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A RE-EXAMINATION OF THE PURE CONSUMPTION LOANS MODEL

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# A RE-EXAMINATION OF THE PURE CONSUMPTION LOANS MODEL[*]

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## I. Introduction

Professor Samuelson's 1958 article on an exact consumption-loan model of interest [8] led to a series of exchanges, thus becoming the basis for one of the controversies which tend to flare up periodically in economics. At issue was the determination and properties of interest rates in a dynamic economic system with no capital. One might have thought that these exchanges, between Samuelson and Lerner [3, 4, 9] on the one hand, and between Samuelson and Meckling [5, 6, 10] on the other, would result, if not in a complete resolution of the disagreements, at least in the emergence of a clear picture of what the issues are and how they might be treated. Unfortunately, such was not the case. The Samuelson-Lerner and the Samuelson-Meckling dialogues leave the reader rather perplexed, as though he had just watched a New Wave film -- executed with brilliance, rather enjoyable while in progress, but not quite clear as to what is happening and never giving one a sense of resolution.

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The 1960's have brought with them an upsurge of interest in capital theory and, more generally, in questions of allocation over time. Now Samuelson's model, even though it has no capital, is of interest to capital theorists (we shall resist the temptation to call this "a paradox") because it has many of the features of a model of capital accumulation with decentralized decision making. Since we are of the opinion that the 1958-60 discussions of this model by Samuelson, Lerner and Meckling left some questions unanswered, we feel that a restudy of the model is in order. The following is an attempt at such a restudy.

We shall take the liberty of deviating somewhat from Samuelson's original notation.

## II. The Model

We shall concentrate throughout most of our discussion on the simpler of Samuelson's two models, namely the one in which people live for two periods, earning a fixed income in the first and earning nothing in the second. In the first period of his lifetime, a person earns one unit of output, where "output" is something usable directly (and exclusively) in consumption, and we do not inquire as to wherefrom it comes.

The generation which is born at time $t$ will be called generation $t$, and we shall assume that there are $(1+n)^t$ people in it. Thus, $n$ is the (relative) rate of growth of population, which is assumed constant. Members of generation $t$ are thought of as being alike in all respects, so one can speak of a member of generation $t$ without specifying the individual. Let the symbol $c_t^1$ stand for the first-period consumption of a member of generation $t$, and let $c_t^2$ stand for his second-period consumption. A member of generation $t$ is assumed to value any given consumption plan $(c_t^1, c_t^2)$

according to the value, $U(c_t^1, c_t^2)$ , taken on by a "regularly shaped"

utility function U at $(c_t^1, c_t^2)$ . The utility function U is assumed to

be the same for all generations.

Continuing in Samuelson's footsteps, we now proceed to assume that

output is nondurable and thus cannot be carried over from one period to

the next. This assumption reduces the production possibilities in the model

to nought. It is clear, furthermore, that the assumptions made so far are

so restrictive as to rule out from the outset any possibility of trades,

markets, or prices. A member of generation t who wishes to engage in a

transaction cannot find anyone willing and able to participate in the

transaction on the opposite side.

Given that production and trade have both been dispensed with, there

remains only one other economic activity to be considered -- distribution.

This function is still open in our economy, for output can be taken from the

young who earn it and given to the old who don't. Thus, our first task will

be the examination of alternative distribution schemes. However, before

proceeding with this examination, we must define the notion of "the rate of

interest." Writing $r_t$ for the rate of interest in period t , we <u>define</u>:

$$r_t = \frac{c_{t-1}^1 + c_{t-1}^2 - 1}{1 - c_{t-1}^1} .$$

Two things should be noted: (a) $1 + r_t$ , with $r_t$ as defined here, is

<u>not</u> a price, i.e., no transactions are ever held using it as a rate of exchange.

It is not even an "implicit price," in the sense of a price which emerges as a

by-product of efficient allocation of resources. It is, rather, an ex post

rate of exchange which is inferred from observation of the consumption pattern

of a member of generation $t-1$, and it has reference neither to trade nor to

efficiency. (b) When $c^1_{t-1} = 1$, $r_t$ is clearly not defined. If $c^1_{t-1} = 1$

and $c^2_{t-1} \neq 0$ we shall say that $r_t$ can be any real number, and if $c^1_{t-1} = 1$

and $c^2_{t-1} > 0$ we shall say that $r_t = + \infty$ .


## III. Distribution of Output

As has already been remarked, the only economic function remaining in

the model of the foregoing section is that of distribution of output. A pair

of sequences

$$\left\{ c^1_t , \quad t = 0, \pm 1, \pm 2, \ldots \right\}$$

$$\left\{ c^2_t , \quad t = 0, \pm 1, \pm 2, \ldots \right\}$$

with non-negative elements will be referred to as a distribution scheme. It

specifies how much each member of any given generation shall consume in each

period of his lifetime. A distribution scheme will be called feasible if it

does not use up more output than is available in any period. In other words,

the scheme $\left( \left\{ c^1_t \right\} , \quad \left\{ c^2_t \right\} \right)$ is feasible if and only if

$$(1+n)^t \, c^1_t + (1+n)^{t-1} \, c^2_{t-1} \leq (1+n)^t$$

*or*

$$c_t^1 + \frac{c_{t-1}^2}{1 + n} \leqq 1$$

for all $t$, where the inequality arises from an assumption of free disposal.

Suppose that in period $t$ no disposal takes place. Then,

$$c_t^1 + \frac{c_{t-1}^2}{1 + n} = 1 .$$

On the other hand, we have, by the definition of the rate of interest, that

$$c_{t-1}^1 + \frac{c_{t-1}^2}{1 + r_t} = 1 .$$

Subtracting the latter from the former, we get

$$c_t^1 - c_{t-1}^1 + \frac{r_t - n}{(1 + r_t)(1 + n)} c_{t-1}^2 = 0 .$$

As an immediate consequence one now obtains that

$$\text{if } c_{t-1}^1 = c_t^1 \neq 1$$

$$\text{then } r_t = n .$$

If $c_{t-1}^1 = c_t^1 = 1$ then $r_t$ can be any real number, and we might as well

agree once again that $r_t = n$ .

This equality of $r_t$ and $n$ in case $c_{t-1}^1 = c_t^1$ is what Samuelson

calls "the biological rate of interest," which he finds "paradoxical," even

"astonishing," [1] and which Meckling finds very hard to accept. (Lerner's

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[1] [8], pp. 471 and 473, respectively.

position will be discussed below.) The main cause for suspicion seems to be the fact that a rate of interest has been determined without any reference to impatience and time-preference or, more generally, to the utility function $U$ . Somehow, the fact that $r_t$ is a completely mechanistic construct, having no reference to markets, seems to have become blurred.

IV. Stationary Schemes

. In order to analyze the relation between the rate of interest and the rate of growth of population somewhat further, let us define a stationary distribution scheme by the requirement

$$\left. \begin{array}{l} c_t^1 = c^1 \\ \\ c_t^2 = c^2 \end{array} \right\} \text{for all } t$$

which implies that

$$r_t = r = \frac{c^1 + c^2 - 1}{1 - c^1} \quad \text{for all } t .$$

We are assuming for the moment that $c^1 < 1$ , so $r$ is well defined. Now let us check the feasibility of a stationary distribution scheme. By direct substitution, we find that

$$c^1 + \frac{c^2}{1 + n} \lesseqgtr 1 \quad \underline{\text{if and only if}} \quad r \lesseqgtr n .$$

Thus, the feasibility of a stationary distribution scheme is equivalent to the statement that the rate of interest is no greater than the rate of growth

of population. Furthermore, the same algebraic operation which yielded the equivalence of the above inequalities also yields the equivalence of the strict inequality $r < n$ and the strict inequality $c^1 + c^2/(1+n) < 1$. With a stationary distribution scheme, a rate of interest which falls short of the rate of growth of population means that some output is being discarded in every period. In other words, the inequality $r < n$ means that the distribution scheme under consideration is _inefficient_ (unless consumers are satiated.) This result has a familiar ring to it. In models where investment and capital accumulation are possible we often find that among all feasible stationary paths, the path which maximizes per-capita consumption is the so-called golden rule path, which is characterized, among other things, by the equality of the rate of interest and the rate of growth of population. Indeed, we know that a stationary path along which the (constant) rate of interest is lower than the rate of growth of population is in fact inefficient in the sense that everybody's consumption can be increased (see Phelps [7]).

Let us recapitulate: Every stationary distribution scheme is characterized by a pair of non-negative real numbers, $c^1$ and $c^2$. The set of all feasible schemes is represented by the shaded area in Figure I, and it corresponds precisely to the set of all schemes with a rate of interest which is no greater than the rate of growth of population. Among the latter, only the schemes which are represented by points on the northeastern boundary line of the shaded area are _efficient_. These efficient schemes are precisely those for which the rate of interest is, in fact, _equal_ to the rate of growth of population.
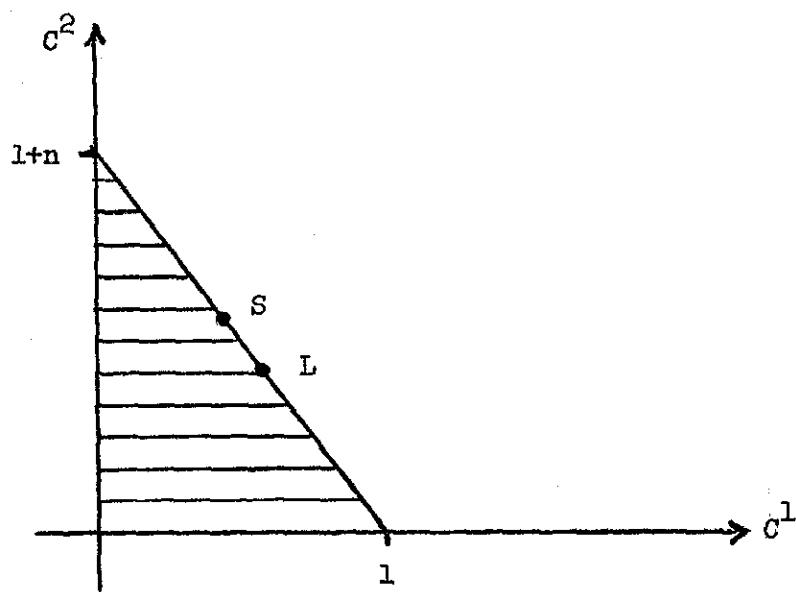
FIGURE I

Note that Figure I actually contains part of Samuelson's Figure 1 on p. 519 of [9]. In that figure, Samuelson marks two specific schemes, represented by the points S and L , which he labels the "Samuelson plan" and the "Lerner plan," respectively. Now both S and L are in fact on the efficient frontier of the shaded set and therefore both represent distribution schemes for which the rate of interest is equal to n . The discussions by both Samuelson and Lerner, in which the point L is referred to as corresponding to a zero rate of interest, therefore seem to be in error. [2]

The only point on the efficient frontier which could possibly be taken to

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[2] Samuelson and Lerner both refer to the case where $n = 1$ and L is the point $(2/3, 2/3)$. Clearly, at that point every person foregoes $1/3$ units of output in the first period and receives $2/3$ in the second period, which corresponds to a rate of interest of 100 percent.

represent a distribution scheme with zero interest is the point $(1, 0)$,
where the rate of interest can be taken to be any real number, including $0$
(and also including $n$).

Let us turn now to the question of choosing between alternative distribution schemes. If we take all the feasible distribution schemes and ask a member of generation $t$ which of these he prefers, he will no doubt say that the schemes which satisfy the requirement

$$c_t^1 = 1, \qquad c_t^2 = 1 + n$$

are highest on his preference scale. These are the schemes which give generation $t$ all the output in period $t$, as well as all the output in period $t + 1$ (under the assumption that within each generation all share alike). He will be indifferent as to which of the schemes satisfying this requirement is picked. It is obvious, therefore, that no single distribution scheme will be preferred to all other feasible schemes by all individuals. In a situation of this sort, the economist usually resorts to one of two things: Either he defines an overall social welfare function and picks the feasible distribution scheme which maximizes it, or he restricts the choice of a scheme to a subclass of the original class of all feasible schemes, a subclass such that individual maximization over it will result in compatible choices. The subclass of all stationary distribution schemes clearly has this property, since stationarity means that everybody has the same lifetime consumption profile. Thus, if we agree to restrict the search for a distribution scheme to the class of all

stationary schemes [3] (and this agreement is extraneous to the analysis, just as the choice of a social welfare function would be) we can find one distribution which maximizes everybody's utility. We write

$$\max \quad U(c^1, c^2)$$

subject to the constraint that the stationary scheme be feasible, i.e. subject to

$$c^1 + \frac{c^2}{1 + n} \leqq 1 .$$

This is indeed Samuelson's maximization problem (leading to the point S in his diagram as the solution) but stated in terms of choice between distribution schemes, rather than in terms of the opportunities open to a "representative man." It should be stressed again that all the efficient points in the set over which the maximization takes place are points which correspond to a rate of interest equal to n (except the point (1, 0) , where the rate of interest is indeterminate).

Before leaving this part of the discussion, let us consider the problem of decentralization. It is clear that the distribution schemes which have been discussed above (whether stationary or not) are not, in general, attainable by having each individual act on his own, in a decentralized fashion. Indeed, the only distribution schemes which are attainable with each individual acting

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[3] It seems that Lerner's concern for equality of income distribution really ought to lead him to stationarity (everybody getting the same consumption profile) and not to equality of consumption for all within each time-period, i.e., $c^1 = c^2$ , which he seems to advocate in [3] and [4].

on his own through the (inactive) market are distribution schemes for which

$$c_t^1 \leq 1$$
$$c_t^2 = 0$$
for all $t$ .

Among these, the only efficient scheme is given by $c_t^1 = 1$ and $c_t^2 = 0$
for all $t$ , which also happens to be a stationary scheme. However, this is
obviously not (in general) the scheme which individuals would pick, among all
stationary schemes, if they had the choice. Thus, as Samuelson points out, we
have here an example in which decentralized (competitive) behavior fails to
lead to an optimum. This conclusion can be sharpened considerably if one drops
the assumption that output is nondurable. (See Section VI below.)

## V. Constant Rate of Interest Paths

In the foregoing section we have seen that the rate of interest is
constant along every stationary path. We now ask whether every path along
which the rate of interest is constant is, in fact, stationary.

Constancy of the rate of interest means that there exists a real
number $r$ such that

$$c_t^1 + \frac{c_t^2}{1 + r} = 1 \quad \text{for all } t .$$

Now, feasibility and efficiency of the distribution scheme mean that

$$c_t^1 + \frac{c_{t-1}^2}{1 + n} = 1 \quad \text{for all } t .$$

Subtracting the latter from the former and rearranging leads to

$$\frac{c_t^2}{c_{t-1}^2} = \frac{1 + r}{1 + n} .$$

Since $c_t^2$ is bounded by

$$c_t^2 \leq 1 + n \quad \text{for all } t ,$$

we find that the only rate of interest which can be constant for all $t = 0, \pm 1$, $\pm 2, \ldots$ is the rate $r = n$ which, indeed, corresponds to a stationary scheme.

However, if we only require the rate of interest to be constant from some point on, say $t = 0, 1, 2, \ldots$, then we find that $r \leq n$ is possible, while $r > n$ is not.[4] If $r < n$ then as $t \to \infty$, $c_t^2$ tends to $0$, and therefore $c_t^1$ tends to $1$. In fact, by a straight-forward calculation one can show that if there exists an $\epsilon > 0$ such that $r_t \leq n - \epsilon$ for $t = 0, 1, 2, \ldots$, then

$$\lim_{t \to \infty} c_t^2 = 0 .$$

As a particular instance, this discussion applies to paths along which the rate of interest is always <u>zero</u>, which is what Lerner seems to advocate.

## VI. <u>Durable Output</u>

Now let us drop the assumption that output is completely perishable and substitute in its stead the opposite assumption, namely that output is completely durable.

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[4] Similarly, if the rate of interest is required to be constant <u>up to</u> some point, then $r \geq n$ is possible, but $r < n$ is not.

The first thing to notice is that trades are still impossible. A member of generation $t$ who wishes to give up a quantity of output at time $t$ in return for a quantity of output at time $t + 1$ can never find anyone who is offering to give up a quantity of output at time $t + 1$ in return for a quantity of output at time $t$. On the other hand, a technology is now available to a member of generation $t$ with which he can transform output at time $t$ into output at time $t + 1$, unit for unit.

Suppose that with this new regime we now require our economy to proceed in an entirely decentralized fashion, each individual acting on his own. The opportunity set available to each individual is now given by the triangle OAB in Figure II. (Recall that when output was nondurable, the individual's opportunity set consisted only of the line segment OA.) An individual of generation $t$ will therefore pick his consumption plan $(c_t^1, c_t^2)$ so as to maximize $U(c_t^1, c_t^2)$ subject to $c_t^1 + c_t^2 = 1$, which will lead him to a point, say $D$, on the boundary of his opportunity set. Clearly, $D$ will not in general coincide with $A$, i.e., the individual will in general choose to consume positive amounts in both periods. Note also that the point $D$ will be optimal for individuals of <u>all</u> generations (since we are assuming a common utility function for all) so that decentralized behavior leads, once again, to a stationary path.

Now let us look at distribution schemes which are not necessarily de-centralized. Obviously, any scheme which was feasible under the assumption that output is nondurable is also feasible under the assumption that output is completely durable. Hence, the set of feasible distribution schemes in our

new regime certainly contains the feasible set of the old regime. In particular, the set of all feasible stationary schemes must contain [5] the shaded set in Figure I, which appears in Figure II as the triangle OAC .
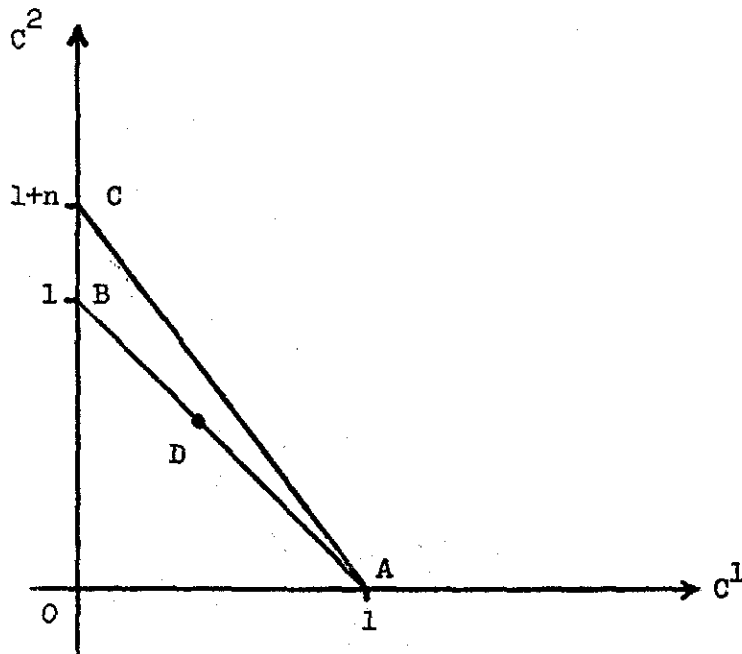


FIGURE II

A brief look at Figure II tells the story: While in the old regime decentralized behavior led to an efficient but (in general) non-optimal distribution scheme, in the new regime decentralization leads (in general) to an inefficient (to say nothing of optimality) distribution scheme.

This phenomenon, the inefficiency of decentralized behavior, does not disappear if we drop yet another assumption and permit capital (which in this

[5] Actually, the two sets coincide.

model is in the form of inert inventories) to become productive. This has been shown recently by P. A. Diamond [2]. [6] In Diamond's model, the inefficiency appears only if people "want to save too much" (in some well-defined sense). In the present model, people in the decentralized scheme always want to save too much, in the sense that the efficient distribution schemes involve zero inventories at every moment of time, while the decentralized scheme (in general) involves positive inventories at every moment of time.

## VII. Financial Intermediation

Thus far, we have avoided the question of how an efficient distribution scheme might be brought about. Is there an economic agent which could be introduced into the model and whose activity would ensure efficiency? Before attempting to answer this question we must probe a little further into the nature of the inefficiency of decentralization in the model of the foregoing section.

At the heart of the inefficiency in our new model (with durable output) lies the fact that decentralization forces people to hoard output in their first period so as to be able to consume what they had hoarded in the second period. The result is that in every period a fraction of total output is put aside in the form of a savings fund, to be carried over to the next period. But when the next period comes around and the older generation consumes its savings, the younger

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[6] Indeed, the present model, with durable output, may be looked upon as a special case of Diamond's model, with the intensive production function (i.e., the function relating the output/labor ratio to the capital/labor ratio) identically equal to unity.

generation establishes its own savings fund and the economy ends up carrying a load of dead weight, in the form of output which is never consumed. Indeed, under the assumption of the same utility function for all, this load of dead weight keeps growing like a geometric progression, because the savings fund of generation $t$ is $1 + n$ the size of the savings fund of generation $t - 1$.

The only way to restore efficiency to our system is to find an arrangement whereby the savings of generation $t$ (when it is young) are used to provide for the consumption of generation $t - 1$ (when it is old). In this way, the situation in which positive amounts of real output are always being carried over from period to period will not arise. But this cannot be done in a decentralized manner within the present framework of assumptions. The first thing which comes to mind at this point is that maybe the trouble lies in the lack of sufficient overlap between generations, which prevents the right trades from being made.

In order to investigate this possibility, we introduce into the picture a financial intermediary of some sort (say a banking system or a system of pension funds) which is assumed to exist concurrently with all generations. People are now able to save by holding the liabilities of this financial intermediary, which they will in fact prefer to do if by holding such liabilities for one period they can earn a positive rate of interest. Now, the intermediary can, in fact, offer holders of its liabilities a rate of interest of $n$ per period, simply by using the output deposited with it by generation $t$ to redeem the liabilities which are held by generation $t - 1$. The result is that all output available in period $t$ is in fact consumed in period $t$, partly by

generation $t$ and partly by generation $t - 1$. Generation $t - 1$ receives whatever generation $t$ decides to save as payment in full (principal as well as interest) of the debt incurred to it one period earlier by the financial intermediary. There will no longer be a carry-over of output from period to period and, as a result, efficiency will be restored. Indeed, optimality will be restored as well, because in response to a rate of interest of $n$ per period people will choose to save exactly the amount which leads to the optimal distribution scheme.

The outlook seems rosy, until one takes a brief look at the balance sheet of our financial intermediary, where things are rather unfortunate: The balance sheet as of the end of period $t$ shows zero assets and liabilities of $s(1+n)^t$, where $s$ is the (stationary) savings ratio of people in their first period of life. This means that at the end of period $t$ the net worth of the intermediary is given by $-s(1+n)^t$. Now by not doing anything (i.e. by shutting down) the intermediary can guarantee itself a net worth of zero, and so one might argue that it will never choose to engage in the aforementioned transactions. On the other hand, it might be argued that the intermediary should be looked upon as a social security system which is not privately owned, so that its net worth position is of no concern (or should not be of any concern). This is what Lerner seems to be saying when he decries ([3] p. 517) "those of the accountants who insist on the 'solvency' of the Social Security Administration." Later, ([4] p. 523) Lerner seems to be taking a $180°$ turn, by insisting on the very accounting practice which he had previously decried:

> "Business is fine," said the optimistic contractor. "It is
> true that I lose money on every contract, but I always start
> a bigger one and get an advance that more than covers the loss
> on the old one."

Be that as it may, it is certainly the case that a privately owned
financial intermediary will not rescue the economy from inefficiency. In
other words, decentralization and competitive behavior still fail to result
in efficient behavior.

VIII.   Lack of Borrowing as a Source of Inefficiency

Let $x_t$ be the amount of output carried over by society from period   t
into period   $t + 1$ .  If we restrict our attention to stationary states then
it is clear that a necessary and sufficient condition for efficiency is   $x_t = 0$
for all   t .  However, if we wish to consider the nonstationary cases as well,
the condition for efficiency becomes somewhat more complicated.  For it is
possible, for instance, for generation   t   to underconsume and for generation
$t + 1$   to overconsume, with the result that   $x_t$   will be positive and yet the
economy will remain efficient.  More generally, the building up of inventories
does not destroy efficiency as long as these inventories are eventually
consumed.  The exact statement in this respect is as follows:  A necessary and
sufficient condition for the economy to be efficient is that there exist a
subsequence  $\left\{ x_{t_k} \right\}$  of the sequence  $\left\{ x_t \right\}$  such that

$$\lim_{k \to \infty} x_{t_k} = 0 .$$

The proof of this assertion will be given in the Appendix.  Roughly speaking,
efficiency requires that inventories return periodically to a level which is
"practically zero."

From the point of view of balance sheets, $x_t$ is clearly <u>net assets</u>

(total assets minus total liabilities) in the economy at the end of period $t$ .

If we concentrate our attention at the time periods $t_k$ , we find that

efficiency requires net assets at the end of these periods to be (practically)

zero. [7]  In other words, in order to have efficiency, it must be the case

that for each outstanding asset in the economy at the end of period $t_k$ there

exists a corresponding liability outstanding at the end of period $t_k$ . But now

let us recall the time structure of our model. People live for two periods,

they consume in both periods but earn income only in the first. This forces

individuals to become net lenders (and never net borrowers) no matter what rate

of interest prevails. [8]  In particular, members of generation $t_k$ will want

to be net lenders, i.e. to have an end-of-period balance sheet which shows just

assets and net worth, and no liabilities. But efficiency requires that

aggregate net worth in the economy be zero, so there must exist someone in the

economy holding liabilities in excess of assets, i.e. having a negative net

worth at the end of period $t_k$ . This puts the unfavorable net worth position

of the financial intermediary of Section VII in perspective. By the same token,

the decentralized economy of Section VI is inefficient precisely because no one

can be a net borrower while everyone wishes to be a net lender.

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[7]  The word "practically" is intended to convey the notion that strictly speaking
net assets at the end of period $t_k$ might be given by some $\epsilon > 0$ , but by
taking $k$ large enough, this $\epsilon$ may be taken as small as we wish. From
here on, we shall neglect to remind the reader of this qualification.

[8]  As long as this rate of interest exceeds -1 .

It is of some interest to investigate Samuelson's discussion concerning "the contrivance of money" in the light of the foregoing remarks. Clearly, if efficiency is to be attained, people have to be dissuaded from holding output as a store of value and persuaded to hold another asset instead. If this other asset bears a positive rate of interest then people will in fact make the desired shift. As Samuelson points out, the role of this other asset can very well be fulfilled by money. People would save by buying the existing money supply and dissave by selling it to next period's savers. On a stationary path, the price of money will rise by a factor of $1 + n$ in each period, which corresponds to a rate of interest of $n$ per period, so individuals will in fact prefer holding money to holding output. Thus, at least with respect to stationary paths, the introduction of money leads to efficiency. (More precisely: Every stationary path in the money economy is efficient.) Samuelson's interpretation of this phenomenon is a philosophical one: An economy is inherently more than a mere mechanical system of particles in motion; it is, in fact, such a system plus something called a "Hobbes-Rousseau social contract" ([8] p. 479). A physical system can operate efficiently without this added aspect, but a social system cannot. [2] Now, it seems to us that the Social Contract is no more involved in Samuelson's money economy than it is in any other general equilibrium model. For this reason we feel that Samuelson's discussion in this area is liable to be misleading. In general equilibrium analysis one thinks of a market convention in which prices

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[2] It is not clear whether Samuelson intends his model to constitute a mathematical proof of the Locke-Hobbes-Rousseau thesis.

are announced and economic agents determine the trades and the productive
activities in which they wish to engage at these prices. If the totality of
all trades clears all markets, then the announced prices are said to be
equilibrium prices. The question which general equilibrium theory asks is the
following: Under the assumption that everybody takes prices as <u>given</u>, is there
a schedule of prices which leads to the clearing of all markets? In Samuelson's
money economy, commodities are time-dated output and time-dated money, and all
that one asks is whether or not a given price schedule is an equilibrium price
schedule. It turns out that the schedule which sets the price of output in
all periods equal to 1 , and the price of money in period t equal to $(1+n)^t$
is, in fact, an equilibrium price schedule. The element of public trust in the
monetary unit is reflected by the fact that a person who buys money in period t ,
at a price of $(1+n)^t$ , assumes that he will be able to sell it in period t + 1 ,
at a price of $(1+n)^{t+1}$ . But this is precisely what is meant in general
equilibrium theory by the phrase "taking prices as <u>given</u>."

At the beginning of this section, we argued that if efficiency is to
be attained, someone will have to have a balance sheet showing liabilities
in excess of assets. For this reason it seems appropriate to look upon money
as a liability of a monetary authority which is committed to paying one
dollar to whoever presents it with one dollar. The balance sheet of this monetary
authority shows only liabilities and no assets, and the value of the authority's
liabilities (quantity of money multiplied by the price of money) is precisely
equal to the excess of assets over liabilities in the private sector. From
this point of view, Samuelson's "contrivance of money" is basically no different

from the financial intermediary of Section VII, except perhaps to the extent of the distinction between "outside money" and "inside money."

We turn now to a brief investigation of a model in which people live for _three_ periods, and in which efficiency can be achieved (under some circumstances) without introducing into the economy a sector with negative net worth.

## IX. A Three-Period Model

We have argued that in order to have efficiency there must be someone in the economy who is, at least periodically, a net borrower. Along stationary paths, this must be the case in _every_ period. However, there was nothing in that argument to suggest that the net borrower must be a net borrower throughout his (or its) lifetime. Indeed, this role may well change hands over time, which was not the case with the financial intermediary of Section VII.

The prime candidate for this state of temporary net borrowership is the consumer himself. To check on this possibility, let us consider the following modification of our model: Assume that people live for three periods rather than two. In the first period of life a person grows up and is educated and therefore earns nothing; in the second period he works and earns one unit of output; in the third he is retired. Every person will now be a net borrower at the end of this first period, a net lender at the end of his second period, and he will be neither a borrower nor a lender at the end of his last period. (We are not using the terms "negative net worth" and "positive net worth" because it is not clear that they are applicable in the present context.) We shall restrict our attention to stationary paths. Along a stationary path, everybody receives

the same lifetime consumption profile, say $(c^1, c^2, c^3)$ . Feasibility and efficiency now mean that the following equation holds:

$$(1+n)c^1 + c^2 + \frac{c^3}{1+n} = 1$$

which is similar to the feasibility and efficiency equation in the two-period case, and is derived in the same way. If we let the (constant) rate of interest along the stationary path be denoted $r$ , then each person's budget constraint is given by

$$(1+r)c^1 + c^2 + \frac{c^3}{1+r} = 1$$

which leads, once again, to $r = n$ . In other words, the only rate of interest which can be established along an efficient stationary path is $n$ . Each person is now viewed as maximizing the utility function $U(c^1, c^2, c^3)$ subject to the budget constraint (with $n$ replacing $r$ ). The maximization will result in an <u>optimal</u> consumption plan, say $(\bar{c}^1, \bar{c}^2, \bar{c}^3)$ . This optimal triple must be examined, to see if it can be brought about by purely competitive (decentralized) trades.

The optimal consumption plan $(\bar{c}^1, \bar{c}^2, \bar{c}^3)$ will be said to be <u>competitively attainable</u> if the following equation holds:

$$\bar{c}^3 = (1+n)^2 \, \bar{c}^1 \ .$$

To see how this condition is obtained, consider a person of generation $t$ . When he is young, he borrows an amount $\bar{c}^1$ from members of generation $t - 1$ , who are in their middle years. Next period, when he is middle aged and generation $t - 1$ is retired, he returns the loan, plus interest. In other words he returns

the amount $(1+n)\bar{c}^1$ . Now aggregate borrowing by generation $t$ (when young)

is given by the amount $(1+n)^t \bar{c}^1$ , and therefore total payment by generation $t$

(when middle aged) to generation $t-1$ (when retired) is given by $(1+n)^{t+1} \bar{c}^1$ .

This quantity, divided by the size of generation $t-1$ , yields the per capita

consumption of the retired, i.e. $\bar{c}^3$ . This is precisely what the foregoing

equation says.

As an example, consider the utility function

$$U(c^1, c^2, c^3) = \sum_{i=1}^{3} \log c^i .$$

Maximizing it subject to the budget constraint leads to

$$(1+n)\bar{c}^1 = \bar{c}^2 = \frac{\bar{c}^3}{1+n} = \frac{1}{3}$$

which is clearly competitively attainable.[10]

Why is the decentralized economy in this example efficient? Surely, it

is possible to attribute this result to "the social contrivance of binding

contracts." It is clearly in the interest of the middle aged to default and

ignore the debt which they have incurred when young. Even if the rules are such

that a person guilty of default is denied access to the capital market as a

lender (and so must lose interest on his savings) it is still true in many

cases that default will result in increased consumption in all periods. Here,

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[10] It is interesting to note that if one introduces the same three-period
configuration into Diamond's model, which has productive capital, then
(with an additive logarithmic utility function) the system is always
efficient. A generalization of Diamond's model is discussed in [1].

once again, is an opportunity to appeal to the Social Contract and here, once again, it would seem to be beside the point, and for very much the same reasons as before: The assumption that contracts are not defaulted upon usually goes without saying in the theory of the competitive mechanism; it does not _explain_ our result, it merely _permits_ it.

But our example is a very lucky one. For it is not, in general, to be expected that the optimal solution of a consumer's lifetime allocation problem will satisfy as stringent a condition as $\bar{c}^3 = (1+n)^2 \bar{c}^1$. In fact, with most utility functions this condition will not hold. If $\bar{c}^3 > (1+n)^2 \bar{c}^1$ then we shall, once again, have too little borrowing in the economy and an additional agent with negative net worth would be needed in order to attain efficiency. However if $\bar{c}^3 < (1+n)^2 \bar{c}^1$ then we shall have _too much_ borrowing in the economy, and it will be possible to introduce a financial intermediary with _positive_ net worth which will guarantee efficiency. The balance sheet of this intermediary will show IOU's of young people on the asset side and nothing on the liabilities side. In order to see how this intermediary would operate, let us concentrate, once again, on period $t$. In this period, a member of generation $t$ offers to sell a quantity $\bar{c}^1$ of IOU's. The total supply of IOU's by generation $t$ is therefore given by $(1+n)^t \bar{c}^1$. However, a member of generation $t-1$ wishes to buy only $\bar{c}^3/(1+n)$ in IOU's, so that the total quantity of IOU's demanded is $(1+n)^{t-2} \bar{c}^3$, which falls short of the quantity supplied. The intermediary now steps in and buys the excess supply, using as payment the resources which generation $t-1$ pays in when it redeems its own IOU's, which it sold to the intermediary in the previous period. Stationarity ensures that these resources will always be precisely adequate to buy the outstanding IOU's. The result

will be that physical output will not be carried over from period to period, so that efficiency will be attained. The aggregate debt of the consumer sector to the financial intermediary will grow like a geometric progression, but each consumer by himself will be balancing expenditures and receipts over his life-time, i.e., at the end of his last period of life he will have zero net worth. Under these circumstances, the intermediary may very well be thought of as a privately owned, competitive, institution.

## X. Conclusion: A Note on Infinity

Both Diamond and Samuelson seem to attribute the possible inefficiency [11] of the competitive mechanism (in models of the sort discussed here) to the mysteries, or pathologies, of infinity. [12] In our opinion, this is somewhat misplaced emphasis. Consider, for instance, the model in which the world ends abruptly at time $T$ , and assume that the end of the world catches people by surprise. In other words, assume that people's expectations are _static_. This is a finite model which behaves exactly like the infinite model of the foregoing discussion. Admittedly, the behavior of this new model depends crucially on the assumption of static expectations, just as the behavior of the old model depends crucially on the assumption that the world is infinite. It merely seems that the phrase "this involves infinity" should not be regarded as an explanation

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[11] Sameulson refers only to non-optimality. But, as has been remarked above (Section VI), his non-optimality transforms into inefficiency if output is durable.

[12] Diamond speaks of "the infinity of decision makers" ([2] p. 17) and Samuelson makes repeated reference to "the infinity paradox" (see, e.g., [8] p. 474, and [9] p. 522.)

of any phenomenon. Suppose it were shown, in fact, for <u>all</u> finite models, that

competition always leads to efficiency. It would still seem faulty to argue

that "since finite models are always efficient, inefficiency must be due to

infinity." It is rather like arguing that "since women never contract

hemophilia, hemophilia must be due to the fact that men are not women."

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## APPENDIX

Let $x_t$ be the level of inventories carried over by the economy from period $t$ into period $t + 1$. In Section VIII, a necessary and sufficient condition for efficiency was stated. The following is an equivalent assertion:

THEOREM: The economy is efficient if and only if

$$\liminf_{t \to \infty} x_t = 0$$

PROOF. (a) Necessity: Suppose $\liminf x_t > 0$. Then there exists an $\epsilon > 0$ and an integer $T$ such that $x_t > \epsilon$ for $t \geqq T$. Let aggregate consumption in period $T$ be increased by $\epsilon$, and let consumption in all subsequent periods remain unchanged. Let the sequence of inventory levels under this new scheme be denoted $\left\{x_t'\right\}$. Clearly, $x_t' = x_t$ for $t \leqq T$ and $x_t' = x_t - \epsilon$ for $t > T$. All that must be verified to show that the new scheme is feasible is $x_t' \geqq 0$ for all $t$. But this follows from the hypothesis.

(b) Sufficiency is proved similarly: Assume $\liminf x_t = 0$. We must show that it is impossible to increase aggregate consumption in any period while keeping aggregate consumption in subsequent periods unchanged. Let aggregate consumption in period $T$ be increased by an amount $\epsilon > 0$, where $T$ is an arbitrary integer. Denote the new sequence of inventory levels by $\left\{x_t'\right\}$. Now for $t > T$, we have $x_t' = x_t - \epsilon$, but by hypothesis there exists a $t > T$ such that $x_t < \epsilon$. Hence $x_t'$ would have to be negative, i.e. the new scheme is not feasible.