A reciprocity inequality for Gaussian Schell-model beams and some of its consequences

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A reciprocity inequality is derived, involving the effective size of a planar, secondary, Gaussian Schell-model source and the effective angular spread of the beam that the source generates. The analysis is shown to imply that a fully spatially coherent source of that class (which generates the lowest-order Hermite–Gaussian laser mode) has certain minimal properties.

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An important class of partially coherent beams are the so-called Gaussian Schell-model beams (Ref. 1, Sect. 5.6.4). They are generated by planar, secondary sources whose intensity distribution $I^{(0)}(\boldsymbol{\rho}, \nu)$ at frequency ν and whose spectral degree of coherence (Ref. 1, Sect. 4.3.2) $\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) \equiv g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu)$ across the source plane $z = 0$ are both Gaussian; i.e., they have the form (see Fig. 1)

$$I^{(0)}(\boldsymbol{\rho}, \nu) = A^2(\nu)\exp[-\boldsymbol{\rho}^2/2\sigma_I^2(\nu)], \tag{1}$$

$$g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \nu) = \exp[-(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2/2\sigma_g^2(\nu)]. \tag{2}$$

In these formulas $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, and $\boldsymbol{\rho}_2$ are position vectors of points in the source plane and $A(\nu)$, $\sigma_I(\nu)$, and $\sigma_g(\nu)$ are positive constants. From now on we will not display the dependence of the various parameters on ν. With a suitable choice of the parameters such a source generates a beam. In the coherent limit ($\sigma_g \to \infty$) the beam is just the lowest-order Hermite–Gaussian laser mode.

Properties of beams of this kind have been extensively studied in the literature. It has been predicted theoretically (Ref. 2 or Ref. 1, Sect. 5.4.2) that a certain trade-off is possible between the parameters σ_I and σ_g, which characterize sources with different intensity distributions and different coherence properties, yet each of these sources will generate the same far-zone intensity distribution as a single-mode laser. This prediction was confirmed experimentally soon afterward.[3,4]

In this Letter we derive a simple reciprocity inequality that involves the angular spread of a Gaussian Schell-model beam and the effective width of the intensity profile of its source, and we derive some interesting consequences from it.

We recall that the radiant intensity in the direction specified by a unit vector $\mathbf{s}$ generated by a Gaussian Schell-model source is given by the expression

$$J(\mathbf{s}, \nu) = \beta^2 \exp(-a\theta^2/2), \tag{3}$$

where θ is the angle that the vector $\mathbf{s}$ makes with the normal to the source plane,

$$\beta = (kA\sigma_I\delta), \qquad a = k^2\delta^2, \tag{4}$$

with

$$\frac{1}{\delta^2} = \frac{1}{(2\sigma_I)^2} + \frac{1}{\sigma_g^2}, \tag{5}$$

$$k = 2\pi\nu/c. \tag{6}$$

Formula (3) follows at once from Eq. (5.4-16) of Ref. 1 in the paraxial approximation ($\cos\theta \approx 1$, $\sin\theta \approx \theta$) appropriate to a beam.

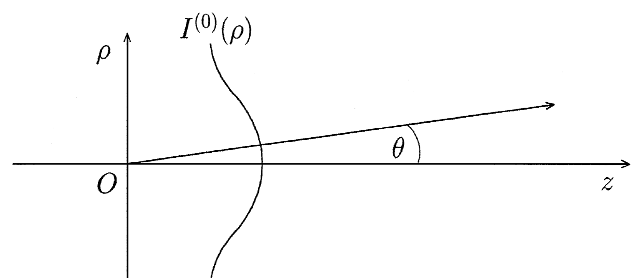

Fig. 1. Illustrating the notation.

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Let us now calculate the angular spread, $\Delta\theta$ say, of the beam, defined by the formula

$$(\Delta\theta)^2 = \frac{\int_0^{\pi/2} \theta^2 [J(\theta)]^2 \mathrm{d}\theta}{\int_0^{\pi/2} [J(\theta)]^2 \mathrm{d}\theta} \cdot \tag{7}$$

Since $J(\theta)$ is sharply peaked around the direction $\theta = 0$, we may extend the range of integration from $(0, \pi/2)$ to $(0, \infty)$ in the integrals in Eq. (7), without introducing an appreciable error. On substituting for $J(\theta)$ from the expression (3), one readily finds that

$$\Delta\theta = \frac{1}{\sqrt{2}} \frac{1}{k\delta} \cdot \tag{8}$$

The effective width $\Delta\boldsymbol{\rho}$ of the intensity profile of the source, defined by the expression

$$(\Delta\boldsymbol{\rho})^2 = \frac{\int \boldsymbol{\rho}^2 [I^{(0)}(\boldsymbol{\rho})]^2 \mathrm{d}^2\rho}{\int [I^{(0)}(\boldsymbol{\rho})]^2 \mathrm{d}^2\rho}, \tag{9}$$

with $I^{(0)}(\boldsymbol{\rho})$ given by the formula (1) and with the integration extending over the whole source plane $z = 0$, is readily found to have the value

$$\Delta\boldsymbol{\rho} = \sigma_I . \tag{10}$$

It follows from Eqs. (8) and (10) that for all Gaussian Schell-model beams

$$(\Delta\theta)(\Delta\boldsymbol{\rho}) = \frac{\sigma_I}{k\delta\sqrt{2}} \tag{11}$$

or, more explicitly, if we substitute for δ from Eq. (5),

$$(\Delta\theta)(\Delta\boldsymbol{\rho}) = \frac{1}{k2\sqrt{2}} \left[1 + 4\left(\frac{\sigma_I}{\sigma_g}\right)^2\right]^{1/2} \cdot \tag{12}$$

Several interesting consequences follow from formula (12). First, we note that if the source is completely spatially coherent, i.e., when $\sigma_g \to \infty$, Eq. (12) gives

$$(\Delta\theta)_{\mathrm{coh}}(\Delta\boldsymbol{\rho}) = \frac{1}{k2\sqrt{2}}, \tag{13}$$

where $(\Delta\theta)_{\mathrm{coh}}$ denotes the angular spread of the coherent Gaussian Schell-model beam. On dividing Eq. (12) by Eq. (13), we obtain the result that

$$(\Delta\theta) = (\Delta\theta)_{\mathrm{coh}} \left[1 + 4\left(\frac{\sigma_I}{\sigma_g}\right)^2\right]^{1/2} \cdot \tag{14}$$

Since the factor multiplying $(\Delta\theta)_{\mathrm{coh}}$ on the right necessarily exceeds unity, it follows that

$$\Delta\theta > (\Delta\theta)_{\mathrm{coh}} \tag{15}$$

for all partially coherent Gaussian Schell-model beams. Stated in words, the inequality (15) asserts that *among all planar, secondary Gaussian Schell-model sources of the same effective width $\Delta\boldsymbol{\rho} \equiv \sigma_I$ of the intensity profile, the completely coherent one will generate the most directional beam*. As we already noted, the limiting, fully coherent case represents the lowest-order Hermite–Gaussian beam.

Next let us consider Gaussian Schell-model beams that have the same effective angular spread $\Delta\theta$ but are generated by sources with different effective widths $\Delta\rho$ of their intensity profiles. For the fully coherent case we have from Eq. (12), on taking the limit $\sigma_g \to \infty$ while

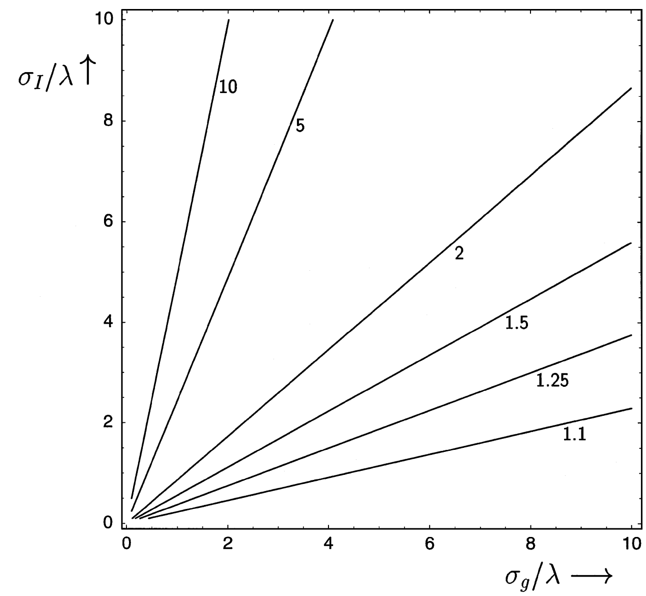

Fig. 2. Contours of the factor

$$\left[1 + 4\left(\frac{\sigma_I}{\sigma_g}\right)^2\right]^{1/2},$$

which represents the ratios $\Delta\theta/(\Delta\theta)_{\mathrm{coh}}$ and $\Delta\boldsymbol{\rho}/(\Delta\boldsymbol{\rho})_{\mathrm{coh}}$. [Eqs. (14) and (17)].

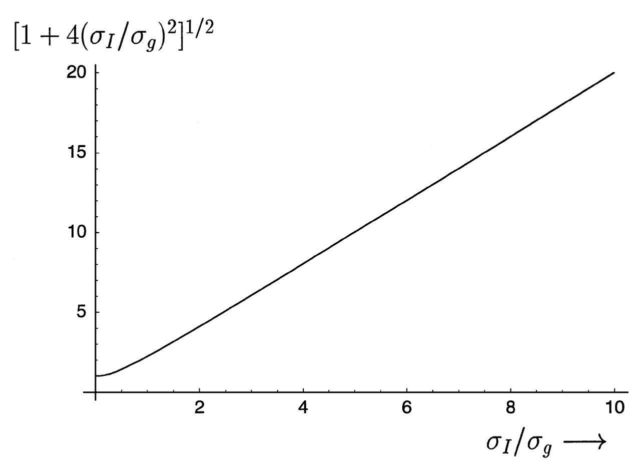

Fig. 3. The factor

$$\left[1 + 4\left(\frac{\sigma_I}{\sigma_g}\right)^2\right]^{1/2}$$

plotted as a function of the parameter σ_I/σ_g.

keeping $\Delta\theta$ fixed,

$$(\Delta\theta)(\Delta\boldsymbol{\rho})_{\mathrm{coh}} \equiv \frac{1}{k2\sqrt{2}} \,. \tag{16}$$

On dividing Eq. (12) by Eq. (16) we find that

$$(\Delta\boldsymbol{\rho}) = (\Delta\boldsymbol{\rho})_{\mathrm{coh}}\left[1 + 4\left(\frac{\sigma_I}{\sigma_g}\right)^2\right]^{1/2} \tag{17}$$

for all Gaussian Schell-model beams. This formula implies that *among all planar, secondary, Gaussian Schell–model sources which generate beams of the same angular spread* $\Delta\theta$, *the fully coherent one has the smallest effective size*. These results are in agreement with some computations presented in Refs. 2 and 5. Figures 2 and 3 show the behavior of the important factor $[1 + 4(\sigma_I/\sigma_g)^2]^{1/2}$ as function of σ_I, σ_g, and σ_I/σ_g.

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