A realization of the quantum Lorentz group

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Abstract

A realization of a deformed Lorentz algebra is considered and its irreducible representations are found; in the limit $q \to 1$, these are precisely the irreducible representations of the classical Lorentz group.

Since the invention of the quantum group as a pure mathematical structure [1] much progress has been made in the realization of deformations of simple Lie algebras [?], and also Lie superalgebras, and in the construction of their representations. Quantization of space-time symmetry groups (Lorentz, Poincare, the conformal group) has remained a problem for incorporating ana applying the structure of quantum groups into physical systems. A deformation of the Lorentz group has been studied in [5] and a six-generator deformed Lorentz algebra has been found in [?] in terms of the chiral $SL(2) \times SL(2)$ generators.

In this paper we consider a realization of deformed Lorentz algebra commutation relations obeyed by the generators - rotations and boosts - and construct its representations which are the exact quantum analogue of the classical Lorentz group representations.

The quantum group is generally defined as a q-deformation of the universal enveloping algebra of the underlying classical group. Thus when introducing a quantum group one should first construct a deformed associative with a non-cocommutative Hopf algebra structure; this is usually done in terms of generators obeying deformed Lie commutation relations.

We recall that the Lorentz group contains the $SU(2)$ subgroup of rotations $M_i, i = 1, 2, 3$, and the boost generators $N_i, i = 1, 2, 3$ form an irreducible $SU(2)$ vector representation. One can form two chiral $SL(2)$ subgroups $I_i^L = M_i + iN_i$ and $I_i^R = M_i - iN_i$ which act only on spinors with undotted and dotted indices. The rotation $SU(2)$ subgroup is the diagonal in $SL^L(2) \times SL^R(2)$. The matrix elements of the two-dimensional fundamental representation generators (and of the conjugated one) satisfy $\langle N_i \rangle = \mp i \langle M_i \rangle$, the latter expressing that $I_i^{L,R}$ vanish when acting on functions of only dotted and undotted indices respectively.

In defining a deformed quantum Lorentz group we wish to generalize the properties of the classical Lorentz group to the q-case. We shall determine the deformed commutation relations as relations imposed on the matrix elements of the fundamental representation generators acting on two-dimensional spinors with undotted indices. We assume a full analogy with the classical case, namely that the quantum Lorentz group contains the $SU_q(2)$ subgroup, the boost operators transform under $SU_q(2)$ as components of irreducible tensor operators, and the fundamental representation has the properties of the two-dimensional classical spinor representation.

Let $M_\pm, M_3$ satisfy the deformed commutation relations of $SU_q(2)$ [7]. In the deformed case the $SU_q(2)$ generators do not transform under the irreducible tensor representation. The q-analogue of an irreducible $SU_q(2)$ tensor operator has been defined [8, 9] as the set of $2l + 1$

components T^l_m, $m = -l, ..., l$ obeying

$$\begin{aligned} [M_3, T^l_m] &= mT^l_m, \\ [M_\pm, T^l_m]_{q^{-m/2}} &= [l \mp m]^{1/2}[l \pm m + 1]^{1/2}T^l_{m\pm1}q^{M_3/2} \end{aligned} \tag{1}$$

where $[A, B]^\alpha_q = AB - q^\alpha BA$. There is an alternative definition to (1), obtained by $q \to q^{-1}$. Accordingly one can construct two q-vector operators S and T:

$$\begin{aligned} S_\pm = \pm q^\mp M_\pm q^{-M_3/2} \\ S_0 = [2]^{-1/2}(q^{-1/2}M_-M_+ - q^{1/2}M_+M_-) \end{aligned} \tag{2}$$

and

$$\begin{aligned} T_\pm = \pm q^\pm M_\pm q^{M_3/2} \\ T_0 = [2]^{-1/2}(q^{1/2}M_-M_+ - q^{-1/2}M_+M_-) \end{aligned} \tag{3}$$

We identify the operators $-iq^{-1/4}M_+q^{-M_3/2}$, $-iq^{-1/4}M_-q^{M_3/2}$ and $-i[M_3]q^{-M_3/2}$ with the generators N_+, N_- and N_3respectively, of a two-dimensional q-deformed Lorentz boost transformation, expressed in aq-tensor form. We identify further the operators $M_\pm$ in a q-tensor form, i.e. $M_\pm \equiv q^{-1/4}M_\pm q^\mp M_3/2$, and the operator M_3 with the rotation generators of a q-deformed Lorentz transformation. The commutation relations of a deformed Lorentz algebra are imposed as the relations obeyed by the generators of the two-dimensional Lorentz transformation. Namely, the rotations $M_\pm, M_3$ satisfy the Lie brackets of the deformed $U_q(su(2))$; the commutation relations between rotations and boosts are determined as the action of the $U_q(su(2))$ generators on the irreducible $su_q(2)$ tensor operators (2) and (3); the commutators between the boosts are such, that in the limit $1 \to 1$ the classical two-dimensional boost generators are recovered. The deformed Lorentz algebra has the form

$$\begin{aligned} M_+M_- - M_-M_+ &= [2M_3], \\ M_3M_\pm - M_\pm M_3 &= \pm M_\pm, \\ N_+N_- - N_-N_+ &= -[2M_3] \\ N_3N_+q^{1/2} - q^{-1/2}N_+N_3 &= -M_+, \\ \tilde{N}_3N_-q^{1/2} - q^{-1/2}N_-\tilde{N}_3 &= M_-, \\ M_3N_\pm - N_\pm M_3 &= \pm N_\pm, \\ M_+N_-q^{-1/2} - q^{1/2}N_-M_+ &= [2]\tilde{N}_3 + (q^{1/2} - q^{-1/2})C'^q_2, \\ M_-N_+q^{-1/2} - q^{1/2}N_+M_- &= -[2]N_3 + (q^{1/2} - q^{-1/2})C'^q_2, \\ M_+\tilde{N}_3q^{1/2} - q^{-1/2}\tilde{N}_3M_+ &= -N_+, \\ M_-N_3q^{1/2} - q^{-1/2}N_3M_- &= N_- \end{aligned} \tag{4}$$

and all other (usual) commutators vanish. The element

$$2C'^q_2 = \frac{M_+N_-q^{-1/2} - N_-M_+q^{1/2} + M_-N_+q^{-1/2} - N_+M_-q^{1/2}}{q^{1/2} - q^{-1/2}} \tag{5}$$

is central in the algebra and is the quantum analogue of the second order Lorentz Casimir $C_2' = -M_i N_i, i = 1,2,3$. In the limit $q \to 1$ the deformed Lorentz algebra (4) contracts to the Lie algebra of the classical Lorentz group. One can show that the operators $M_\pm, M_3, N_\pm(q^{-1}), N_3$ satisfy the q-deformed Lorentz algebra (4) with q replaced by q^{-1} and $N \leftrightarrow \tilde{N}_3$. Then a q-adjoint involution on the deformed Lorentz group can be defined

$$\begin{aligned} (M_\pm)^* &= M_\mp, \qquad M_3^* = M_3, \\ (N_\pm^* &= N_\mp(q^{-1}), \quad N_3^* = N_3 \end{aligned} \tag{6}$$

The q-deformed Lie-brackets (4) and the corresponding ones with q replaced by q^{-1} and $N_3 \leftrightarrow \tilde{N}_3$ define a six-generator quantum Lorentz group.

We note an interesting novelty, the appearance of the central element in the defining commutation relations of the deformed Lorentz algebra (16). It is a property of the deformed relations obeyed by the generators of the quantized universal enveloping algebra which is an associative algebra with a unit and with a Poincare-Birkoff-Witt basis.

To construct the irreducible representations of the deformed Lorentz algebra we follow the classical procedure. Namely, the representations are realized in the space of the $U_q(su(2))$ irreducible representations with canonical basis $|j,m\rangle_q$ with j integer or half integer and $m = -j, ..., j$. The action of the rotation and boost generators on the basis vectors is given by:

$$M_\pm|j,m\rangle_q = [j \mp m]^{1/2}[j \pm m + 1]^{1/2} q^{-1/4} q^{\mp m/2} |j, m \pm 1\rangle_q \tag{7}$$

$$M_3|j,m\rangle_q = m|j,m\rangle_q \tag{8}$$

$$\begin{aligned} N_\pm|j,m\rangle_q &= \pm c_j[j \mp m]^{1/2}[j \mp m - 1]^{1/2} q^{-1/4} q^{-(j \pm m)/2} |j-1, m\pm\rangle_q \\ &- a_j[j \mp m]^{1/2}[j \pm m + 1]^{1/2} q^{-1/4} q^{\mp} m/2 |j, m \pm 1\rangle_q \\ &\pm c_{j+1}[j \pm m + 1]^{1/2}[j \pm m + 2]^{1/2} q^{1/4} q^{(j \mp m)/2} |j+1, m \pm 1\rangle_q \end{aligned} \tag{9}$$

$$\begin{aligned} N_3|j,m\rangle_q &= c_j[j-m]^{1/2}[j+m]^{1/2} q^{-m/2} |j-1, m\rangle_q \\ &- a_j[m] q^{-m/2} |j,m\rangle_q \\ &- c_{j+1}[j+m+1]^{1/2}[j-m+1]^{1/2} q^{-m/2} |j+1, m\rangle_q \end{aligned} \tag{10}$$

The action of the operator $C_2'^q$ on the basis vectors is given by

$$C_2'^q |j,m\rangle_q = i[l_0][l_1]|j,m\rangle_q \tag{11}$$

The coefficients a_j, c_j can be determined by using the commutators between the generators N_+ and N_- and $N_\pm$, and N_3 which results in the pair of difference equations

$$(a_{j+1}[j+2] - a_j[j])c_{j+1} = 0, \tag{12}$$

$$c_j^2[2j-1] - a_j^2 - c_{j+1}^2[2j+3] = 1 \tag{13}$$

We first note that since $j \geq 0$ there is a minimal (integer or half integer) value $j_{min} = l_0$ and hence $j = l_0 + 1, l_0 + 2, ...$ Assuming that the coefficient $c_{l_0} = 0$ we have two possibilities, either

$$c_{l_0} = 0, \quad c_{l_0+1} \neq 0, ...c_{l_0+n} \neq 0, \quad c_{l_0+n+1} = 0, \tag{14}$$

and the representation is finite-dimensional, or

$$c_{l_0} = 0, \quad c_j \neq 0 \quad for \quad any \quad j > l_0, \tag{15}$$

and the representation is infinite-dimensional. Eqs.(12, 13) for the coefficient can be easily solved and the result, being dependent on two constants l_0, l_1 is

$$a_j = \frac{i[l_0][l_1]}{[j][j+1]}, \quad c_j = \frac{i}{[j]}\sqrt{\frac{([j]^2-[l_0]^2)([j]^2-[l_1]^2)}{[2j-1][2j+1]}} \tag{16}$$

for any $j > l_0$. Since $j = l_0 + n$, where n is a natural number, the representation will be finite-dimensional if for some n

$$[l_1]^2 = [l_0 + n + 1]^2 \tag{17}$$

Due to the property of the quantity $[A]$ the above equation is satisfied for $\pm l_1 = \pm(l_0 + n + 1)$. The parameter l_1 is in general a complex number, but $l_0 + n + 1$ is a real positive number, so that the representation series will terminate if, for some real l_1,

$$|l_1| = l_0 + n + 1. \tag{18}$$

hence the spin content of the irreducible finite-dimensional representation Lorentz q-representation is determined by

$$j = l_0, l_0 + 1, ..., |l_1| - 1, \qquad m = -j, -j+1, ..., j. \tag{19}$$

We summarize the result: The irreducible representation of the deformed Lorentz algebra is determined by the pair $([l_0], [l_1]$, where l_0 is a non-negative integer or half-integer real number and l_1 is a complex number. The irreducible representation corresponding to a given pair $([l_0], [l_1])$ in the $U_q(su(2))$ canonical basis $|j, m\rangle_q$ is given by formulae (7-11) with the coefficients (16).
If, for some natural number n,

$$[l_1]^2 = [l_0 + n + 1]^2 \tag{20}$$

then the representation is finite-dimensional with the possible values of j and m given by (19). If

$$[l_1]^2 \neq [l_0 + n + 1]^2, \tag{21}$$

then the representation is infinite-dimensional. In the limit $q \to 1$, (7-11) and (16) reproduce exactly the irreducible Lorentz group [10] representations.

We now consider the conditions under which the representations of the deformed Lorentz algebra are unitary. Since the generator N_3 is self-q-adjoint, according to (6), then

$$\begin{aligned} \langle j, m|N_3|j, m\rangle &= \langle j, m|N_3^*|j, m\rangle, \\ \langle j-1, m|N_3|j-1, m\rangle &= \langle j-1, m|N_3^*|j-1, m\rangle \end{aligned} \tag{22}$$

Hence $a_j = \bar{a_j}$ and $c_j = -\bar{c_j}$. From the first of the formulae (16), it follows that the condition $a_j = \bar{a_j}$ is satisfied if either $[l_1]$ is arbitrary and $[l_0] = 0$, or $[l_0] = 0$ is arbitrary and $i[l_1]$ is real. The second possibility with q real means that l_1 should be pure imaginary

$$l_1 = i\rho, \tag{23}$$

with ρ real. The condition $c_j = -\bar{c}_j$ for the second of the formulae (16) means that the expression under the square root must be positive, and this is obviously the case if, only

$$[j]^2 - [l_1]^2 > 0 \tag{24}$$

We have to consider two possibilities:
(a) $l_0 \neq 0$ and $[l_1]$ pure imaginary, which coincides with (23).
(b) $[j]^2 \geq [l_1]^2$ with $[l_1]$ real.
The latter expression has to be satisfied for all j and this is only possible if $[l_1]^2 \leq [1]$. Hence the possible values of $[l_1]$ are

$$0 < |l_1| \leq 1 \tag{25}$$

The relations between $N_\pm$ and their q-adjoint yield the same values for l_0 and l_1.

We thus conclude: The irreducible representations of the deformed Lorentz algebra determined by the pair $[l_0], [l_1]$ is unitary if either l_1 is pure imaginary and l_0 is an arbitrary non-negative integer or half-integer, or $l_0 = 0$ and l_1 is a real number in the interval $0 < |l_1| \leq 1$. In the limit $q \to 1$ the corresponding representations (7-11) reproduce exactly the infinite-dimensional Lorentz group representations [10] of the principal and complementary series respectively.

To analyze the Hopf structure of the quantum lorentz group we need to generalize to the q-case the classical picture of forming two $SL(2)$ groups from Lorentz rotations and boosts. For this purpose we consider the operators

$$\begin{aligned}
I^L_\pm &= M_\pm + iN_\pm, \\
I^R_\pm & \quad M_\pm - iN_\pm, \\
I^L_3 &= [M_3]q^{-M_3/2} + iN_3, \\
I^R_3 &= [M_3]q^{-M_3/2} - iN_3, \\
\tilde{I}^L_3 &= [M_3]q^{M_3/2} + i\tilde{N}_3 \\
\tilde{I}^R_3 &= [M_3]q^{M_3/2} - i\tilde{N}_3
\end{aligned} \tag{26}$$

These operators satisfy the algebra

$$\begin{aligned}
I^L_+ I^L_- - I^L_- I^L_+ &= 2(I^L_3 + \tilde{I}^L_3), \\
I^L_3 I^L_+ q^{1/2} - q^{-1/2} I^L_+ I^L_3 &= 2I^L_+, \\
\tilde{I}^L_3 I^L_- q^{1/2} - q^{-1/2} I^L_- \tilde{I}^L_3 &= -2I^L_-, \\
\tilde{I}^L_3 I^L_+ q^{-1/2} - q^{1/2} I^L_+ \tilde{I}^L_3 &= 2I^L_+, \\
I^L_3 I^L_- q^{-1/2} - q^{1/2} I^L_- I^L_3 &= -2I^L_-, \\
I^R_+ I^R_- - I^R_- I^R_+ &= 2(I^R_3 + \tilde{I}^R_3), \\
I^R_3 I^R_+ q^{1/2} - q^{-1/2} I^R_+ I^R_3 &= 2I^R_+, \\
\tilde{I}^R_3 I^R_- q^{1/2} - q^{-1/2} I^R_- \tilde{I}^R_3 &= -2I^R_-, \\
\tilde{I}^R_3 I^R_+ q^{-1/2} - q^{1/2} I^R_+ \tilde{I}^R_3 &= 2I^R_+, \\
I^R_3 I^R_- q^{-1/2} - q^{1/2} I^R_- I^R_3 &= -2I^R_-,
\end{aligned} \tag{27}$$

The generators $I_\pm^L, I_3^L, I_3^{\tilde{L}}$ simply commute with $I_\pm^R, I_3^R, I_3^{\tilde{R}}$. It seems that the algebra (27) has more than six generators. Due to the relations

$$\begin{aligned} 1+\alpha I_3^{\tilde{L}}+\alpha I_3^{\tilde{L}} &= (1-\alpha I_3^L-\alpha I_3^R)^{-1}, \\ I_3^{\tilde{L}}-I_3^{\tilde{R}} &= (1+\alpha I_3^{\tilde{L}}+\alpha I_3^{\tilde{L}})(I_3^L-I_3^R) \end{aligned} \tag{28}$$

with $\alpha=(q^{1/2}-q^{-1/2})/2$, the quantum algebra (27) is, in fact, generated by two raising, two lowering and two diagonal operators.

There exists a q-adjoint involution in the algebra (27), defined as

$$\begin{gathered} (I_\pm^L(q))^* = I_\mp^R(q^{-1}), \\ I_3^{L*}=I_3^R, \qquad I_3^{\tilde{L}*}=I_3^{\tilde{R}} \end{gathered} \tag{29}$$

Denoting the two-dimensional q-deformed Lorentz representations

$$\begin{aligned} |1/2,m;l_0=1/2,l_1=3/2\rangle_q &= \tau_{1/2}, \\ |1/2,m;l_0=1/2,l_1=-3/2\rangle_q &= \tau_{\tilde{1/2}}, \end{aligned} \tag{30}$$

we observe that $I_\pm^L, I_3^L)$ (respectively $I_\pm^R, I_3^R)$) vanish when acting on $\tau_{\tilde{1/2}}$ (respectively $\tau_{1/2}$).

The algebra (27) is the q-analogue of the chiral decomposition of the classical Lorentz group, which is exactly reproduced in the limit $q\to 1$.

We further introduce the shifted diagonal generators

$$\begin{gathered} T_3^{L,R}=2-(q^{1/2}-q^{-1/2})I_3^{L,R}, \\ T_3^{\tilde{L},R}=2+(q^{1/2}-q^{-1/2})I_3^{\tilde{L},R} \end{gathered} \tag{31}$$

which does not change the structure of the algebra (27). The co-product for the deformed algebra (27) is given by which amounts to a non-cocommutative Hopf algebra structure.

$$\begin{aligned} \Delta(I_+^L) &= I_+^L\otimes 1+T_3^L\otimes I_+^L, \\ \Delta(I_-^L) &= I_-^L\otimes T_3^{\tilde{L}}+1\otimes I_-^L, \\ \Delta(I_+^R) &= I_+^R\otimes T_3^{\tilde{R}}+1\otimes I_+^R, \\ \Delta(I_-^R) &= I_-^R\otimes 1+T_3^L\otimes I_-^R, \\ \Delta(T_3^{L,R}) &= T_3^{L,R}\otimes T_3^{L,R}, \\ \Delta(T_3^{\tilde{L},R}) &= T_3^{\tilde{L},R}\otimes T_3^{\tilde{L},R}. \end{aligned} \tag{32}$$

To summarize, we have defined a quantized Lorentz algebra and found that every irreducible classical Lorentz group representation labelled by l_0 and l_1 can be q-deformed to an irreducible q-representation of the deformed Lorentz algebra labelled by $[l_0]$ and $[l_1]$. The possible values of l_0 and l_1 are exactly the same as in the classical case.

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