatively clean fluids, a bubble or drop has a peak terminal velocity at transition from the viscosity-dependent regime 1. to the viscosity-independent regime 2. (Fig. 1).

Where this peak velocity at transition was evident in the reported experimental data, calculated and observed values of such velocity have been compared. The second velocity for comparison has been mostly that at particle diameters approximately two to three times as large as those at peak velocity, i.e. particle mass 8 to 27 times as large.

**Table 1** shows 90 sets of calculated and observed values of  $U$  for drops, **Table 2** shows 12 sets for bubbles. **Fig. 2** shows observed values of  $U$  for both drops and bubbles in juxtaposition to values calculated by Eq. (15).

To obtain an indication of accuracy of prediction, values calculated by Eqs. (4), (5), (6), and (15) have been compared by using the following criteria:

$$s(U) = \sqrt{\frac{\sum (U_{calc} - U_{obs})^2}{(\text{number of observations} - 1)}} \quad (16)$$

$$V(U) = \frac{s(U) 100}{\sum U_{obs} / \text{number of observations}} \quad (17)$$

The deviations and coefficients of variation calculated by Eqs. (16) and (17) respectively have been tabulated for comparison in **Table 3**.

## Conclusions

An equation has been formulated that describes terminal velocity of single bubbles and drops in the viscosity-independent regime,—i.e. at intermediate and high Reynolds number,—which are moving freely, i.e. “infinite” continuous phase is approximated.

This equation (15) is based on the premise that a system will tend to minimum potential energy. The prediction accuracy of Eq. (15) has been compared with that of previously published equations using 90 observed data for drops in liquid, and 12 for bubbles in liquid (Table 3). For the drop data here, Eq. (15) is superior to others, but for the bubble data here, Eq. (6) is somewhat more accurate. For drop and bubble data here combined, Eq. (15) is more accurate than Eq. (6). Eq. (15) is easily memorised and is based on rational argument.

**Table 1 Drop velocities in Regime 2 (Fig. 1), terminal velocity  $U$ , comparison of calculated values with reported observed values**

Fluid system	equivalent diameter $D$ [cm]	Terminal velocities, $U$ [cm·sec <sup>-1</sup> ]				Fluid system	equivalent diameter $D$ [cm]	Terminal velocities, $U$ [cm·sec <sup>-1</sup> ]			
		Observed	Calculated					Observed	Calculated		
			This work Eq. (15)	Klee & Treybal <sup>(7)</sup> Eq. (4)	Thorsen <i>et al.</i> <sup>(14)</sup> Eq. (5)				This work Eq. (15)	Klee & Treybal <sup>(7)</sup> Eq. (4)	Thorsen <i>et al.</i> <sup>(14)</sup> Eq. (5)
by Krishna <sup>(8)</sup> Table 2 & Fig. 2-8					by Klee and Treybal <sup>(7)</sup> Fig. 1						
<i>n</i> -amyl phthalate	1.4 3.77	7.7 6.5	7.5 7.0	6.1 6.1	7.0 4.3	benzene drops in water	0.6 1.4	12.0 11.2	13.5 11.9	11.2 11.2	14.6 9.6
aniline	0.75 2.15	6.28 4.3	5.7 5.3	5.0 5.0	5.5 3.7	Kerosine drops in water	0.6 1.35	14.8 13.6	16.1 14.7	13.7 13.7	18.7 12.5
bromoforn	0.19	24.2 & 22.91	28.5	25.5	30.6	SAE 10 W oil drops in water	0.6 1.2	11.5 10.9	11.5 11.2	10.7 10.7	11.8 8.3
<i>n</i> -butyl phthalate	0.525 1.0 2.57	22.2 9.8 8.0	26.6 10.7 9.2	25.5 8.2 8.2	18.4 9.1 5.7	nonyl alcohol drops in water	0.32 0.6	9.0 9.2	8.6 8.7	9.1 9.1	8.8 6.4
carbon disulfide	0.58 1.49	16.9 14.0	17.6 16.8	15.1 15.1	17.4 10.9	secondary butyl alcohol in water	0.4	6.0	5.1	5.7	2.6
carbon tetrachloride	0.335 0.98	19.95 17.8	22.3 20.5	18.8 18.8	23.6 13.8	water drops in turfural	0.4	6.8	5.7	6.5	3.8
chlorobenzene	0.8 2.15	13.1 11.0	13.2 12.4	10.9 10.9	12.7 7.8	methyl ethyl ketone drops in water	0.4	5.5	5.2	5.3	1.9
1-chloronaphthalene	0.63	15.8	16.2	13.6	15.9	benzene drops in 20% sucrose solution	0.7 1.4	13.7 13.0	13.5 13.5	12.8 12.8	14.3 10.1
<i>m</i> -cresol	0.62 1.06	5.6 4.7	5.4 5.2	5.3 5.3	4.8 3.7	methyl isobutyl ketone drops in water	0.4 0.88	10.3 10.2	10.5 10.6	10.3 10.3	11.2 7.5
$\alpha$ -chlorophydrin	0.28 0.898	12.3 9.0	11.9 10.6	10.2 10.2	12.1 6.7	pentachloroethane drops in water	0.4 0.8	20.0 19.2	21.3 20.6	19.4 19.4	21.2 15.0
ethyl chloroacetate	0.31 1.19	13.56 10.0	13.1 11.0	10.2 10.2	13.5 6.9	<i>n</i> -heptylic acid drops in water	0.55	7.8	7.7	7.7	7.1
ethyl cinnamate	0.86 2.49	10.1 7.5	9.7 8.9	7.9 7.9	9.4 5.5	by Thorsen <i>et al.</i> <sup>(14)</sup> Table 2					
ethyl phthalate	0.57 1.19	11.9 9.7	10.6 10.6	10.1 10.1	9.6 6.6	methylene bromide	0.150 0.201	30.8 26.7	28.9 26.2	23.4 23.4	32.6 28.2
1,2 dibromoethylene	0.24 0.63	22.3 19.8	24.4 23.2	22.0 22.0	25.9 16.0	carbon tetrachloride	0.200 0.339	25.85 22.55	26.1 21.6	18.5 18.5	29.5 22.7
eugenol	0.60 1.61	9.3 7.2	8.9 8.4	7.8 7.8	7.7 4.7	bromoforn	0.137	35.5	31.9	25.6	36.0
isoeugenol	0.64 1.19	9.3 7.8	8.4 8.5	8.3 8.3	7.2 5.3	<i>o</i> -dichlorobenzene	0.26 0.439	22.35 18.8	22.2 18.3	15.3 15.3	24.3 18.7
methyl phthalate	0.425 0.92	12.3 9.0	11.2 11.1	10.6 10.6	10.4 7.1	ethyl bromide	0.217 0.316	23.0 20.0	21.9 19.1	16.3 16.3	24.1 20.0
nitrobenzene	0.45 1.26	13.4 12.0	14.5 13.5	12.5 12.3	14.4 8.6	ethylene bromide	0.153 0.230	30.3 25.65	26.8 23.5	21.5 21.5	30.3 24.8
<i>m</i> -nitrotoluene	0.485 1.51	15.2 11.0	14.6 13.1	12.0 12.0	14.7 8.3	tetrabromoethane	0.192 0.273	30.7 26.55	28.2 26.3	25.7 25.7	29.8 25.0
<i>o</i> -nitrotoluene	0.59 1.45	13.3 11.5	13.4 12.9	11.6 11.6	12.8 8.2	by Hu and Kintner <sup>(9)</sup> Table 2					
diphenyl ether	0.92 2.69	12.5 9.5	12.9 11.8	10.1 10.1	12.5 7.3	tetrabromoethane	0.2	25.9	27.1	25.6	27.9
1,2 dichloropropene	0.59 1.62	13.4 11.3	14.2 13.3	11.9 11.9	14.0 8.4	dibromoethane	0.23	22.1	23.4	21.4	24.8
1,1,2,2 tetrabromoethane	0.196 0.465	25.63 22.5	26.5 25.7	24.6 24.6	27.3 17.7	ethyl bromide	0.345	17.4	18.4	16.2	18.8
1,1,2,2 tetrachloroethane	0.306 0.807	18.35 16.3	19.6 18.5	17.3 17.3	20.3 12.5	nitrobenzene	0.485	13.5	14.0	12.4	13.7
tetrachloroethylene	0.35 0.944	21.0 18.5	21.9 20.6	18.7 18.7	22.8 13.9	bromobenzene	0.395	19.2	19.6	17.3	19.8

**Table 2 Bubble velocities in Regime 2 (Fig. 1), terminal velocity  $U$ , comparison of calculated values with reported observed values**

Continuous phase liquid	Equivalent bubble diameter $D$ [cm]	Terminal velocities $U$ [cm·sec <sup>-1</sup> ]		
		Observed Haberman & Morton Fig. 2 & 4	Calculated	
			This work Eq. (15)	Mendel- son <sup>(12)</sup> Eq. (6)
filtered water	0.14	34.5	40.4	33.3
	0.20	30.0	34.5	28.8
	0.95	24.5	26.4	24.9
	2.0	31.5	33.0	32.5
	6.0	55	54.6	54.5
turpentine	0.16	26	26.1	21.9
mineral oil	0.60	16.5	20.3	19.3
methyl alcohol	0.10	27.5	29.8	24.6
	0.14	25	25.8	21.6
	0.40	18	20.1	18.3
	0.70	21	21.5	20.6
varsol	0.14	25.5	27.2	22.7

**Table 3 Comparison of prediction accuracy of equations for terminal velocity  $U$**

Particle, & number of data	Prediction equation number	Deviation $s(U)$ Eq. (16) [cm·s <sup>-1</sup> ]	Coefficient of variation $V(U)$ , Eq. (17) [%]
drops, 90 bubbles, 12	(15)	1.63	10.7
	(15)	2.84	10.2
drops & bubbles combined, 102	(15)	1.79	10.8
	(4)	2.64	17.4
drops, 90 drops, 90	(5)	2.47	16.3
	(6)	2.43	16.0
bubbles, 12 drops & bubbles combined, 102	(6)	2.30	8.2
	(6)	2.40	14.4

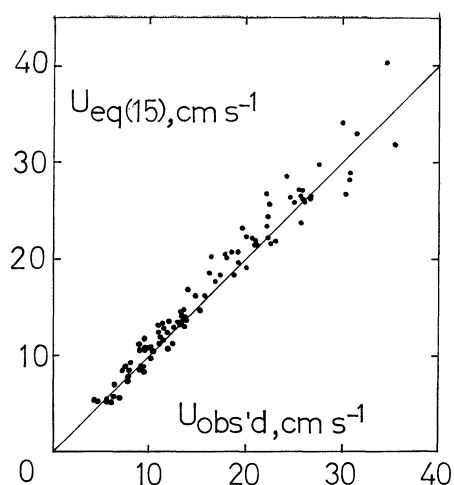


Fig. 2 Terminal velocity  $U$  calculated by Eq. (15) vs. observed terminal velocity, observed values from reports by Krishna *et al.*,<sup>8)</sup> Klee and Treybal,<sup>7)</sup> Thorsen *et al.*,<sup>14)</sup> Hu and Kintner,<sup>6)</sup> Haberman and Morton<sup>4)</sup>

#### Nomenclature

$a$	= acceleration	[ $Lt^{-2}$ ]
$C_D$	= drag coefficient	
$D$	= diameter of sphere of same volume as the bubble or drop	[ $L$ ]
$g$	= gravity acceleration	[ $Lt^{-2}$ ]
$s(U)$	= standard deviation of $U$	[ $Lt^{-1}$ ]
$PE$	= potential energy	[ $ML^2t^{-2}$ ]
$t$	= time for conversion of all $PE$ to kinetic energy	[ $t$ ]
$U$	= terminal velocity of bubble or drop	[ $Lt^{-1}$ ]
$v$	= velocity of fluid at time $t$	[ $Lt^{-1}$ ]
$V(U)$	= coefficient of variation of $U$	[%]

$\Delta$	= difference	
$\rho$	= density	[ $ML^{-3}$ ]
$\sigma$	= interfacial tension	[ $Mt^{-2}$ ]
$\mu$	= viscosity	[ $ML^{-1}t^{-1}$ ]

#### <Subscripts>

$c$	= continuous phase
$d$	= dispersed phase

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