A Recursively Enumerable Kripke Complete First-Order Logic Not Complete with Respect to a First-Order Definable Class of Frames

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Abstract

It is well-known that every quantified modal logic complete with respect to a first-order definable class of Kripke frames is recursively enumerable. Numerous examples are also known of "natural" quantified modal logics complete with respect to a class of frames defined by an essentially second-order condition which are not recursively enumerable. It is not, however, known if these examples are instances of a pattern, i.e., whether every recursively enumerable, Kripke complete quantified modal logic can be characterized by a first-order definable class of frames. While the question remains open for normal logics, we show that, in the context of quasi-normal logics, this is not so, by exhibiting an example of a recursively enumerable, Kripke complete quasi-normal logic that is not complete with respect to any first-order definable class of (pointed) frames.

Keywords: first-order modal logic, recursive enumerability, Kripke completeness, first-order definability

1 Introduction

Some important (first-order) quantified modal logics are based on propositional logics characterized by classes of frames defined by essentially second-order conditions on their accessibility relations. Among them are the quantified provability logics **QGL** (Quantified Gödel-Löb) and **QGrz** (Quantified Grzegorczyk), as well as their "linear" counterparts **QGL.3** and **QGrz.3**; quantified counterparts **QPDL**, **QCLT**, and $\mathbf{CTL}^*$ of propositional logics **PDL**, **CTL**, and $\mathbf{CTL}^*$; quantified epistemic logics with the common knowledge operator; and the quantified logic of finite Kripke frames.

A Kripke complete propositional modal logic can be extended to a (first-order) quantified one in essentially two ways. Given a propositional logic L complete with respect to a class of frames $\mathfrak{C}$, we can either consider the set of quantified formulas true on all frames from $\mathfrak{C}$, or alternatively, add to L, considered as a logical calculus, axioms and inference rules of the classical first-order logic. If class $\mathfrak{C}$ is defined by an essentially second-order condition, then in either case, we obtain quantified logics with undesirable properties. If we consider the set of quantified formulas true on all the frames of a propositional logic with essentially second-order Kripke semantics, we obtain logics that are not recursively enumerable, and thus cannot be represented as logical calculi—this holds, for example, for logics of frames with the condition of non-existence of infinite ascending chains, such as **QGL**, **QGrz**, **QGL.3**, and **QGrz.3** [4]; for logics of frames where one binary relation is the reflexive and transitive closure of another, such as **QPDL** and **QCLT** [7]; and for quantified logics of finite Kripke frames [5]. If, on the other hand, we consider extending such a propositional logic with classical first-order axioms and rules of inference, we obtain logics that are Kripke incomplete, i.e., are not complete with respect to any class of Kripke frames,—the proofs for **QGL**, **QGrz**, **QGL.3**, and **QGrz.3** can be found in [3], [4]; similar arguments apply to all the other logics mentioned above.

It would thus appear that quantified extensions of propositional logics with essentially second-order Kripke semantics are either Kripke incomplete or not recursively enumerable. In other words, Kripke completeness with respect to semantics with essentially second-order conditions and recursive enumerability do not seem to sit well together for quantified modal logics. Whether this is indeed so has, however, not been established. More precisely, it has not been established whether every recursively enumerable, Kripke complete quantified modal logic can be characterized by a class of frames defined by a classical first-order condition. We note here that the converse is known to be true, i.e., every quantified modal logic Kripke complete with respect to a class of frames defined by a classical first-order condition is recursively enumerable,—this is a straightforward consequence of the fact that such a logic can be embedded into the classical first-order logic through the so-called standard translation (see, e.g., [2]).

For normal logics, the above question still remains open. In the present paper, we show that for logics that are not required to be normal, i.e., closed under necessitation, the answer is negative,—there do exist quasi-normal quantified modal logics that are both recursively enumerable and Kripke complete, but are not complete with respect to any first-order definable class of (pointed) frames.

The paper is structured as follows. In section 2, we briefly introduce quantified modal logic and the associated Kripke semantics. In section 3, we present an example of a recursively enumerable, Kripke complete quasi-normal logic not complete with respect to any first-order definable class of pointed frames. We conclude in section 4.

2 Quantified modal logic

A (first-order) quantified modal language contains countably many individual variables; countably many predicate letters of every arity; Boolean connectives $\wedge$ and $\neg$; a modal connective $\Box$; and a quantifier $\forall$. Formulas as well as the symbols $\vee$, $\to$, $\exists$, and $\Diamond$ are defined in the usual way.

For every formula φ, we denote by $\mathbf{md}(\varphi)$ the modal depth of φ, which is defined inductively, as follows:

$$\begin{aligned}\mathbf{md}(P(y_1,\dots,y_n)) &= 0;\\ \mathbf{md}(\varphi_1 \wedge \varphi_2) &= \max\{\mathbf{md}(\varphi_1), \mathbf{md}(\varphi_2)\};\\ \mathbf{md}(\neg\varphi_1) &= \mathbf{md}(\varphi_1);\\ \mathbf{md}(\forall x\, \varphi_1) &= \mathbf{md}(\varphi_1);\\ \mathbf{md}(\Box\varphi_1) &= \mathbf{md}(\varphi_1) + 1.\end{aligned}$$

Modal formulas can be interpreted using Kripke semantics. A (Kripke) frame is a tuple $\mathfrak{F} = \langle W, R\rangle$, where W is a non-empty set (of worlds) and R is a binary (accessibility) relation on W. A predicate (Kripke) frame is a tuple $\mathfrak{F}_D = \langle W, R, D\rangle$, where $\langle W, R\rangle$ is a frame and D is a function from W into a set of non-empty subsets of some set (the domain of $\mathfrak{F}_D$), satisfying the condition that wRw' implies $D(w) \subseteq D(w')$. We call the set $D(w)$ the domain of w. If a predicate frame satisfies the condition that wRw' implies $D(w) = D(w')$, we refer to it as a frame with constant domains.

A (Kripke) model is a tuple $\mathfrak{M} = \langle W, R, D, I\rangle$, where $\langle W, R, D\rangle$ is a predicate Kripke frame and I is a function assigning to a world $w \in W$ and an n-ary predicate letter P an n-ary relation $I(w, P)$ on $D(w)$. We refer to I as the interpretation of predicate letters with respect to worlds in W.

An assignment in a model is a function g associating with every individual variable y an element of the domain of the underlying frame.

The truth of a formula φ at a world w of a model $\mathfrak{M}$ under an assignment g is inductively defined as follows:

- $\mathfrak{M}, w \models^g P(y_1,\dots,y_n)$ if $\langle g(y_1),\dots,g(y_n)\rangle \in I(w, P)$;
- $\mathfrak{M}, w \models^g \varphi_1 \wedge \varphi_2$ if $\mathfrak{M}, w \models^g \varphi_1$ and $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \neg\varphi_1$ if $\mathfrak{M}, w \not\models^g \varphi_1$;
- $\mathfrak{M}, w \models^g \Box\varphi_1$ if wRw' implies $\mathfrak{M}, w' \models^g \varphi_1$, for every $w' \in W$;
- $\mathfrak{M}, w \models^g \forall y\, \varphi_1$ if $\mathfrak{M}, w \models^{g'} \varphi_1$, for every assignment g' such that g' differs from g in at most the value of y and such that $g'(y) \in D(w)$.

Note that, given a Kripke model $\mathfrak{M} = \langle W, R, D, I\rangle$ and $w \in W$, the tuple $\mathfrak{M}_w = \langle D_w, I_w\rangle$, where $D_w = D(w)$ and $I_w(P) = I(w, P)$, is a classical predicate model.

We say that φ is true at world w of model $\mathfrak{M}$ and write $\mathfrak{M}, w \models \varphi$ if $\mathfrak{M}, w \models^g \varphi$ holds for every g assigning to free variables of φ elements of $D(w)$. We say that φ is true in $\mathfrak{M}$ and write $\mathfrak{M} \models \varphi$ if $\mathfrak{M}, w \models \varphi$ holds for every world w of $\mathfrak{M}$. We say that φ is true on predicate frame $\mathfrak{F}_D$ and write $\mathfrak{F}_D \models \varphi$ if φ is true in every model based on $\mathfrak{F}_D$. We say that φ is true on frame $\mathfrak{F}$ and write $\mathfrak{F} \models \varphi$ if φ is true on every predicate frame of the form $\mathfrak{F}_D$. Finally, we say that a formula is true on a class of frames if it is true on every frame from the class.

Let $\mathfrak{M} = \langle W, R, D, I\rangle$ be a model, $w \in W$, and $a_1, \ldots, a_n \in D(w)$. Let $\varphi(y_1, \ldots, y_n)$ be a formula whose free variables are among $y_1, \ldots, y_n$. We write $\mathfrak{M}, w \models \varphi[a_1, \ldots, a_n]$ to mean $\mathfrak{M}, w \models^g \varphi(y_1, \ldots, y_n)$, where $g(y_1) = a_1, \ldots, g(y_n) = a_n$.

Sometimes, semantics based on pointed frames, rather than frames, is useful. A pointed Kripke frame is a tuple $(\mathfrak{F}, w_0)$, where $\mathfrak{F} = \langle W, R\rangle$ is a Kripke frame and $w_0 \in W$ is a distinguished world. A formula φ is true on a pointed frame $(\mathfrak{F}, w_0)$ if it is true at w_0 in every model based on $\mathfrak{F}$.

A (first-order) quantified (quasi-normal) modal logic is a set L of formulas containing the validities of the classical first-order logic as well as the formula $\Box(p \to q) \to (\Box p \to \Box q)$, and closed under predicate substitution, modus ponens, and generalization (if $\varphi \in L$, then $\forall x\, \varphi \in L$). A normal modal logic is a quasi-normal modal logic L that is closed under necessitation (if $\varphi \in L$, then $\Box\varphi \in L$).

If $\mathfrak{C}$ is a class of frames, then the set of formulas true on every frame in $\mathfrak{C}$ is denoted by $L(\mathfrak{C})$. If $\mathfrak{C}$ is a class of pointed frames, then the set of formulas true at the distinguished world of every frame in $\mathfrak{C}$ is denoted by $rL(\mathfrak{C})$. If $\mathfrak{C}$ is a class of frames, then $L(\mathfrak{C})$ is a normal modal logic; if $\mathfrak{C}$ is a class of pointed frames, then $rL(\mathfrak{C})$ is a quasi-normal modal logic.

A quasi-normal logic is sound and complete with respect to a class of pointed frames $\mathfrak{C}$ if $L = rL(\mathfrak{C})$ for some class of pointed Kripke frames. We say that a quasi-normal logic is Kripke complete if it is sound and complete with some class of pointed frames. Analogously for normal logics and Kripke frames.

A class $\mathfrak{C}$ of (pointed) frames is first-order definable if there exists a first-order formula φ (for frames, φ contains no free variables; for pointed frames, a single free variable), containing binary predicate letters R and $=$ (and no other predicate letters), such that $\mathfrak{F} \in \mathfrak{C}$ if, and only if, φ is true in $\mathfrak{F}$ considered as a classical model (for pointed frames, the free variable of φ is interpreted as the distinguished world).

Remark 2.1 If a class $\mathfrak{C}$ of pointed Kripke frames is first-order definable, say, by formula $\varphi(x)$, then $\mathfrak{C}$ considered as a class of frames—i.e., disregarding the roots of the frames—is also first-order definable, namely by the formula $\exists x\, \varphi(x)$.

The following proposition is well-known (see, for example, [2], Proposition 3.12.8).

Proposition 2.2 *Let $\mathfrak{C}$ be a first-order definable class of frames. Then, $L(\mathfrak{C})$ is recursively enumerable.*

A similar proposition holds for pointed frames.

Proposition 2.3 *Let $\mathfrak{C}$ be a first-order definable class of pointed frames. Then, $rL(\mathfrak{C})$ is recursively enumerable.*

In the strict sense, the converses of Proposition 2.2 and Proposition 2.3 are known not to be true, as some recursively enumerable logics are not complete with respect to any class of frames (i.e., are Kripke incomplete), and thus, not complete with respect to any first-order definable class. Examples for normal logics have been mentioned in the Introduction. An example of a Kripke incomplete quasi-normal logic that is not normal is the quantified counterpart of Solovay's logic (see, e.g., [1]), which is obtained from the syntactically defined **QGL** by adding the axiom $\Box p \to p$ and removing the requirement of closure under necessitation. A more interesting question, therefore, as noted above, is whether every recursively enumerable Kripke complete logic is a logic of a first-order definable class of frames. The main contribution of this paper is to show, which we do in the next section, that this is not so for quasi-normal logics—namely, we exhibit an example of a recursively enumerable Kripke complete quasi-normal logic that is not complete with respect to any first-order definable class of pointed frames.

3 Construction of the main counterexample

In this section, we present an example of a quasi-normal quantified modal logic L_0 that is recursively enumerable, Kripke complete, but not complete with respect to any first-order definable class of frames. The logic L_0 is defined as the set of formulas true at the distinguished world of the following pointed Kripke frame $\mathfrak{F}$. Let

$$\begin{array}{rcl} W_0 & = & \{w_0^0\}; \\ W_{k+1} & = & W_k \cup \{w_0^{k+1}, \ldots, w_{k+1}^{k+1}\}. \end{array}$$

For every $n \in \mathbb{N}$, let R_n be a binary relation on W_n such that, for every $w_m^k, w_s^t \in W_n$,

$$w_m^k R_n w_s^t \iff t = k+1 \text{ and } m = 0.$$

Let $\mathfrak{F}_n$ denote the frame $\langle W_n, R_n \rangle$. Finally, let $\mathfrak{F} = \langle W, R \rangle$, where

$$W = \bigcup_{i=0}^{\infty} W_i, \quad R = \bigcup_{i=0}^{\infty} R_i.$$

The frame $\mathfrak{F}$ is depicted in Fig. 1. We define L_0 to be the set of formulas that are true at w_0^0 in $\mathfrak{F}$; thus, L_0 is a quasi-normal modal logic. By definition, L_0 is complete with respect to a class of pointed Kripke frames (namely, the class containing a single frame, $\mathfrak{F}$). We next show that L_0 is recursively enumerable and not complete with respect to any first-order definable class of frames.

To show that L_0 is recursively enumerable, we effectively embed it into the classical first-order logic with equality **QClE**.

First, notice that, since $\mathfrak{F}_n$ is a finite frame, we can effectively construct a classical first-order formula F_n describing $\mathfrak{F}_n$,—all we need to do is say what worlds exist in $\mathfrak{F}_n$, that those worlds are pairwise distinct, that there are no other worlds in $\mathfrak{F}_n$, and describe which worlds are related by the accessibility relation.

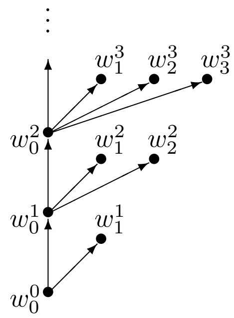

Fig. 1. The pointed frame $\mathfrak{F}$

Now, let R and D be binary, and W unary, predicate letters not occurring in φ; intuitively, $W(x)$ means "x is a world", $D(x, y)$ means "y is an element of the domain of world x", and $R(x, y)$ means "y is accessible from x". Note that w_0^0 is the only world in $\mathfrak{F}_n$ that satisfies the property $Root(x)$, defined as follows:

$$Root(x) \;=\; \forall y\, \neg R(y, x).$$

Let $ST_x(\varphi)$ be the standard translation of the formula φ into classical first-order logic, defined as follows:

$$\begin{aligned}
ST_x(P(y_1, \ldots, y_m)) &= P'(y_1, \ldots, y_m, x);\\
ST_x(\varphi_1 \wedge \varphi_2) &= ST_x(\varphi_1) \wedge ST_x(\varphi_2);\\
ST_x(\neg\varphi_1) &= \neg ST_x(\varphi_1);\\
ST_x(\Box\varphi_1) &= \forall y\, (W(y) \wedge R(x, y) \to ST_y(\varphi_1));\\
ST_x(\forall y\, \varphi_1) &= \forall y\, (\neg W(y) \wedge D(x, y) \to ST_x(\varphi_1)),
\end{aligned}$$

where the arity of P' is one greater than P, letter P' is distinct from letter Q' if, and only if, P is distinct from Q, and all the newly introduced individual variables are distinct from the previously used ones. Let M be the formula

$$\begin{aligned}
M \;=\; & \exists x\, W(x) \wedge \forall x\, [W(x) \to \exists y\, D(x, y)] \wedge\\
& \wedge \forall x \forall y \forall z\, [W(x) \wedge W(y) \wedge \neg W(z) \wedge D(x, z) \wedge R(x, y) \to D(y, z)].
\end{aligned}$$

Intuitively, M describes general properties of predicate Kripke frames.

Lastly, for an arbitrary classical first-order formula with equality θ, inductively define the formula θ^* as follows:

$$\begin{aligned}
(x = y)^* &= (x = y);\\
(R(x, y))^* &= R(x, y);\\
(\theta_1 \wedge \theta_2)^* &= \theta_1^* \wedge \theta_2^*;\\
(\neg\theta_1)^* &= \neg\theta_1^*;\\
(\forall x\, \theta_1)^* &= \forall x\, (W(x) \to \theta_1^*).
\end{aligned}$$

Lemma 3.1 *For every closed modal formula φ with $\mathbf{md}(\varphi) = n$, the following holds:*

$$(\mathfrak{F}_n, w_0^0) \models \varphi \iff M \wedge F_n^* \to \forall x\, [W(x) \wedge Root(x) \to ST_x(\varphi)] \in \mathbf{QClE}.$$

Proof. It is well-known (see, e.g., [2]) that the standard translation has the property that $\mathfrak{M}, w \models \varphi$ if, and only if, $\mathfrak{M} \models ST_x(\varphi)[w]$. Thus, if $(\mathfrak{F}_n, w)$ is a (finite) pointed frame, then $(\mathfrak{F}_n, w) \models \varphi$ if, and only if, $\mathfrak{M} \models ST_x(\varphi)[w]$ holds for all first-order models "based on" $\mathfrak{F}_n$ (i.e., their domain and the interpretation of the binary relation R coincides with the set of worlds and the accessibility relation, respectively, of $\mathfrak{F}_n$). In turn, the latter holds if, and only if, the formula $M \wedge F_n^* \to \forall x\,[W(x) \wedge Root(x) \to ST_x(\varphi)]$, which claims that $ST_x(\varphi)$ holds provided we evaluate it in a model that looks like $\mathfrak{F}_n$ with x assigned to the "root" node, is valid. □

Proposition 3.2 *L_0 is recursively enumerable.*

Proof. As, for a modal formula φ with $\mathbf{md}(\varphi) = n$,

$$(\mathfrak{F}, w_0^0) \models \varphi \iff (\mathfrak{F}_n, w_0^0) \models \varphi,$$

it immediately follows from Lemma 3.1 that, given an arbitrary $n \in \mathbb{N}$, the set of theorems of L_0 with modal depth n is recursively enumerable. Thus, using the standard technique from recursion theory, we can recursively enumerate all the theorems of L_0. □

It remains to show that L_0 is not complete with respect to any first-order definable class of pointed Kripke frames. To prove this, we need an auxiliary Lemma, whose statement is a slight modification of a result from [5].

Lemma 3.3 *Let L be a normal modal logic that is sound and complete with respect to a class $\mathfrak{C}$ of frames that satisfies the following conditions:*

(i) *if $\mathfrak{F} \in \mathfrak{C}$, then every world in $\mathfrak{F}$ can see only finitely many worlds;*

(ii) *for every $n \in \mathbb{N}$, there exist $\mathfrak{F} = \langle W, R\rangle$ in $\mathfrak{C}$ and $w \in W$ such that w can see at least n worlds.*

Then, L is not recursively enumerable.

Proof. Let φ be an arbitrary classical first-order formula and let T be a unary, and E a binary, predicate letter not occurring in φ. Let $Congr$ be a formula saying that E is a congruence relation with respect to all predicate letters in φ and let

$$A = \forall x \Diamond T(x) \wedge \forall x \forall y\,(\Diamond(T(x) \wedge T(y)) \to E(x, y)).$$

Let **QClFin** be the classical first-order logic of finite models and let $\varphi^* = (Congr \wedge A) \to \varphi$. We can then show that

$$\varphi \in \mathbf{QClFin} \iff \varphi^* \in L.$$

Indeed, assume that $\varphi \notin \mathbf{QClFin}$; that is, there exists a classical model $\mathfrak{M}$ with a finite domain $D = \{a_1, \ldots, a_n\}$ such that $\mathfrak{M} \not\models \varphi$. We construct a model $\mathfrak{M}^*$, based on a frame from $\mathfrak{C}$, falsifying φ^*. By assumption, $\mathfrak{C}$ contains a frame $\mathfrak{F} = \langle W, R\rangle$ such that some $w_0 \in W$ can see at least n worlds; select exactly n out of those and bijectively map them to the elements of D; let the world $\bar{a}_i$ correspond to element a_i, where $i \in \{1, \ldots, n\}$. Let $D^*(w) = D$ for every $w \in W$. Let $\mathfrak{M}, w \models T[a_i]$ if, and only if, $w = \bar{a}_i$, for $i \in \{1, \ldots, n\}$. Let $I^*(w, E)$, for every $w \in W$, be the identity relation and let all the predicate letters in φ be defined in every $w \in W$ exactly as they are defined in $\mathfrak{M}$. It is then easy to check that $\mathfrak{M}^*, w_0 \not\models \varphi^*$. As $\mathfrak{M}^*$ is based on a frame from $\mathfrak{C}$, we conclude that $\varphi^* \notin L$.

Assume, on the other hand, that $\varphi^* \notin L$; that is, there exists a frame $\mathfrak{F} = \langle W, R\rangle$ in $\mathfrak{C}$, $w_0 \in W$, and a model $\mathfrak{M}^*$ based on $\mathfrak{F}$ such that $\mathfrak{M}^*, w_0 \not\models \varphi^*$. We construct a finite classical model $\mathfrak{M}$ falsifying φ. By assumption, w_0 can see only finitely many worlds, say $w_1, \ldots, w_n$. As $\mathfrak{M}^*, w_0 \models Congr \wedge A$, for every $b \in D^*(w_0)$, we have $w' \models T[b]$ for at least one w' accessible from w_0, and for every w accessible from w_0, T holds for the elements of only one equivalence class with respect to E. As w_0 can see only finitely many worlds, $D^*(w_0)$ contains only finitely many equivalence classes $a_1, \ldots, a_n$ with respect to E. Let $D = \{a_1, \ldots, a_n\}$ and let $I(P) = I^*(w_0, P)$ for every predicate letter P occurring in φ. Let $\mathfrak{M} = \langle D, I\rangle$. As $\mathfrak{M} \not\models \varphi$ and D is finite, $\varphi \notin$ **QClFin**.

As **QClFin** is not recursively enumerable [6], the statement of the Lemma follows.

□

Corollary 3.4 *Let $\mathfrak{C}$ be a class of frames satisfying the following conditions:*

(i) *if $\mathfrak{F} \in \mathfrak{C}$, then every world in $\mathfrak{F}$ can see only finitely many worlds;*

(ii) *for every $n \in \mathbb{N}$, there exist $\mathfrak{F} = \langle W, R\rangle$ in $\mathfrak{C}$ and $w \in W$ such that w can see at least n worlds.*

Then, $\mathfrak{C}$ is not first-order definable.

Proof. Immediately follows from Lemma 3.3 and Proposition 2.2. □

Proposition 3.5 *Let $\mathfrak{C}$ be a class of pointed frames such that L_0 is sound and complete with respect to $\mathfrak{C}$. Then, $\mathfrak{C}$ is not first-order definable.*

Proof. Let $rL(\mathfrak{C})$ be the set of formulas true at the distinguished world of every frame in $\mathfrak{C}$. By the statement of the proposition, $rL(\mathfrak{C}) = L_0$. For every $i, n \in \mathbb{N}^+$ such that $i \leqslant n$, let

$$\alpha_n^i = \Diamond(p_1 \wedge \ldots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \ldots \wedge p_n).$$

For every $n \in \mathbb{N}$, let

$$\beta_n = \Box^n \neg(\alpha_{n+3}^1 \wedge \ldots \wedge \alpha_{n+3}^{n+3}).$$

As from every world w_k^n of $\mathfrak{F}$, we can reach either 0 (if $k > 0$) or $n+2$ (if $k = 0$) worlds, we have $(\mathfrak{F}, w_0^0) \models \beta_n$ and, thus, $\beta_n \in L_0$. As $rL(\mathfrak{C}) = L_0$, for every pointed frame $(\mathfrak{F}', w) \in \mathfrak{C}$, we have $(\mathfrak{F}', w) \models \beta_n$. Therefore, each world in $\mathfrak{F}'$ reachable from w in n steps can see no more than $n+2$ worlds. As $\mathfrak{F}'$ is a pointed frame with distinguished world w, every world in $\mathfrak{F}'$ can thus see only finitely many worlds, and hence $\mathfrak{C}$ satisfies the first condition of Corollary 3.4.

On the other hand, $(\mathfrak{F}, w_0^0) \not\models \Box\beta_n$; hence, $\Box\beta_n \notin L_0$, and thus $\mathfrak{C}$ contains a pointed frame $(\mathfrak{F}', w)$ such that $(\mathfrak{F}', w) \not\models \Box\beta_n$. Therefore, $\mathfrak{F}'$ contains a world

w' that can see at least $n+3$ worlds, and hence $\mathfrak{C}$ satisfies the second condition of Corollary 3.4.

Thus, in view of Corollary 3.4, $\mathfrak{C}$ considered as a class of frames is not first-order definable. Then, in view of Remark 2.1, $\mathfrak{C}$ considered as a class of pointed frames is not first-order definable, either, which concludes the proof. $\Box$

4 Discussion

We have exhibited an example of a quasi-normal quantified modal logic L_0 such that (1) L_0 is recursively enumerable, (2) L_0 is Kripke complete, and (3) L_0 is not complete with respect to any class of pointed frames defined by a classical first-order condition. The logic L_0 was defined as the logic of a particular pointed Kripke frame, $\mathfrak{F}$; it is not, however, an isolated example. Recall that the frame $\mathfrak{F}$ is a tree where, for every $n \in \mathbb{N}$, the nth level contains $n+1$ worlds $w_0^n, \ldots, w_{n+1}^n$ and where the world w_0^n can see all the worlds on level $n+1$. This construction can be generalized to use an arbitrary computable function f not bound above by any $n \in \mathbb{N}$ so that the nth level of the tree contains $f(n)$ worlds, thus giving us countably many quasi-normal quantified logics satisfying properties (1) through (3). We could also work with logics whose Kripke semantics involves constant, rather than varying, domains.

The most important question for future research remains the one posed at the beginning of the present paper—whether there exists a normal quantified modal logic satisfying properties (1) through (3). It is not clear whether the technique used in the present paper is transferable to normal logics.

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