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A Rational Trigonometric Spline to Visualize Positive Data

Uzma Bashir and Jamaludin Md. Ali

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia

Abstract. In this paper, we construct a cubic trigonometric Bézier curve with two shape parameters on the basis of cubic trigonometric Bernstein-like blending functions. The proposed curve has all geometric properties of the ordinary cubic Bézier curve. Later, based on these trigonometric blending functions a C^1 rational trigonometric spline with four shape parameters to preserve positivity of positive data is generated. Simple data dependent constraints are developed for these shape parameters to get a graphically smooth and visually pleasant curve.

Keywords: cubic trigonometric blending functions, cubic trigonometric Bézier curve, positive data.
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INTRODUCTION

Computer aided geometric design (CAGD) studies the construction and manipulation of curves and surfaces using polynomial, rational, piecewise polynomial or piecewise rational methods. Among many generalizations of polynomial splines, the trigonometric splines are of particular theoretical interest and practical importance. In recent years, trigonometric splines with shape parameters have gained wide spread application in particular in curve design [1-3]. Bézier form of parametric curve is frequently used in CAD and CAGD applications like data fitting and font designing, because it has a concise and geometrically significant presentation. A recent development in CAGD is the introduction of Bernstein-like basis functions using trigonometric functions [4-6].

Smooth curve representation of scientific data is also of great interest in the field of data visualization. Key idea of data visualization is the graphical representation of information in a clear and effective manner. When data arises from a physical experiment, prerequisite for the interpolating curve is to incorporate the inherit feature of the data like positivity, monotonicity, and convexity. Various authors have worked in the area of shape preserving using ordinary and trigonometric rational splines [7-9].

This paper has two objectives. Firstly, cubic trigonometric Bernstein-like blending functions with two shape parameters are developed and a cubic trigonometric Bézier curve is constructed. Secondly, a piecewise C^1 rational trigonometric spline with four shape parameters is presented to preserve the positivity of positive data. Two of these shape parameters are constrained and two are left free to control the smoothness of the interpolating curve.

The work is organized as: cubic trigonometric Bernstein-like basis functions and a cubic trigonometric Bézier curve are given in Section 2. In Section 3 a piecewise C^1 rational trigonometric spline with four shape parameters is constructed which is then used to generate a positive curve interpolation scheme. Finally conclusion of the work with some future work is given in Section 4.

Cubic Trigonometric Bernstein-like Basis Functions

Definition 1: For $u \in [0, \frac{\pi}{2}]$, cubic trigonometric basis functions with two shape parameters m and n, $-1 \le m, n \le 2$ are defined as:

$$\left.\begin{aligned}
f_0(u) &= (1-\sin u)^2(1+(1-m)\sin u) \\
f_1(u) &= \sin u(1-\sin u)(m(1-\sin u)+(1+\sin u)) \\
f_2(u) &= \cos u(1-\cos u)(n(1-\cos u)+(1+\cos u)) \\
f_3(u) &= (1-\cos u)^2(1+(1-n)\cos u)
\end{aligned}\right\} \qquad (1)$$

Theorem 1: Cubic trigonometric polynomials defined in (1) have the following properties:

(a) Non-negativity: $f_i(u) \ge 0$, $i = 0,1,2,3$

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(b) Partition of unity: $\sum_{i=0}^{3} f_i(u)=1$

(c) Monotonicity: For the given value of the parameters m and n, $f_0(u)$ is monotonically decreasing and $f_3(u)$ is monotonically increasing.

(d) Symmetry: $f_i(u;m,n)=f_{3-i}(\frac{\pi}{2}-u;n,m),\ i=0,1,2,3$

Proof: (a) For $u\in[0,\frac{\pi}{2}]$ and $m,n\in[-1,2]$, $(1\pm\sin u)\geq 0$, $(1+(1-m)\sin u)\geq 0$, $(1\pm\cos u)\geq 0$, $(1+(1-n)\cos u)\geq 0$, $\cos(u)\geq 0$, $\sin(u)\geq 0$. It immediately follows that $f_i(u)\geq 0,\ i=0,1,2,3$.
The remaining properties are obvious.
Figure 1 shows the curves of cubic trigonometric basis functions for $m=n=0.5$.

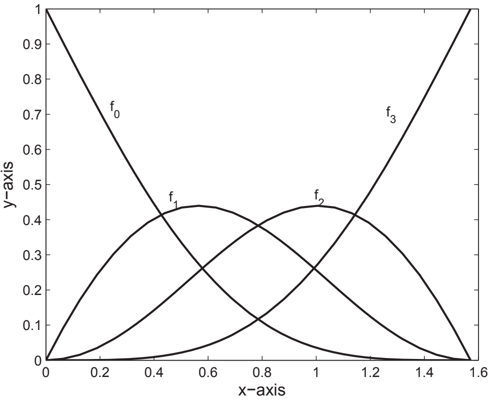

FIGURE 1. Cubic trigonometric basis functions

Cubic Trigonometric Bézier Curve

Definition 2: For the control points $P_i\left(i=0,1,2,3\right)$ in $\Re^2$ or $\Re^3$, a cubic trigonometric Bézier curve with two shape parameters m and $n\in[-1,2]$ is defined as:

$$f(u)=\sum_{i=0}^{3} f_i\left(u\right)P_i \tag{2}$$

The following theorem shows that the curve defined in (2) has the geometric properties of the ordinary cubic Bézier curve.

Theorem 2: Cubic trigonometric Bézier curve upholds the following properties:

End point properties

$$\left.\begin{aligned}
&f(0)=P_0, \quad f(\frac{\pi}{2})=P_3\\
&f'(0)=(1+m)(P_1-P_0), \quad f'(\frac{\pi}{2})=(1+n)(P_3-P_2)\\
&f''(0)=2(P_2-2m(P_1-P_0)-P_0), \quad f''(\frac{\pi}{2})=2(P_1+2n(P_3-P_2)-P_3)
\end{aligned}\right\} \tag{3}$$

Symmetry

Since $f_i(u;m,n)=f_{3-i}(\frac{\pi}{2}-u;n,m)$, the control points of cubic trigonometric Bézier curve define the same curve in different parameterization. Thus curve (2) satisfies the following equation:

$$f(u;m,n,P_0,P_1,P_2,P_3)=f(\frac{\pi}{2}-u;\,n,m,P_3,P_2,P_1,P_0)$$

Geometric invariance

The shape of the curve (2) is independent of the choice of its control points. i.e., it satisfies the following two equations:

$$f(u;m,n,P_0+r,P_1+r,P_2+r,P_3+r)=f(u;m,n,P_0,P_1,P_2,P_3)+r$$

$$f(u;m,n,P_0*T,P_1*T,P_2*T,P_3*T)=f(u;m,n,P_0,P_1,P_2,P_3)*T$$

where r is any arbitrary vector in $\Re^2$ or $\Re^3$ and T is an arbitrary $d\times d$ matrix, $d=2$ or 3

Convex hull Property

From $\sum_{i=0}^{3} f_i(u)=1$ and $0\le f_i(u)\le 1,\ u\in[0,\frac{\pi}{2}]$, it implies that the whole curve is located in the convex hull of its defining control points.

Figure 2 shows the effect of the shape parameters on the shape of the curve. In Figure 2(a) solid lines show the effect of changing m while keeping n fixed $(n=0.5)$, where as broken lines show the effect of varying the values of n with fixed value of $\mathrm{m}=0.5$. In Figure 2(b) curves are drawn by altering the values of m and n simultaneously.

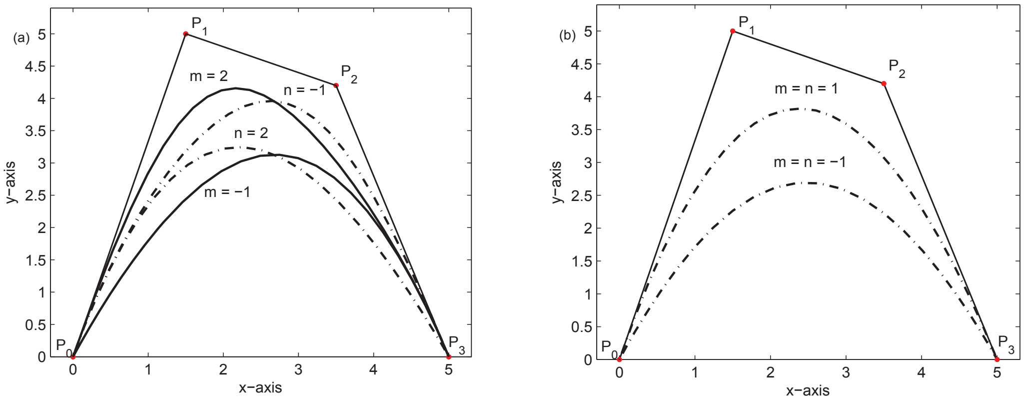

FIGURE2. The effect on the shape of cubic trigonometric Bézier curve for different values of *m* and *n*

Piecewise C^1 Rational Trigonometric Spline

In this section, we use Bernstein-like basis functions defined in (1) for fixed values of m and n $(m=n=0)$ to develop a C^1 rational cubic trigonometric spline with four shape parameters.

For $m=n=0$, basis functions take the form

$$\left.\begin{aligned} g_0&=(1-\sin u)\cos^2 u\\ g_1&=\sin u\cos^2 u\\ g_2&=\cos u\sin^2 u\\ g_3&=(1-\cos u)\sin^2 u \end{aligned}\right\}\qquad (4)$$

Let $\{(t_i, f_i, d_i) : i = 0,1,2,...,n\}$ be a given set of data points over an arbitrary interval $[a,b]$, where $a = t_0 < t_1 < t_2 < ... < t_n = b$, f_i are function values and d_i are the derivative at the knots of the function being interpolated. A piecewise rational trigonometric function with four positive shape parameters over each sub-interval $[t_{i,}\ t_{i+1}], i = 0,1,2,...,n-1$, is defined as:

$$P(t) \equiv P_i(t) = \frac{M_0 g_0 + M_1 g_1 + M_2 g_2 + M_3 g_3}{\alpha_i g_0 + \beta_i g_1 + \gamma_i g_2 + \delta_i g_3} \tag{5}$$

$u = \frac{\pi}{2}\left(\frac{t - t_i}{h_i}\right)$, $h_i = t_{i+1} - t_i$ and $g_j, j = 0,1,2,3$ are as defined in equation (4).

The necessary conditions for interpolating spline to be of class $C^1[a,b]$ are:

$$\left.\begin{array}{ll} P(t_i) = f_i, & P(t_{i+1}) = f_{i+1} \\ P'(t_i) = d_i, & P'(t_{i+1}) = d_{i+1} \end{array}\right\} \tag{6}$$

where $P'(t)$ denotes the derivative with respect to $'t'$. The derivatives d_i at the knots are either given or can be computed by some numerical method.

Using Conditions (6) the values of unknowns $M_i, i = 0,1,2,3$ are:

$$M_0 = \alpha_i f_i,\ M_1 = \beta_i f_i + \frac{2h_i d_i \alpha_i}{\pi},\ M_1 = \gamma_i f_{i+1} - \frac{2h_i d_{i+1}\delta_i}{\pi} \text{ and } M_3 = \delta_i f_{i+1} \tag{7}$$

Substituting these values of unknowns into (5) reduces it to a C^1 piecewise rational cubic trigonometric spline given as:

$$P(t) \equiv P_i(t) = \frac{p_i(u)}{q_i(u)} \tag{8}$$

where

$$p(u) = \alpha_i f_i\, g_0 + \left(\beta_i f_i + \frac{2h_i d_i \alpha_i}{\pi}\right) g_1 + \left(\gamma_i f_{i+1} - \frac{2h_i d_{i+1}\delta_i}{\pi}\right) g_2 + \delta_i f_{i+1}\, g_3$$

$$q(u) = \alpha_i g_0 + \beta_i g_1 + \gamma_i g_2 + \delta_i g_3$$

It is to mention that if the values of the shape parameters are chosen on trial basis, the shape characteristics of the data are not preserved always. Thus there arises a need of some conditions to be imposed on these shape parameters.

Positive Curve Interpolation

Piecewise C^1 rational cubic trigonometric spline given by Equation (8) is used to achieve the positivity of positive data.

Theorem 3: A C^1 piecewise rational cubic trigonometric spline defined in Equation (8) preserves the positivity of the positive data in each subinterval $[t_{i,}\ t_{i+1}], i = 0,1,2,...,n-1$ if the shape parameters β_i, γ_i satisfy the following conditions.

$$\left.\begin{array}{l} \beta_i > \max\left\{0, \dfrac{-2d_i h_i \alpha_i}{\pi f_i}\right\} \\ \gamma_i > \max\left\{0, \dfrac{2d_{i+1} h_i \delta_i}{\pi f_{i+1}}\right\} \end{array}\right\} \tag{9}$$

Proof: Consider a positive data set $\{(t_i, f_i) : i = 0,1,2,...,n\}$ such that $t_i < t_{i+1}$ and $f_i > 0,\ i = 0,1,2,...,n-1$

A piecewise rational cubic trigonometric spline given in (8) preserves the positivity through positive data if $P_i(t) > 0$ if

$$p_i(u), q_i(u) > 0$$

Since positivity of shape parameters assures strictly positive denominator, thus the problem of positivity preserving interpolating curve reduces to find out suitable values of shape parameters that make the trigonometric function $p_i(u)$ positive.

Note that $p_i(u) > 0$ if

$$\beta_i > \frac{-2d_i h_i \alpha_i}{\pi f_i} \text{ and } \gamma_i > \frac{2d_{i+1} h_i \delta_i}{\pi f_{i+1}} \tag{10}$$

But $\beta_i, \gamma_i > 0$, which yields

$$\beta_i > \max\left\{0, \frac{-2d_i h_i \alpha_i}{\pi f_i}\right\} \text{ and } \gamma_i > \max\left\{0, \frac{2d_{i+1} h_i \delta_i}{\pi f_{i+1}}\right\}, \ \alpha_i, \delta_i > 0 \tag{11}$$

This proves the desired result.

The developed scheme has been implemented on positive data sets. The curves in Figure 3 and Figure 5 are drawn by using C^1 piecewise rational cubic trigonometric spline for 2D positive data sets given in Table 1 and Table 2 respectively by taking the values of shape parameters on trial and error basis. These figures clearly show that the resulting curves do not preserve the positivity. To obtain the desired shape feature, the scheme developed in Section 3 is used and positivity preserving curves thus obtained are shown in Figure 4 and Figure 6 respectively.

TABLE (1). A 2D positive data set

i	1	2	3	4	5	6	7	8	9	10	11	12
t_i	0	0.4	0.5	0.6	0.7	0.8	1.1	1.2	1.3	1.4	1.5	1.9
f_i	1.4	1.2	0.9	0.6	0.3	0.05	0.11	0.3	0.6	0.9	1.2	1.3
d_i	1.50	-1.75	-3.0	-3.0	-2.75	-1.15	1.05	2.45	3.0	3.0	1.625	-1.95

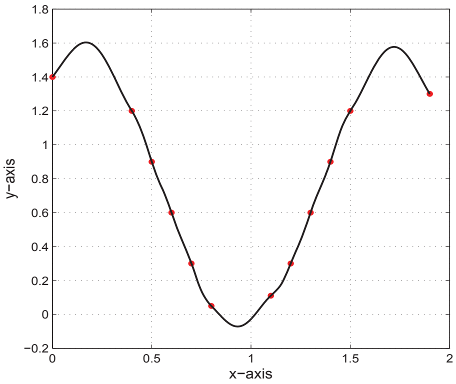

FIGURE 3. C^1 rational cubic trigonometric spline with $\alpha_i = \beta_i = \gamma_i = \delta_i = 1$

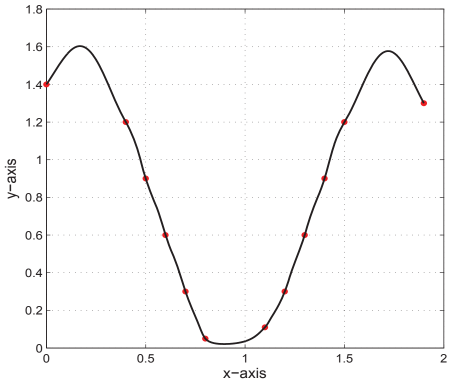

FIGURE 4. C^1 positive rational cubic trigonometric spline with $\alpha_i = \delta_i = 1$

TABLE (2). A 2D positive data set

i	1	2	3	4	5	6	7
t_i	1	3	8	10	11	12	16
f_i	14	2	0.8	0.65	0.75	0.70	0.69
d_i	-6.65	-3.12	-0.16	0.0125	0.0250	-0.0263	0.0355

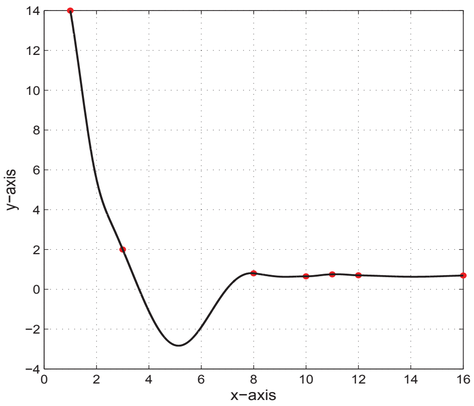

FIGURE 5. C^1 rational cubic trigonometric spline with $\alpha_i = 0.8, \beta_i = 0.6, \gamma_i = 0.6, \delta_i = 0.8$

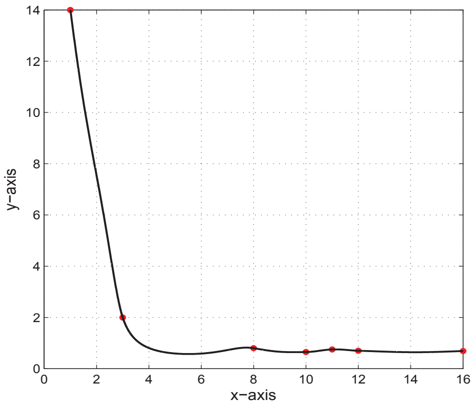

FIGURE 6. C^1 positive rational cubic trigonometric spline with $\alpha_i = \delta_i = 1$

Conclusion and Future Work

A cubic trigonometric Bézier curve with two shape parameters based on Bernstein-like cubic trigonometric basis functions is presented in this paper. The proposed curve holds all the geometric properties of the ordinary cubic Bézier curve but is more flexible as it includes shape parameters in its description. These basis functions are further used to develop a piecewise C^1 rational trigonometric interpolating scheme to visualize a positive data. The scheme is tested for positive data sets and result is shown in the form of graphically smooth and visually pleasant positive curves. In future this scheme will be used to visualize monotonicity and convexity through monotone and convex data respectively.

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REFERENCES

1. X. Han. *Computer Aided Geometric Design* **19**(7), 503-512 (2002).
2. X. Han. *Mathematics of Computation* **72**(243), 1369-1378 (2003).
3. X. Han. *Computer Aided Geometric Design* **21**(6), 535-548 (2004).
4. X.A.Han. Y.C. Ma and X.L. Huang. *Applied Mathematics Letters* **22**(2), 226-231 (2009).
5. L.Yang, J. Li and Z. Chen. In proceeding of international conference on Multimedia Technology (ICMT). 2011. IEEE.
6. U. Bashir, M. Abbas, and J.M. Ali. In proceeding of ninth international conference on Computer Graphics, Imaging and Visualization (CGIV). 2012. IEEE.
7. M. Z. Hussain and M. Sarfraz. *Journal of Computational and Applied Mathematics* **218**(2), 446-458 (2008).
8. Y. Zhu, X. Han, and J. Han. *Journal of Computational Information Systems* **8**(2), 905-914 (2012).
9. F. Ibraheem et al. *Journal of Applied Mathematics* **2012**, 19 pages (2012), Article ID 247120.