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A reduction of order two for infinite-order Lagrangians

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Given a Lagrangian system depending on the position derivatives of any order, and assuming that certain conditions are satisfied, a second-order differential system is obtained such that its solutions also satisfy the Euler equations derived from the original Lagrangian. A generalization of the singular Lagrangian formalism permits a reduction of order keeping the canonical formalism in sight. Finally, the general results obtained in the first part of the paper are applied to Wheeler-Feynman electrodynamics for two charged point particles up to order $1/c^4$.

I. INTRODUCTION

There are in the literature several attempts to derive a Lagrangian function—or a Hamiltonian—for action-at-a-distance electrodynamics, including relativistic corrections up to a certain order of approximation. Historically, the first of these attempts is due to Darwin,[1] who obtained the first-order relativistic corrections to a pure Coulomb Lagrangian (i.e., one containing terms up to $1/c^2$).

Almost 40 years later, Golubenkov and Smorodinski[2] pushed Darwin's method further, and obtained a Lagrangian function for the second post-Coulomb approximation (i.e., containing all terms up to $1/c^4$). Their study was restricted to the special case of two equal charges because, in that case, radiation effects were not expected to contribute until the next order of approximation, namely, $1/c^5$. However, the Lagrangian they obtained was acceleration dependent. But, since accelerations only occurred in the highest-order terms, they substituted them by means of the Coulomb force law—the lowest order in the approximation method they were working in—so obtaining a final Lagrangian function which only depended on positions and velocities.

However, as has been known for many years, the Euler equations derived from the latter Lagrangian neither reproduce the correct equations of motion for two equal charges in classical electrodynamics nor are relativistic invariant up to the order of approximation considered. It is also well known that these objections do not apply to the Darwin Lagrangian. It is therefore apparent that these troubles genuinely arise in the second post-Coulomb approximation. This has become clear, with the no-interaction theorems[3,4] for approximated Lagrangians.

It seems that the failure of the Lagrangian function in Ref. 2 comes from the fact that the accelerations have been removed by substituting the Coulomb law straight into the Lagrangian. Although there is no objection to this substitution from the viewpoint of the approximation scheme used, it actually breaks down some basic assumptions of the Lagrangian formalism. Indeed, substitution implies the introduction of some constraints that modify the very nature of the variational principle underlying the Lagrangian formalism, so converting it into a "conditioned" variational principle. The usual Euler method to derive the equations of motion no longer holds after the constraints have been substituted, and some Lagrange multipliers, giving account of the constraints, would have to be introduced.

It is also worthwhile to mention here an attempt by Kerner[5] to obtain a Hamiltonian formalism for Wheeler-Feynman electrodynamics.[6] Although the final aim is not achieved in that paper, it contains several remarkable results. One of them is the "translation" of the path-dependent Lagrangian of Wheeler and Feynman into an infinite-order Lagrangian, i.e., one depending on the positions and their derivatives of any order. A Hamiltonian is then derived by merely using a generalization of Ostrogradski transformation.[7] Nevertheless, this Hamiltonian turns out to be useless for practical purposes because of the infinitely many dimensions of the phase space. Indeed, to determine the future evolution of the system, infinitely many initial data are needed. Moreover, canonical quantization would be meaningless, as it would involve wave functions depending not only on position, but on the higher-order derivatives also.

This drawback was not new at that time; other dynamical systems were known whose description involved an infinite number of degrees of freedom (e.g., purely retarded electrodynamics, and Wheeler-Feynman electrodynamics itself). The way out proposed by Kerner is similar to one pointed out by Bhabha:[8] namely, the requirement must be added that the motions of particles become free when the coupling constant goes to zero.

Within the too-broad solution of the infinite-order system, that apparently innocuous condition picks out a family of motions that only depends on the initial position and velocities, being thus ascribable to second-order differential systems.

A similar result also holds if, on one hand, the inverse speed of light is taken instead of the coupling constant, and, on the other, the motion derived from the Coulomb law is substituted for free motions.

In the infinite-order Lagrangian obtained by Kerner[5] for Wheeler-Feynman electrodynamics written as a power series of $1/c$, it clearly appears that accelerations do not occur in terms lower than $1/c^4$, and, generally, that the nth derivatives of coordinates come multiplied by $1/c^{2n}$ at least. (Odd-order terms vanish for Wheeler-Feynman electrodynamics and other time-symmetric interactions.)

This general rule, which also holds for the infinite-order Lagrangians obtained in Ref. 9 from a wide family of Fokker-type action principles, will be one of the cornerstones of the methods developed in the present work.

From a mathematical viewpoint, an infinite-order Lagrangian is a rather obscure object, especially because such things as the convergence of the series involved is not well established at all. However, as far as we can understand from the literature, the expansions in powers of some ϵ (either $1/c$ or the coupling constant) have only an algebraic meaning in the kind of problems we are considering. Topological issues, such as convergence of the infinite sums, are usually left aside at this level.

Therefore, in order to avoid these obscurities, an infinite-order Lagrangian will be replaced by a hierarchy of Lagrangians "approximated to (ϵ^{n+1})," which we shall write as $L_n+O(\epsilon^{n+1})$, $n\in N$, each L_n being a polynomial of degree n in the variable ϵ, its coefficients being some functions of coordinates, velocities, and higher-order derivatives. These L_n, $n\in N$ are said to form a hierarchy because any two contiguous terms are related by $L_{n+1}=L_n+O(\epsilon^{n+1})$; i.e., they are equal modulo terms ϵ^{n+1}. Hereafter, "approximated to order ϵ^{n+1}" will mean "equal modulo ϵ^{n+1}," that is, equal, provided that terms multiplied by ϵ^r, $r\geq n+1$, are neglected. The word "approximated" will not presume that some difference or error is small. It is commonly used in the literature with precisely the meaning stated above.

Since n is arbitrary, this hierarchical approach allows us to deal with the same amount of information as the infinite-order approach. Furthermore, advantage is gained in two senses. First, we can avoid the unpleasant feature that, in the infinite order formalism, the configuration space and the phase space are actually isomorphic. And second, some algebraic properties of the "equality modulo ϵ^{n+1}" will enable us to cast the order-reduction process onto the singular Lagrangian formalism.[10]

The latter will be most significant, since it will permit reduction of the order of the equations of motion (e.g., substitute the Coulomb law for the accelerations) without "forgetting" the Hamiltonian structure underlying the Lagrangian formalism we have started from. The canonical structure will be supplied by some Dirac brackets[10,11] which we shall define according to the constraints involved in the order-reduction process.

The present paper is organized as follows. Section II is devoted to some general points concerning higher-order Lagrangian systems: all those results can be found in Ref. 7, but, for the sake of notation, we spend some space in summarizing them. In Sec. III a special set of Lagrangians approximated to order ϵ^{n+1} is considered. Although these Lagrangians are of order n, they are also singular, and the Lagrangian primary and secondary constraints enable us to remove all derivatives higher than accelerations from the equations of motion. The Hamiltonian counterpart of this development provides a canonical structure for the reduced equations of motion (the second-order ones), which is given by the corresponding Dirac brackets.

II. SOME GENERAL RESULTS CONCERNING LAGRANGIAN SYSTEMS OF ANY ORDER

A. Euler equations

A Lagrangian function of order n is one depending on the coordinates and their derivatives up to nth order. It can be written as

$$L=L(t;q_\alpha,q_\beta^{(1)},\ldots,q_\nu^{(n)}),\quad \alpha,\beta,\nu,\ldots=1,\ldots,N\ ,$$

where $q_\alpha^{(s)}$, $s\geq 1$ is the sth time derivative of q_α. Occasionally, $q_\alpha^{(0)}$ means q_α.

Associated to this Lagrangian function, we have the action principle

$$\delta\int_{t_1,P}^{t_2,Q} L(t;q_\alpha,q_\beta^{(1)},\ldots,q_\nu^{(n)})dt \tag{1}$$

which is posed in the configuration space, spanned by $q_\alpha\cdots q_\beta^{(n-1)}$, to which the points P and Q belong.

The solutions of the variational problem (1) are also the solutions of Euler equations

$$\mathscr{L}_\alpha[L]\equiv\sum_{k=0}^{n}(-1)^kD^k\left[\frac{\partial L}{\partial q_\alpha^{(k)}}\right]=0\ , \tag{2}$$

where D is the total time derivative, that is,

$$D\equiv\frac{\partial}{\partial t}+\sum_{r=0}^{2n-1}q_\sigma^{(r+1)}\frac{\partial}{\partial q_\sigma^{(r)}}\ . \tag{3}$$

Thus, Eq. (2) is a $2n$th-order differential system, and all derivatives of $2n$th order can be expressed in terms of those of lower order if, and only if, the Hessian matrix $H_{\alpha\beta}\equiv\partial^2L/\partial q_\alpha^{(n)}\partial q_\beta^{(n)}$ is regular, i.e., has rank N.

B. Hamiltonian formalism

The Hamiltonian formalism is made up from the Lagrangian by means of the Ostrogradski transformation. The conjugate momentum $\Pi_{j\alpha}$, corresponding to the configuration space variable $q_\alpha^{(j)}$, is defined by

$$\Pi_{j\alpha}\equiv\sum_{l=0}^{n-j-1}(-D)^l\frac{\partial L}{\partial q_\alpha^{(l+j+1)}},\quad j=0,1,\ldots,n-1\ ,\qquad \alpha=1,\ldots,N\ . \tag{4}$$

A glance at the dependence of $\Pi_{j\alpha}$ on the higher-order derivatives shows that it depends on $q_\beta^{(2n-j-1)}$, at most, and in a very simple way. Indeed, we have that

$$\begin{aligned}\Pi_{j\alpha}&=\Pi_{j\alpha}(q,q^{(1)},\ldots,q^{(2n-j-1)};t)\\&=q_\beta^{(2n-j-1)}H_{\alpha\beta}+K_{j\alpha}(q,\ldots,q^{(2n-j-2)};t)\ .\end{aligned} \tag{5}$$

Hence, the Ostrogradski transformation can be inverted if, and only if, the Hessian matrix $H_{\alpha\beta}$ is regular. In such a case, the actual inversion of the transformation (5) implies a finite iterative procedure. Indeed, starting with $j=n-1$, one obtains $q^{(n)}(q,q^{(1)},\ldots,q^{(n-1)},\Pi_{n-1};t)$;

then substitutes it into the expression for $j=n-2$, one obtains $q^{(n+1)}(q,\ldots,q^{(n-1)};\Pi_{n-1},\Pi_{n-2};t)$, and so on.

As usual, the phase space is spanned by the configuration-space variables $q_\alpha, q_\beta^{(1)},\ldots,q_\rho^{(n-1)}$, plus their conjugate momenta $\Pi_{c\alpha},\Pi_{1\beta},\ldots,\Pi_{n-1\rho}$. In terms of these variables, the time evolution is ruled by the following first-order ordinary differential system on $2nN$ variables:

$$\frac{dq_\alpha^{(k)}}{dt}=q_\alpha^{(k+1)},\quad k=0,\ldots,n-2 \tag{6a}$$

$$\frac{dq_\alpha^{(n-1)}}{dt}=q_\alpha^{(n)}(q,\ldots,q^{(n-1)};\Pi_{n-1};t)\ , \tag{6b}$$

$$\frac{d\Pi_{j\alpha}}{dt}=-\Pi_{j-1\alpha}+\frac{\partial L}{\partial q_\alpha^{(j)}}(t;q,\ldots,q^{(n)})\ ,\quad j=0,\ldots,n-1\ , \tag{6c}$$

where the right-hand side of (6b) is obtained by inverting the Ostrogradski transformation, and (6c) is obtained by differentiating Eq. (4). It must be understood also that, in Eq. (6c) $\Pi_{-1\alpha}$ means zero (indeed, if we take $j=-1$ in Eq. (4), the right-hand side turns out to be $\mathscr{L}_\alpha[L]$, which vanishes for the solutions of the dynamical system).

As can be easily proven, Eqs. (6) can be derived from the Hamiltonian function:

$$H=-L(t;q,q^{(1)},\ldots,q^{(n-1)},q^{(n)}(q,\ldots,q^{(n-1)};\Pi_{n-1};t))+\sum_{j=0}^{n-2}\Pi_{j\alpha}q_\alpha^{(j+1)}+\Pi_{n-1\alpha}q_\alpha^{(n)}(q,\ldots,q^{(n-1)};\Pi_{n-1};t)\ . \tag{7}$$

Sum over repeated greek indices will be always understood.

The elementary Poisson brackets corresponding to this Hamiltonian formalism are

$$\{q_\alpha^{(k)},\Pi_{j\beta}\}=\delta_{\alpha\beta}\delta_j^k,\quad \{q_\alpha^{(k)},q_\beta^{(j)}\}=\{\Pi_{k\alpha},\Pi_{j\beta}\}=0\ , \tag{8}$$

$k,j=0,1,\ldots,n-1$; $\alpha,\beta=1,\ldots,N$; the Hamilton equation of motion can be written as

$$\frac{df}{dt}=\{f,H\}+\frac{\partial f}{\partial t} \tag{9}$$

where $f(t;q,q^{(1)},\ldots,q^{(n-1)};\Pi_0,\ldots,\Pi_{n-1})$ is any function on the extended phase space.

III. LAGRANGIAN SYSTEMS APPROXIMATED TO $O(\epsilon^{n+1})$

We deal hereafter with certain Lagrangian functions that depend polynomially on a given parameter ϵ, and can be written as

$$L=\tfrac{1}{2}\sum_{\alpha=1}^{N}m_\alpha(q_\alpha^{(1)})^2+\sum_{s=0}^{n}\epsilon^s V_s(q,q^{(1)},\ldots,q^{(s)})+O(\epsilon^{n+1})\ , \tag{10}$$

where m_α, $\alpha=1,\ldots,N$, are some given constants standing for different "masses" associated with the several degrees of freedom. We shall also assume that the matrices $\partial^2 V_s/\partial q_\alpha^{(s)}\partial_\beta^{(s)}$ are regular.

In what follows, terms containing powers higher than ϵ^n will be neglected, and put into the abbreviated form $O(\epsilon^{n+1})$ which we shall write on the right-hand side of our equations. To state the latter in a more algebraical language, the domain of our fundamental quantities will not be the field $\mathbf{R}$ of real numbers, but the quotient ring $\mathbf{R}[\epsilon]/(\epsilon^{n+1})$ (i.e., polynomials in one variable ϵ with real coefficients, modulo ϵ^{n+1}). As a consequence of this assumption, those solutions of the equations of motion that are ill behaved in the limit $\epsilon\to 0$ are excluded from the very beginning. Indeed, according to the choice of our fundamental quantities, the solution corresponding to a given set of initial data (ID) is assumed to be shaped according to the general form

$$q_\alpha(t;\mathrm{ID},\epsilon)=\sum_{s=0}^{n}\epsilon^s q_{\alpha s}(t;\mathrm{ID})+O(\epsilon^{n+1})\ .$$

As another consequence of this choice, we must keep in mind that $\mathbb{R}[\epsilon]/(\epsilon^{n+1})$ is a ring containing divisors of zero (e.g., any multiple of ϵ), and that these elements are not invertible. Such a simple fact will enable us to cast the order-reduction process into the singular Lagrangian formalism.[10]

Since the symbol $O(\epsilon^{n+1})$ has the algebraic meaning stated above, the results that follow hold regardless of the size ϵ might be. Nonetheless, it is necessary to consider the size of ϵ in most practical applications, where it must be estimated whether the terms included in $O(\epsilon^{n+1})$ can be neglected without distorting too much the final results.

In spite of the restrictions on the functions V_s, the Lagrangian (10) is general enough to encompass the Lagrangians obtained in Ref. 9 from certain Fokker-type relativistic action principles.

According to (2), the Euler equations derived from the Lagrangian (10) are

$$-m_\alpha q_\alpha^{(2)}+\sum_{s=0}^{n}\epsilon^s A_{\alpha s}(q,\ldots,q^{(2s)})=O(\epsilon^{n+1}) \tag{11}$$

with

$$A_{\alpha s}=\sum_{r=0}^{s}(-D)^r\frac{\partial V_s}{\partial q_\alpha^{(r)}}\ . \tag{12}$$

As can be immediately seen, the Lagrangian (10) is a singular one. Indeed, writing Eq. (11) in such a way that the highest-order derivatives appear explicitly, we have

$$q_\beta^{(2n)}\epsilon^n\frac{\partial^2 V_n}{\partial q_\beta^{(n)}\partial q_\alpha^{(n)}}+f_\alpha(q,\ldots,q^{(2n-1)};\epsilon)=O(\epsilon^{n+1}) \tag{13}$$

and, since ϵ is a divisor of zero, the "Hessian matrix" $\epsilon^n\partial^2 V_n/\partial q_\beta^{(n)}\partial q_\alpha^{(n)}$ cannot be inverted within $\mathbb{R}[\epsilon]/(\epsilon^{n+1})$.

In the usual singular Lagrangian formalism, primary constraints would be generated by the inner product of Eq. (13) by the null vectors of the Hessian matrix. In the present case, things are a little bit simpler, because every vector that is a multiple of ϵ, is a null vector of $\epsilon^n \partial^2 V_n / \partial q_\beta^{(n)} \partial q_\alpha^{(n)}$. The primary constraints are therefore

$$\epsilon f_\alpha(q,\ldots,q^{(2n-1)};\epsilon)=O(\epsilon^{n+1})$$

which, according to (11) and (12), can be written as

$$\epsilon\left[-m_\alpha q_\alpha^{(2)}+\sum_{s=0}^{n-1}\epsilon^s A_{\alpha s}(q,\ldots,q^{(2s)})\right]=O(\epsilon^{n+1})\ . \tag{14}$$

Since the matrix $\partial^2 V_n/\partial q_\alpha^{(n)}\partial q_\beta^{(n)}$ has been assumed to be regular, there are no more primary constraints.

Other constraints, the secondary ones, are obtained by repeatedly differentiating (14) with respect to time. The iterative algorithm generating secondary constraints is developed in Appendix A. From (A10) we have that a minimal set of constraints is

$$-m_\alpha q_\alpha^{(r)}+\sum_{s=0}^{n}\epsilon^s B_{\alpha rs}(q,q^{(1)})=O(\epsilon^{n+1})\ ,\qquad r=2,\ldots,2n-1\ , \tag{15}$$

where the functions $B_{\alpha rs}(q,q^{(1)})$ are obtained from $A_{\alpha j}(q,\ldots,q^{(2j)})$, as is specified in Appendix A.

To cast the equations of motion into a Hamiltonian formalism, we perform the Ostrogradski transformation

$$\Pi_{j\alpha}\equiv m_\alpha q_\alpha^{(1)}\delta_j^0+\epsilon^{j+1}\Phi_{j\alpha}(q,\ldots,q^{(2n-j-1)})+O(\epsilon^{n+1}) \tag{16}$$

with

$$\Phi_{j\alpha}=\sum_{s=0}^{n-j-1}\epsilon^s\sum_{l=0}^{s}(-D)^l\frac{\partial V_{s+j+1}}{\partial q_\alpha^{(l+j+1)}}\ . \tag{17}$$

The canonical Hamiltonian is

$$H=-L+\sum_{k=0}^{n-1}\Pi_{k\alpha}q_\alpha^{(k+1)}+O(\epsilon^{n+1})\ . \tag{18}$$

Translated into this formalism,[12] the constraints (15) read

$$\omega_{r\alpha}\equiv q_\alpha^{(r)}-\frac{1}{m_\alpha}B_{\alpha r}(q,q^{(1)})=O(\epsilon^{n+1})\ ,\qquad r=2,\ldots,n-1\ , \tag{19a}$$

$$\chi_{j\alpha}\equiv\Pi_{j\alpha}-\epsilon^{j+1}\phi_{j\alpha}(q,q^{(1)})=O(\epsilon^{n+1})\ ,\qquad j=1,\ldots,n-1\ , \tag{19b}$$

$$\omega_{1\alpha}\equiv q_\alpha^{(1)}-\frac{1}{m_\alpha}[\Pi_{0\alpha}-\epsilon\phi_{0\alpha}(q,q^{(1)})]=O(\epsilon^{n+1})\ , \tag{19c}$$

where

$$B_{\alpha r}(q,q^{(1)})\equiv\sum_{s=0}^{n}\epsilon^s B_{\alpha rs}(q,q^{(1)})\ , \tag{20}$$

$$B_{\alpha 1}(\Pi_0,q,q^{(1)})\equiv\Pi_{0\alpha}-\epsilon\phi_{0\alpha}(q,q^{(1)})\ , \tag{21}$$

and $\phi_{j\alpha}(q,q^{(1)})$ is obtained by substituting (15) into the $q_\beta^{(r)}$, $r\geq 2$, occurring in $\Phi_{j\alpha}(q,q^{(1)},\ldots,q^{(2n-j-1)})$.

These constraints define a submanifold Γ_{2N} [labeled by the coordinates $(q_\alpha,q_\beta^{(1)})$ or, equivalently, $(q_\alpha,\Pi_{0\beta})$] of the phase space $\mathscr{E}_{2Nn}$ (coordinated by $q_\alpha,\ldots,q_\nu^{(n-1)}$, $\Pi_{0\beta},\ldots,\Pi_{n-1\rho}$). The motions take place in this submanifold Γ_{2N}, which will be called *reduced phase space.*

As is proven in Appendix B, the set of constraints (19) is second class; i.e., the matrix $\mathbb{D}$ of their mutual Poisson brackets is regular. Hence, according to the general theory of constrainted Hamiltonian systems, the equations of motion are given by[10]

$$\frac{df}{dt}=\{f,H_R\}^*+\frac{\partial f}{\partial t}\ , \tag{22}$$

where $f(q_\alpha,\Pi_{0\beta},t)$; H_R is the *reduced Hamiltonian,* which is obtained by substituting the constraints into the canonical Hamiltonian (18); and $\{\ ,\ \}^*$ are the Dirac brackets.

To define the latter, the inverse matrix $\mathbb{D}^{-1}$ is needed. Writing it as (see Appendix B)

$$\mathbb{D}^{-1}=\begin{bmatrix}\mathbb{X} & \mathbb{Y}\\ -\mathbb{Y}^T & \mathbb{Z}\end{bmatrix} \tag{23}$$

the Dirac brackets between any two functions f ang g defined on Γ_{2N} are given by[10]

$$\{f,g\}^*=\{f,g\}+\sum_{k,l=1}^{n-1}\sum_{\alpha,\beta=1}^{N}[\mathbb{X}_{k\alpha,l\beta}\{f,\chi_{k\alpha}\}\{g,\chi_{l\beta}\}+\mathbb{Z}_{k\alpha,l\beta}\{f,\omega_{k\alpha}\}\{g,\omega_{l\beta}\}+\mathbb{Y}_{k\alpha,l\beta}(\{f,\chi_{k\alpha}\}\{g,\omega_{l\beta}\}-\{g,\chi_{k\alpha}\}\{f,\omega_{l\beta}\})]\ , \tag{24}$$

where, as is usual in this formalism, the functions f and g appearing on the right-hand side of (24) must be understood as whatever two functions $\tilde{f}$ and $\tilde{g}$ extending, respectively, f and g into $\mathscr{E}_{2Nn}$.

For the sake of convenience, and since $q_\alpha^{(1)}$ can be obtained as a function of $q_\beta,\Pi_{0\rho}$, from (19c), we take q_α and $\Pi_\beta\equiv\Pi_{0\beta}$ as fundamental variables on Γ_{2N}. In terms of them, the elementary Dirac brackets are

$$\{q_\alpha,q_\beta\}^*=\Delta_{\alpha\beta},\quad\{q_\alpha,\Pi_\beta\}^*=\delta_{\alpha\beta}+\Omega_{\alpha\beta},\quad\{\Pi_\alpha,\Pi_\beta\}^*=\Lambda_{\alpha\beta}\ , \tag{25}$$

where

$$\Delta_{\alpha\beta}=\frac{1}{m_\alpha m_\beta}\mathbb{Z}_{1\alpha,1\beta}+O(\epsilon^{n+1})\ , \tag{26a}$$

$$\Omega_{\alpha\beta}=\sum_{l=1}^{n-1}\sum_{\nu}\left[\frac{-1}{m_\alpha m_\nu}Z_{1\alpha,l\nu}\frac{\partial\beta_{\nu l}}{\partial q_\beta}+\frac{1}{m_\alpha}\epsilon^{l+1}\frac{\partial\phi_{l\nu}}{\partial q_\beta}Y_{l\nu,1\alpha}\right]+O(\epsilon^{n+1}) , \tag{26b}$$

$$\Lambda_{\alpha\beta}=\sum_{k,l=1}^{n-1}\sum_{\mu,\nu}\left[X_{k\mu,l\nu}\epsilon^{k+l+2}\frac{\partial\phi_{k\mu}}{\partial q_\alpha}\frac{\partial\phi_{l\nu}}{\partial q_\beta}+\frac{1}{m_\mu m_\nu}Z_{k\mu,l\nu}\frac{\partial B_{\mu k}}{\partial q_\alpha}\frac{\partial B_{\nu l}}{\partial q_\beta}+Y_{k\mu,l\nu}\epsilon^{k+1}\left[\frac{\partial\phi_{k\mu}}{\partial q_\alpha}\frac{\partial B_{\nu l}}{\partial q_\beta}-\frac{\partial\phi_{k\mu}}{\partial q_\beta}\frac{\partial B_{\nu l}}{\partial q_\alpha}\right]\right]$$
$$+O(\epsilon^{n+1}) . \tag{26c}$$

Although relative to the Poisson brackets the coordinates q_α and momenta Π_β were part of a canonical set of variables on $\mathscr{E}_{2Nn}$, they do not necessarily keep this property after having undergone the complete order-reduction process.

Let us now have a glance at what some few lowest-order terms in $\Delta_{\alpha,\beta}$, $\Omega_{\alpha\beta}$, and $\Lambda_{\alpha\beta}$, look like. Using the formulas (B16) of Appendix B, together with (19), (21), and (26), we obtain

$$^0\Delta_{\alpha\beta}={}^1\Delta_{\alpha\beta}={}^0\Omega_{\alpha\beta}={}^1\Omega_{\alpha\beta}={}^0\Lambda_{\alpha\beta}={}^1\Lambda_{\alpha\beta}={}^2\Lambda_{\alpha\beta}=0 , \tag{27}$$

$$^2\Delta_{\alpha\beta}=\frac{1}{m_\alpha m_\beta}\left[\frac{\partial^0\phi_{1\beta}}{\partial q_\alpha^{(1)}}-\frac{\partial^0\phi_{1\alpha}}{\partial q_\beta^{(1)}}\right],\quad {}^2\Omega_{\alpha\beta}=\frac{1}{m_\alpha}\frac{\partial^0\phi_{1\alpha}}{\partial q_\beta} ,$$

or

$$\{q_\alpha,q_\beta\}^*=\epsilon^2\frac{1}{m_\alpha m_\beta}\left[\frac{\partial^0\phi_{1\beta}}{\partial q_\alpha^{(1)}}-\frac{\partial^0\phi_{1\alpha}}{\partial q_\beta^{(1)}}\right]+O(\epsilon^3) ,$$
$$\{q_\alpha,\Pi_\beta\}^*=\delta_{\alpha\beta}+\epsilon^2\frac{1}{m_\alpha}\frac{\partial^0\phi_{1\alpha}}{\partial q_\beta}+O(\epsilon^3) , \tag{28}$$
$$\{\Pi_\alpha,\Pi_\beta\}^*=O(\epsilon^3) ,$$

whence, as can be easily checked, it follows that

$$\tilde{q}_\alpha=q_\alpha-\epsilon^2\frac{1}{m_\alpha}{}^0\phi_{1\alpha}(q,q^{(1)})+O(\epsilon^3) \tag{29}$$

and Π_β $(\alpha,\beta=1,\ldots,N)$ form a set of canonical coordinates and momenta on the reduced phase space Γ_{2N}, with respect to the canonical structure defined by the Dirac brackets (24).

IV. APPLICATION: WHEELER-FEYNMAN ELECTRODYNAMICS FOR TWO POINT CHARGES UP TO $1/c^4$

According to what is proven in Refs. 5 and 9, the Fokker-type Lagrangian of Wheeler-Feynman electrodynamics is equivalent to an infinite-order Lagrangian, provided that some regularity conditions are satisfied by the particle world lines.

In the case of two particles, the infinite-order Lagrangian is

$$L=L_f-U \tag{30}$$

with

$$L_f=\sum_{a=1}^{2}m_ac^2[1-(1-v_a{}^2/c^2)^{1/2}] \tag{31a}$$

and

$$U=-e_1e_2\sum_{s=0}^{\infty}\frac{(-1)^s}{c^{2s}(2s)!}D_1^sD_2^s\left[\left[1-\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right]r^{2s-1}\right] , \tag{31b}$$

where

$$\mathbf{v}_a=\dot{\mathbf{x}}_a,\quad a=1,2;\quad \mathbf{r}=\mathbf{x}_1-\mathbf{x}_2 ,$$
$$r=|\mathbf{r}|,\quad v_a=|\mathbf{v}_a| ,$$

and D_a differentiates with respect to time the dynamical variables of particle a only.

As has been already discussed, dealing with this infinite-order Lagrangian is equivalent to working within a formalism approximated modulo ϵ^{n+1}, with n arbitrary. It can be easily checked that the Lagrangian L has the special shape (10), provided we put $\epsilon=1/c^2$. Hence, the general framework developed in Sec. III applies to this particular case. That is what we are going to do in this section, approximating up to terms $1/c^4$, i.e., modulo $O(\epsilon^3)=O(1/c^6)$.

The approximated Lagrangian is

$$L=L_0+\frac{1}{c^2}L_1+\frac{1}{c^4}L_2+O(1/c^6) \tag{32}$$

with

$$L_0=\tfrac{1}{2}\sum_{a=1}^{2}m_av_a{}^2-\frac{e_1e_2}{r} , \tag{33a}$$

$$L_1=\tfrac{1}{8}\sum_{a=1}^{2}m_av_a{}^4+\frac{e_1e_2}{2r}[(\mathbf{v}_1\cdot\mathbf{v}_2)+(\mathbf{v}_1\cdot\hat{\mathbf{n}})(\mathbf{v}_2\cdot\hat{\mathbf{n}})] , \tag{33b}$$

and

$$L_2=\tfrac{1}{16}\sum_{a=1}^{2} m_a v_a{}^6-\frac{e_1e_2}{8}\Bigg\{r[3(\mathbf{a}_1\cdot\mathbf{a}_2)-(\mathbf{a}_1\cdot\hat{\mathbf{n}})(\mathbf{a}_2\cdot\hat{\mathbf{n}})]+2[(\mathbf{a}_2\cdot\mathbf{v}_1)(\mathbf{v}_1\cdot\hat{\mathbf{n}})-(\mathbf{a}_1\cdot\mathbf{v}_2)(\mathbf{v}_2\cdot\hat{\mathbf{n}})]$$
$$+(\mathbf{a}_1\cdot\hat{\mathbf{n}})[v_2{}^2-(\mathbf{v}_2\cdot\hat{\mathbf{n}})^2]+(\mathbf{a}_2\cdot\hat{\mathbf{n}})[(\mathbf{v}_1\cdot\hat{\mathbf{n}})-v_1{}^2]$$
$$+\frac{1}{r}[-2(\mathbf{v}_1\cdot\mathbf{v}_2)^2+v_1{}^2v_2{}^2-v_1{}^2(\mathbf{v}_2\cdot\hat{\mathbf{n}})^2-v_2{}^2(\mathbf{v}_1\cdot\hat{\mathbf{n}})^2+3(\mathbf{v}_1\cdot\hat{\mathbf{n}})^2(\mathbf{v}_2\cdot\hat{\mathbf{n}})^2]\Bigg\}\,, \tag{33c}$$

where

$$\hat{\mathbf{n}}=\frac{\mathbf{r}}{r}\,.$$

At the lowest order of approximation we have the Coulomb law of motion

$$m_b\mathbf{a}_b=\eta_b\frac{e_1e_2}{r^2}\hat{\mathbf{n}},\quad \eta_b\equiv(-1)^{b+1},\quad b=1,2\,. \tag{34}$$

According to (12) and (13), the Ostrogradski transformation is defined by

$$\mathbf{\Pi}_a=m_a\mathbf{v}_a+\frac{1}{c^2}\mathbf{\Phi}_{0a}(\mathbf{x}_b,\mathbf{v}_c,\mathbf{a}_d,\dot{\mathbf{a}}_e)+O(1/c^6),\quad \mathbf{\Pi}_{1a}=\frac{1}{c^4}\mathbf{\Phi}_{1a}(\mathbf{x}_b,\mathbf{v}_c,\mathbf{a}_d)+O(1/c^6)\,, \tag{35}$$

with

$$\mathbf{\Phi}_{1b}=-\frac{e_1e_2}{8}\{r[3\mathbf{a}_{b'}-(\mathbf{a}_{b'}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]+\eta_b[-2(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})\mathbf{v}_{b'}+v_{b'}{}^2\hat{\mathbf{n}}-(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})^2\hat{\mathbf{n}}]\}\,, \tag{36a}$$

$$\mathbf{\Phi}_{0b}=\tfrac{1}{2}m_bv_b{}^2\mathbf{v}_b+\frac{e_1e_2}{2r}[\mathbf{v}_{b'}+(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]$$
$$+\frac{1}{c^2}\Bigg\{\frac{3}{8}m_bv_b{}^4\mathbf{v}_b+\frac{e_1e_2}{8r}\{[v_{b'}{}^2+2(\mathbf{v}_1\cdot\mathbf{v}_2)+2(\mathbf{v}_1\cdot\hat{\mathbf{n}})(\mathbf{v}_2\cdot\hat{\mathbf{n}})-(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})^2]\mathbf{v}_{b'}+[(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})^2-v_b{}^2]\mathbf{v}_b$$
$$+[v_{b'}{}^2(\mathbf{v}_b\cdot\hat{\mathbf{n}})+3v_{b'}{}^2(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})-2(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})(\mathbf{v}_1\cdot\mathbf{v}_2)-3(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})^3-3(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})^2(\mathbf{v}_b\cdot\hat{\mathbf{n}})]\hat{\mathbf{n}}\}$$
$$+\frac{e_1e_2}{8}\{r[3\dot{\mathbf{a}}_{b'}-(\dot{\mathbf{a}}_{b'}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]+\eta_b[\hat{\mathbf{n}}\mathbf{v}_b-5(\hat{\mathbf{n}}\cdot\mathbf{v}_{b'})]\mathbf{a}_{b'}$$
$$+\eta_b(\mathbf{a}_{b'}\cdot\hat{\mathbf{n}})(\mathbf{v}_b-\mathbf{v}_{b'})+\eta_b[3(\mathbf{a}_{b'}\cdot\mathbf{v}_{b'})-3(\mathbf{a}_{b'}\cdot\mathbf{v}_b)-(\mathbf{v}_b\cdot\hat{\mathbf{n}})(\mathbf{a}_{b'}\cdot\hat{\mathbf{n}})-3(\mathbf{v}_{b'}\cdot\hat{\mathbf{n}})(\mathbf{a}_{b'}\cdot\hat{\mathbf{n}})]\hat{\mathbf{n}}\}\Bigg\}\,, \tag{36b}$$

where b' means "not b." (Note that Φ_{0b} contains terms of order $1/c^2$.)

As has become obvious along this section, the α-type indices labeling the N degrees of freedom in Secs. II—IV are now written (ai), (bj), and so on; $a,b,\ldots=1,2$; $i,j,\ldots=1,2,3$. Furthermore, the spatial indices are, in most cases, hidden under the vector notation.

The constraints defining the reduced phase space Γ_{12} are

$$\mathbf{x}_{1a}\equiv\mathbf{\Pi}_{1a}-\frac{1}{c^4}\boldsymbol{\phi}_{1a}(\mathbf{x},\mathbf{v})+O(1/c^6),\quad \boldsymbol{\omega}_{1a}\equiv\mathbf{v}_a-\frac{1}{m_a}\mathbf{B}_{a1}(\mathbf{\Pi},\mathbf{x},\mathbf{v})+O(1/c^6)\,, \tag{19'}$$

where

$$\boldsymbol{\phi}_{1a}=\frac{e_1e_2}{8}\eta_a[2(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})\mathbf{v}_{a'}-v_{a'}{}^2\hat{\mathbf{n}}+(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})^2\hat{\mathbf{n}}]+\frac{e_1{}^2e_2{}^2}{4m_{a'}r}\eta_a\hat{\mathbf{n}}\,, \tag{37a}$$

$$\mathbf{B}_{a1}=\mathbf{\Pi}_{1a}-\frac{1}{c^2}\left[\tfrac{1}{2}m_a v_a{}^2\mathbf{v}_a+\frac{e_1e_2}{2r}[\mathbf{v}_{a'}+(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]\right]$$
$$+\frac{1}{c^4}\left[\tfrac{3}{8}m_a v_a{}^4\mathbf{v}_a+\frac{e_1e_2}{8r}\{[v_{a'}{}^2+2(\mathbf{v}_1\cdot\mathbf{v}_2)+2(\mathbf{v}_1\cdot\hat{\mathbf{n}})(\mathbf{v}_2\cdot\hat{\mathbf{n}})-(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})^2]\mathbf{v}_{a'}+[(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})^2-v_{a'}{}^2]\mathbf{v}_a\right.$$
$$+[v_{a'}{}^2(\mathbf{v}_a\cdot\hat{\mathbf{n}})+3v_{a'}{}^2(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})-2(\mathbf{v}_1\cdot\mathbf{v}_2)(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})-3(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})^3-3(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})^2(\mathbf{v}_a\cdot\hat{\mathbf{n}})]\hat{\mathbf{n}}\}$$
$$\left.-\frac{e_1{}^2e_2{}^2}{4m_{a'}r^2}[2(\mathbf{v}_a-\mathbf{v}_{a'})-5(\mathbf{v}_a\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+(\mathbf{v}_{a'}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]\right], \tag{37b}$$

which have been obtained by substituting (34) and its time derivative into (36).

Now, applying the results of Sec. III we obtain, for the elementary Dirac brackets,

$$\{\chi_{ai},\chi_{bj}\}^*=\frac{e_1e_2}{4m_1m_2c^4}\delta_{a'b}\left[-\eta_a\left[\frac{(\mathbf{\Pi}_1\cdot\hat{\mathbf{n}})}{m_1}+\frac{(\mathbf{\Pi}_2\cdot\hat{\mathbf{n}})}{m_2}\right](\delta_{ij}-\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j)+\left[\frac{\Pi_{1i}}{m_1}-\frac{\Pi_{2i}}{m_2}\right]\hat{\mathbf{n}}_j-\left[\frac{\Pi_{1i}}{m_1}-\frac{\Pi_{2j}}{m_2}\right]\hat{\mathbf{n}}_i\right]+O(1/c^6)\,, \tag{38a}$$

$$\{\chi_{ai},\Pi_{bj}\}^*=\delta_{ab}\delta_{ij}+\eta_a\eta_b\frac{e_1e_2}{8m_1m_2rc^4}\left[\frac{1}{m_{a'}}\{2\Pi_{a'i}\Pi_{a'j}+\delta_{ij}((\mathbf{\Pi}_{a'}\cdot\hat{\mathbf{n}})^2-\Pi_{a'}{}^2)+[\Pi_{a'}{}^2-3(\mathbf{\Pi}_{a'}\cdot\hat{\mathbf{n}})^2]\hat{\mathbf{n}}_i\hat{\mathbf{n}}_j\right.$$
$$\left.+2(\mathbf{\Pi}_{a'}\cdot\hat{\mathbf{n}})(\Pi_{a'j}\hat{\mathbf{n}}_i-\Pi_{a'i}\hat{\mathbf{n}}_j)\}+\frac{2e_1e_2}{r}(\delta_{ij}-2\hat{\mathbf{n}}_i\hat{\mathbf{n}}_j)\right]+O(1/c^6)\,, \tag{38b}$$

$$\{\Pi_{ai},\Pi_{bj}\}^*=O(1/c^6)\,, \tag{38c}$$

which agree with previously known results: namely, the position coordinates $\mathbf{x}_a$ can be taken as canonical ones in approximations up to the post-Coulomb order, but not in further ones.

Moreover, the nonvanishing $1/c^4$ terms in the elementary Dirac brackets (38) agree with what could be expected from the results of Ref. 4, concerning the noninteraction theorem for Lagrangians approximated to $1/c^n$.

Then, according to (29), we can readily change the variables $\mathbf{x}_a$, $\mathbf{\Pi}_b$, which are not canonical relatively to the Dirac brackets, into the new ones:

$$\mathbf{q}_a=\mathbf{x}_a-\frac{e_1e_2}{4m_1m_2c^4}\eta_a\left[\frac{(\mathbf{\Pi}_{a'}\cdot\hat{\mathbf{n}})}{m_{a'}}\mathbf{\Pi}_{a'}+\left[\frac{(\mathbf{\Pi}_{a'}\cdot\hat{\mathbf{n}})^2-\Pi_{a'}{}^2}{2m_{a'}}+\frac{e_1e_2}{r}\right]\hat{\mathbf{n}}\right]+O(1/c^6)\,, \tag{39a}$$

$$\mathbf{p}_b=\mathbf{\Pi}_b+O(1/c^6)\,, \tag{39b}$$

which, besides being canonical as referred to the Dirac brackets, share with the former x's and Π's, the good behavior under the Euclidean group, do not select a single set of coordinates and momenta. Indeed, any new variables defined by

$$\mathbf{q}'_a=\mathbf{q}_a+\frac{1}{c^4}\frac{\partial F}{\partial \mathbf{p}_a}+O(1/c^6),\quad \mathbf{p}'_a=\mathbf{p}_a-\frac{1}{c^4}\frac{\partial F}{\partial \mathbf{q}_a}+O(1/c^6)\,, \tag{40}$$

with $F[q^2,\mathbf{q}\cdot\mathbf{p}_a,\mathbf{p}_{a'}\cdot\mathbf{p}_b,\mathbf{q}\cdot(\mathbf{p}_1\times\mathbf{p}_2)]$, also satisfy both requirements (where $\mathbf{q}=\mathbf{q}_1-\mathbf{q}_2$ and $q=|\mathbf{q}|$).

Finally, according to (18), and in terms of the canonical coordinates and momenta (39), the reduced Hamiltonian is

$$H_R=H_0+\frac{1}{c^2}H_1+\frac{1}{c^4}H_2+O(1/c^6)\,, \tag{41}$$

$$H_0=\frac{1}{2}\sum_a\frac{p_a{}^2}{m_a}+\frac{e_1e_2}{q}\,, \tag{42a}$$

$$H_1=-\frac{1}{8}\sum_a\frac{p_a{}^4}{m_a{}^3}-\frac{e_1e_2}{2qm_1m_2}[(\mathbf{p}_1\cdot\mathbf{p}_2)+(\mathbf{p}_1\cdot\hat{\mathbf{q}})(\mathbf{p}_2\cdot\hat{\mathbf{q}})]\,, \tag{42b}$$

$$H_2=\frac{1}{16}\sum_a \frac{p_a{}^6}{m_a{}^5}+\frac{e_1e_2}{4m_1m_2q}\left[\frac{1}{m_1m_2}\left[\frac{p_1{}^2p_2{}^2}{2}-(\mathbf{p}_1\cdot\mathbf{p}_2)^2+\frac{3}{2}(\mathbf{p}_1\cdot\hat{\mathbf{q}})^2(\mathbf{p}_2\cdot\hat{\mathbf{q}})^2\right]\right.$$
$$\left.+\sum_a \frac{p_a{}^2}{m_a}\left[\frac{(\mathbf{p}_a\cdot\mathbf{p}_{a'})}{m_a}+\frac{(\mathbf{p}_a\cdot\hat{\mathbf{q}})(\mathbf{p}_{a'}\cdot\hat{\mathbf{q}})}{m_a}-\frac{1}{2}\frac{(\mathbf{p}_a\cdot\hat{\mathbf{q}})^2}{m_{a'}}\right]\right]$$
$$+\frac{3}{16}\frac{e_1{}^2e_2{}^2}{m_1m_2q^2}\left[\sum_a \frac{1}{m_a}[p_a{}^2+(\mathbf{p}_a\cdot\hat{\mathbf{q}})^2]\right]. \tag{42c}$$

V. CONCLUSION

Starting from a Fokker-type Lagrangian, for a relativistic system of directly interacting particles, and by means of a $1/c$ expansion, an infinite-order Lagrangian can be obtained.[5,9] The corresponding equations of motion are of "infinite order," and infinitely many initial data are needed in order to determine the future evolution.

Several justifications can be found in the literature for providing some supplementary criteria[5,8,13] that permit selection of some few "physically significant" solutions among the whole set of solutions of the infinite-order system. One among these criteria consists in requiring the particle trajectories to depend analytically on some coupling constant (i.e., the particles become free once the interaction has been switched off). Another possible condition requires analytical dependence on $1/c$, namely, the solutions admit a nonrelativistic limit. Even another possibility consists in expanding in the ratio of the masses of two particles.[14]

Usually, and in most of physically interesting cases, the solutions selected by any of these additional criteria, can be parametrized by a Newton-type set of initial data (namely, six times as many parameters as the number of particles, for pointlike particle systems). They therefore satisfy an ordinary second-order differential system, which is said to be a reduction of order 2 of the original infinite-order system.

It could then happen that some reductions or order 2, for some physically interesting infinite-order systems were actually significant, even beyond the purely esthetical motivations that could drive our search for a Newtonian set of equations of motion. This can be especially hoped after the spontaneous predictivization of hereditary systems has been proven.[15] The later consists in that retarded difference-differential systems admit some reductions of order 2 which act as attractors (i.e., special trajectories to which other solutions converge asymptotically).

The introduction of supplementary conditions usually affects the Hamiltonian framework associated with the Lagrangian we have started from. This is basically the reason why the results of Ref. 2 are not correct. The main contribution of the present paper is that a method is provided to introduce those supplementary conditions keeping the canonical formalism in sight. It has been possible thanks to the concepts of singular Lagrangian system and Dirac brackets. These are tools that had been used already in relativistic dynamics of directly interacting particles, although in a quite different context. The methods developed here are especially useful in the framework of $1/c$ expansions, and this has been done in applying them to the electrodynamics of Wheeler and Feynman for two charged particles. Although in that specific case no problem arises, it can be easily seen that, when passing to a case with more than two particles, the $1/c^4$ term in the Lagrangian (and in the Hamiltonian too) will exhibit a part increasing with the distance between particles. It could therefore seem as if the interaction did not admit cluster decomposition. However, this seems to be due to a limitation inherent to $1/c$ expansions rather than to the interaction itself.[16]

Owing to that unpleasant feature of $1/c$ expansions, it would be desirable to develop an alternative treatment based on coupling constant expansions. However, the method here developed depends crucially on the following property of the infinite-order Lagrangians considered: the higher the order of the derivative, the higher is the least order in the $1/c$ expansion where it occurs. This property does not hold in the framework of coupling constant expansions.

Moreover, it would be also desirable to deal with the whole problem starting straight from Fokker-type Lagrangians, thus avoiding the intermediate step of infinite-order Lagrangians, that are unpleasant from a mathematical viewpoint, and also irrelevant as far as the final reduction of order is concerned.

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APPENDIX A: THE SECONDARY CONSTRAINTS

Once the primary constraints are known, we must look for the minimal set of constraints such that (i) it is stable by time differentiation and (ii) it contains the primary ones.

We thus start from the primary constraints:

$$\left[q_\alpha^{(2)} - \frac{1}{m_\alpha} \sum_{s=0}^{n-1} \epsilon^s A_{\alpha s}(q \cdots q^{(2s)}) \right] \epsilon = O(\epsilon^{n+1}) , \quad \text{(A1)}$$

where the highest-order derivatives are $q^{(2n-2)}$ and $q^{(2n-3)}$, and occur in the $A_{\alpha n-1}$, term, thus being multiplied by ϵ^n.

Multiplying (A1) by ϵ^{n-1}, we have

$$\epsilon^n q_\alpha^{(2)} = \frac{1}{m_\alpha} A_{\alpha 0}(q) \epsilon^n + O(\epsilon^{n+1}) \quad \text{(A2)}$$

which, differentiated r times with respect to time, and using (A2) itself everywhere that $\epsilon^n q_\beta^{(2)}$ occurs on the right-hand side, leads to

$$\epsilon^n q_\alpha^{(r+2)} = \frac{1}{m_\alpha} B_{\alpha,r+2,0}(q,q^{(1)}) \epsilon^n + O(\epsilon^{n+1}) . \quad \text{(A3)}$$

We can then substitute (A3), with $r=2n-4$ and $r=2n-5$, into (A1), so eliminating the dependence on $q^{(2n-2)}$ and $q^{(2n-3)}$ in the term $A_{\alpha,n-1}$.

After that, the highest-order derivatives are $q^{(2n-4)}$ and $q^{(2n-5)}$, and occur in the terms $A_{\alpha,n-1}$ and $A_{\alpha,n-2}$, thus being multiplied by ϵ^{n-1}.

Multiplying (A1) by ϵ^{n-2}, we have

$$\epsilon^{n-1} q_\alpha^{(2)} = \frac{1}{m_\alpha} [A_{\alpha 0}(q) + \epsilon A_{\alpha 1}(q,q^{(1)},q^{(2)})] \epsilon^{n-1} + O(\epsilon^{n+1}) . \quad \text{(A4)}$$

The $q^{(2)}$ dependence on the right-hand side of (A4) can be eliminated by using (A3) with $r=0$, so obtaining

$$\epsilon^{n-1} q_\alpha^{(2)} = \frac{1}{m_\alpha} [A_{\alpha 0}(q) + \epsilon \widetilde{A}_{\alpha 1}(q,q^{(1)})] \epsilon^{n-1} + O(\epsilon^{n+1}) \quad \text{(A5)}$$

which, after differentiating r times with respect to time, and using (A5) itself wherever $\epsilon^{n-1} q_\beta^{(2)}$ occurs on the right-hand side, yields

$$\epsilon^{n-1} q_\alpha^{(2+r)} = \frac{1}{m_\alpha} [B_{\alpha,r+2,0}(q,q^{(1)}) + \epsilon B_{\alpha,r+2,1}(q,q^{(1)})] \epsilon^{n-1} + O(\epsilon^{n+1}) . \quad \text{(A6)}$$

Pushing forward the iterative algorithm, whose first two steps have been displayed above, we finally arrive at the set of constraints

$$\left[q_\alpha^{(r)} - \frac{1}{m_\alpha} \sum_{s=0}^{n-1} \epsilon^s B_{\alpha,r,s}(q,q^{(1)}) \right] \epsilon = O(\epsilon^{n+1}) , \quad r=2,\ldots,2n-1 . \quad \text{(A7)}$$

Owing to how this expression has been derived, it is obvious that the constraints (A7) are more restrictive than (A1).

The time derivative of (A7), with $r=2n-1$, is

$$\epsilon \left[q_\alpha^{(2n)} - \frac{1}{m_\alpha} \sum_{s=0}^{n-1} \epsilon^s B_{\alpha,2n,s}(q,q^{(1)}) \right] = O(\epsilon^{n+1}) \quad \text{(A8)}$$

which, properly speaking, is not a constraint yet, because $q_\alpha^{(2n)}$ is not an independent variable on the initial-data space.

But, using (A7) and (A8) to remove the accelerations and higher-order derivatives from the equations of motion (17), we obtain the new constraint

$$q_\alpha^{(2)} = \frac{1}{m_\alpha} \sum_{s=0}^{n} \epsilon^s B_{\alpha,2,s}(q,q^{(1)}) + O(\epsilon^{n+1}) \quad \text{(A9)}$$

which, by differentiating with respect to time, and using itself wherever $q_\beta^{(2)}$ occurs on the right-hand side, finally yields

$$q_\alpha^{(r)} = \frac{1}{m_\alpha} \sum_{s=0}^{n} \epsilon^s B_{\alpha,r,s}(q,q^{(1)}) + O(\epsilon^{n+1}) , \quad \text{(A10)}$$

$r=2,\ldots,2n-1$.

It is also obvious that the constraints (A10) are more restrictive than (A7). Furthermore, they are stable under time differentiation. Indeed, since $q_\alpha^{(2n)}$ occurs multiplied by ϵ^n in the Euler equations (11), only $B_{\alpha 2s}(q,q^{(1)})$ is significant, and (A10) with $r=2,\ldots,n$ into the Euler equations (11) yields (A9) again, so proving the stability of the constraints (A10).

APPENDIX B: ARE THE CONSTRAINTS (19) OF SECOND CLASS?

Let us consider the skew-symmetric $2N(n-1) \times 2N(n-1)$ matrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{S} & \mathbf{T} \\ -\mathbf{T}^T & \mathbf{U} \end{bmatrix} \quad \text{(B1)}$$

defined by the Poisson brackets between pairs of constraints. Its components $\mathbb{S}$, $\mathbb{T}$, and $\mathbb{U}$ are three $N(N-1) \times N(n-1)$ matrices defined as

$$\begin{aligned} \mathbb{S}_{k\alpha,r\beta} &= \{\chi_{k\alpha}, \chi_{r\beta}\} , \\ \mathbb{T}_{k\alpha,r\beta} &= \{\chi_{k\alpha}, \omega_{r\beta}\} , \\ \mathbb{U}_{k\alpha,r\beta} &= \{\omega_{k\alpha}, \omega_{r\beta}\} , \end{aligned} \quad \text{(B2)}$$

and, according to (19) and (21), we have also that

$$\mathbb{S}_{k\alpha,r\beta} = \epsilon^{r+1} \frac{\partial \phi_{r\beta}}{\partial q_\alpha^{(1)}} \delta_{k1} - \epsilon^{k+1} \frac{\partial \phi_{k\alpha}}{\partial q_\beta^{(1)}} \delta_{r1} + O(\epsilon^{n+1}) , \quad \text{(B3)}$$

$$\mathbb{T}_{k\alpha,r\beta} = -\delta_{kr}\delta_{\alpha\beta} + \frac{1}{m_\beta} \left[\delta_{k1} \frac{\partial B_{\beta r}}{\partial q_\alpha^{(1)}} + \delta_{r1} \epsilon^{k+1} \frac{\partial \phi_{k\alpha}}{\partial q_\beta} \right] + O(\epsilon^{n+1}) , \quad \text{(B4)}$$

$$\mathbb{U}_{k\alpha,r\beta} = \frac{1}{m_\alpha m_\beta} \left[\delta_{r1} \frac{\partial B_{\alpha k}}{\partial q_\beta} - \delta_{k1} \frac{\partial B_{\beta r}}{\partial q_\alpha} \right] + O(\epsilon^{n+1}) . \quad \text{(B5)}$$

It hence follows from (B3)—(B5) that

$$\mathbb{T}=-\mathbb{1}+\mathbb{M} \tag{B6}$$

and that the matrix elements in $\mathbb{S}$, $\mathbb{M}$, and $\mathbb{U}$ all vanish, but those corresponding to rows and columns, respectively, labeled (1α) and (1β).

To determine the inverse matrix $\mathbb{D}^{-1}$, we first expand $\mathbb{D}$ as

$$\mathbb{D}=\sum_{s=0}^{n}\epsilon^{s\,s}\mathbb{D}+O(\epsilon^{n+1}) . \tag{B7}$$

Then, provided that $\mathbb{D}$ is regular, we can write (B7) as

$$\mathbb{D}={}^0\mathbb{D}\sum_{s=0}^{n}\epsilon^{s\,0}\mathbb{D}^{-1\,s}\mathbb{D}+O(\epsilon^{n+1})$$

whose inverse is

$$\mathbb{D}^{-1}=\left[\mathbb{1}+\sum_{s=1}^{n}\epsilon^{s\,s}\mathbb{N}\right]{}^0\mathbb{D}^{-1}+O(\epsilon^{n+1}) \tag{B8}$$

${}^sN$ being obtained by means of the iterative law:

$${}^s\mathbb{N}=-{}^0\mathbb{D}^{-1\,s}D-\sum_{r=1}^{s-1}{}^0\mathbb{D}^{-1\,r-s}\mathbb{D}\,{}^r\mathbb{N} . \tag{B9}$$

We still have to prove that ${}^0\mathbb{D}$ is regular. Aiming for this, we write ${}^r\mathbb{D}$ as

$${}^r\mathbb{D}=\begin{bmatrix} {}^r\mathbb{S} & {}^r\mathbb{T} \\ -{}^r\mathbb{T}^T & {}^r\mathbb{U}\end{bmatrix} . \tag{B10}$$

Then, according to (B3)—(B6), we have for $r=0$, that

$$\begin{aligned} &{}^0\mathbb{S}=0,\quad {}^0\mathbb{M}_{k\alpha,r\beta}=\frac{1}{m_\beta}\delta_{k1}\frac{\partial B_{\beta r0}}{\partial q_\alpha^{(1)}} ,\\ &{}^0\mathbb{U}_{\kappa\alpha,r\beta}=\frac{1}{m_\alpha m_\beta}\left[\delta_{r1}\frac{\partial \beta_{\alpha k0}}{\partial q_\beta}-\delta_{k1}\frac{\partial B_{\beta r0}}{\partial q_\alpha}\right] . \end{aligned} \tag{B11}$$

Hence, the shape of ${}^0\mathbb{D}$ is

$${}^0\mathbb{D}=\begin{bmatrix} 0 & -\mathbb{1}+{}^0\mathbb{M} \\ \mathbb{1}-{}^0\mathbb{M}^T & {}^0\mathbb{U}\end{bmatrix}$$

and, since ${}^0\mathbb{M}^2=0$, it follows immediately that

$${}^0\mathbb{D}^{-1}=\begin{bmatrix} (\mathbb{1}+{}^0\mathbb{M}^T)\,{}^0\mathbb{U}(\mathbb{1}+{}^0\mathbb{M}) & \mathbb{1}+{}^0\mathbb{M}^T \\ -\mathbb{1}-{}^C\mathbb{M} & 0\end{bmatrix} . \tag{B12}$$

Let us now have a glance at what the lowest order terms in $\mathbb{D}^{-1}$ look like, essentially aiming to obtain the lower-order terms of the elementary Dirac brackets (25). As usual we write $\mathbb{D}^{-1}$ as

$$\mathbb{D}^{-1}=\begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ -\mathbb{Y}^T & \mathbb{Z}\end{bmatrix} \tag{23}$$

and, according to (B12), we have that

$$\begin{aligned} &{}^0X=(\mathbb{1}+{}^0\mathbb{M}^T)\,{}^0\mathbb{U}(\mathbb{1}+{}^0\mathbb{M}) ,\\ &{}^0\mathbb{Y}=\mathbb{1}+{}^0\mathbb{M}^T;\quad {}^0\mathbb{Z}=0 . \end{aligned} \tag{B13}$$

In order to obtain the elementary Dirac brackets (25) up to the order ϵ^2, only ${}^0\mathbb{Z}$, ${}^1\mathbb{Z}$, ${}^2\mathbb{Z}$, and ${}^0\mathbb{Y}$ are needed. Therefore, we shall not care for other matrices (e.g., ${}^1\mathbb{Y}$ or ${}^0\mathbb{X}$).

Working out (B8) up to ϵ^2, and taking (B9) into account, we arrive at

$$\begin{aligned} \mathbb{D}^{-1}={}&{}^0\mathbb{D}^{-1}-\epsilon\,{}^0\mathbb{D}^{-1\,1}\mathbb{D}\,{}^0\mathbb{D}^{-1}\\ &+\epsilon^2({}^0\mathbb{D}^{-1\,1}\mathbb{D}\,{}^0\mathbb{D}^{-1\,1}\mathbb{D}-{}^0\mathbb{D}^{-1\,2}\mathbb{D})\,{}^0\mathbb{D}^{-1}+O(\epsilon^3) , \end{aligned} \tag{B14}$$

whence, taking into account that ${}^1\mathbb{S}=0$, we obtain

$${}^1\mathbb{Z}=0,\quad {}^2\mathbb{Z}=(\mathbb{1}+{}^0\mathbb{M})\,{}^2\mathbb{S}(\mathbb{1}+{}^0\mathbb{M}^T) . \tag{B15}$$

Finally, the explicit expressions to be used in Sec. III are

$$\begin{aligned} &{}^0\mathbb{Y}_{k\mu,l\nu}=\delta_{kl}\delta_{\mu\nu}+\frac{1}{m_\mu}\delta_{l1}\frac{\partial B_{\mu k0}}{\partial q_\nu^{(1)}} ,\\ &{}^0\mathbb{Z}_{k\mu,l\nu}={}^1\mathbb{Z}_{k\mu,l\nu}=0 ,\\ &{}^2\mathbb{Z}_{k\mu,l\nu}=\delta_{k1}\delta_{l1}\left[\frac{\partial\,{}^0\phi_{1\nu}}{\partial q_\mu^{(1)}}-\frac{\partial\,{}^0\phi_{1\mu}}{\partial q_\nu^{(1)}}\right] , \end{aligned} \tag{B16}$$

where use has been made of the fact that $\partial B_{\alpha 10}(\Pi,q,q^{(1)})/\partial q_\beta^{(1)}=0$, as follows immediately from (19) and (21).

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