SIAM J. MATH. ANAL.
Vol. 34, No. 1, pp. 64–86

A REFINED GLOBAL WELL-POSEDNESS RESULT FOR SCHRÖDINGER EQUATIONS WITH DERIVATIVE*

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Abstract. In this paper we prove that the one-dimensional Schrödinger equation with derivative in the nonlinear term is globally well-posed in H^s for $s > \frac{1}{2}$ for data small in L^2. To understand the strength of this result one should recall that for $s < \frac{1}{2}$ the Cauchy problem is ill-posed, in the sense that uniform continuity with respect to the initial data fails. The result follows from the method of almost conserved energies, an evolution of the "I-method" used by the same authors to obtain global well-posedness for $s > \frac{2}{3}$. The same argument can be used to prove that any quintic nonlinear defocusing Schrödinger equation on the line is globally well-posed for large data in H^s for $s > \frac{1}{2}$.

Key words. almost conserved energies, global well-posedness, Schrödinger equation with derivative

AMS subject classifications. 35Q53, 42B35, 37K10

PII. S0036141001394541

1. Introduction.

In this paper, using the method of almost conserved energies, we establish a sharp result on global well-posedness for the derivative nonlinear Schrödinger IVP

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda\partial_x(|u|^2 u), \\ u(x,0) = u_0(x), \qquad x \in \mathbb{R},\ t \in \mathbb{R}, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$.

The first result of this kind was obtained in the context of the KdV and the modified KdV (mKdV) IVPs [11], also using almost conserved energies. Below we will discuss in more detail the "almost conservation method" and its relationship with the "I-method" which was applied to (1) in [9] (see also [10, 11, 20, 21]).

From the point of view of physics the equation in (1) is a model for the propagation of circularly polarized Alfvén waves in magnetized plasma with a constant magnetic field [25, 26, 29].

*Received by the editors August 28, 2001; accepted for publication (in revised form) March 1, 2002; published electronically August 15, 2002.
http://www.siam.org/journals/sima/34-1/39454.html

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It is natural to impose the smallness condition

$$\|u_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}} \tag{2}$$

on the initial data, as this will force the energy to be positive via the sharp Gagliardo–Nirenberg inequality [36]. Note that the L^2 norm is conserved by the evolution. In this paper, we prove the following global well-posedness result.

THEOREM 1.1. *The Cauchy problem* (1) *is globally well-posed in* H^s *for* $s > \frac{1}{2}$, *assuming the smallness condition* (2).

We present here once again [9] a summary of the well-posedness story for (1). Scattering and well-posedness for this Cauchy problem has been studied by many authors [14, 15, 16, 17, 18, 19, 27, 28, 30, 34, 35]. The best local well-posedness result is due to Takaoka [30], where a gauge transformation and the Fourier restriction method are used to obtain local well-posedness in $H^s,\ s \geq \frac{1}{2}$. In [31], Takaoka showed this result is sharp in the sense that, when $s < \frac{1}{2}$, the nonlinear evolution $u(0) \mapsto u(t)$, thought of as a map from H^s to H^s for some fixed t, fails to be C^3 or even uniformly C^0 in this topology, even when t is arbitrarily close to zero and the H^s norm of the data is small (see also Bourgain [5] and Biagioni–Linares [2]). Therefore, we see that Theorem 1.1 is sharp, in the sense described above, except for the endpoint.

In [27], global well-posedness is obtained for (1) in H^1 assuming the smallness condition (2). The argument there is based on two gauge transformations performed in order to remove the derivative in the nonlinear term and the conservation of the Hamiltonian. This was improved by Takaoka [31], who proved global well-posedness in H^s for $s > \frac{32}{33}$ assuming (2). His method of proof is based on the idea of Bourgain [4, 6] of estimating separately the evolution of low frequencies and of high frequencies of the initial data. In [9], we used the "I-method" to further push the Sobolev exponent for global well-posedness down to $s > \frac{2}{3}$. The main idea of the "I-method" consists of defining a modified H^s norm permitting us to capture some nonlinear cancellations in frequency space during the evolution (1). These cancellations allow us to prove that the modified $H^s(\mathbb{R})$ norm is nearly conserved in time, and an iteration of the local result proves global well-posedness provided $s > \frac{2}{3}$. In this paper, an algorithmic procedure, first developed in the KdV context [11], is applied to better capture the cancellations in frequency space. Successive applications of the algorithm generate higher-order-in-u but lower-order-in-scaling corrections to the modified H^s norm. After one application of our algorithm, we show that the modified H^s norm *with* the generated correction terms changes less in time than the modifed H^s norm itself, so the first application of the algorithm produces an *almost conserved energy.* The improvement obtained allows us to iterate the local result and prove global well-posedness in $H^s(\mathbb{R})$ provided $s > \frac{1}{2}$. In principle, the algorithm may itself be iterated to generate a sequence of almost conserved energies giving further insight into the dynamical properties of (1). The endpoint $s = \frac{1}{2}$ is not obtained here. We speculate, however, that a further refinement of the "almost conservation method" could be a possible way to approach this question.

We conclude this section with the following remark.

Remark 1.2. Consider the one-dimensional quintic nonlinear Schrödinger

$$i\partial_t u = \partial_{xx} u + iau\bar{u}\partial_x u + ibu^2\partial_x \bar{u} + cu^3\bar{u}^2, \tag{3}$$

where a, b, and c are fixed real numbers. If $(a+b)(3a-5b)/48+c/3 < 0$ the equation in (3) is defocusing and, as was remarked in [9], the techniques used to prove Theorem 1.1

apply here too, and one can prove global well-posedness for initial data in $H^s, s > \frac{1}{2}$. Moreover, if $a = b = 0$, we expect our method to give global well-posedness[1] even below $s = 1/2$.

We should point out that Clarkson and Cosgrove [8] (see also [1]) proved that (3) fails the Painlevé test for complete integrability when

$$c \neq \frac{1}{4}b(2b - a).$$

In particular, this shows that our techniques, which do not depend on a, b, c, do not rely on complete integrability.

2. Notation and known facts. To prove Theorem 1.1 we may assume $\frac{1}{2} < s \leq \frac{2}{3}$, since for $s > \frac{2}{3}$ the result is contained in [27, 31] and [9]. Henceforth $\frac{1}{2} < s \leq \frac{2}{3}$ shall be fixed. Also, by rescaling u, we may assume $\lambda = 1$.

We use C to denote various constants depending on s; if C depends on other quantities as well, this will be indicated by explicit subscripting; e.g., $C_{\|u_0\|_2}$ will depend on both s and $\|u_0\|_2$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$, and $A \sim B$ for $cB \leq A \leq CB$, where c and C are absolute constants. We also use $A \ll B$ if $A \leq \epsilon B$, where ϵ is a very small absolute constant. We use $a+$ and $a-$ to denote expressions of the form $a + \varepsilon$ and $a - \varepsilon$, where $0 < \varepsilon \ll 1$ depends only on s.

We use $\|f\|_p$ to denote the $L^p(\mathbb{R})$ norm and $L^q_t L^r_x$ to denote the mixed norm

$$\|f\|_{L^q_t L^r_x} := \left(\int \|f(t)\|_r^q \, dt \right)^{1/q}$$

with the usual modifications when $q = \infty$.

We define the spatial Fourier transform of $f(x)$ by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx$$

and the spacetime Fourier transform $u(t, x)$ by

$$\widetilde{\mathcal{F}}(u)(\tau, \xi) := \tilde{u}(\tau, \xi) := \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x\xi + t\tau)} u(t, x) \, dt dx.$$

Note that the derivative ∂_x is conjugated to multiplication by $i\xi$ by the Fourier transform.

We shall also define D_x to be the Fourier multiplier with symbol $\langle \xi \rangle := 1 + |\xi|$. We can then define the Sobolev norms H^s by

$$\|f\|_{H^s} := \|D_x^s f\|_2 = \|\langle \xi \rangle^s \hat{f}\|_{L^2_\xi}.$$

We also define the spaces $X^{s,b}(\mathbb{R} \times \mathbb{R})$ (first introduced in the context of the Schrödinger equation in [3]; see also [22, 23]) on $\mathbb{R} \times \mathbb{R}$ by

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})} := \|\langle \xi \rangle^s \langle \tau - |\xi|^2 \rangle^b \hat{u}(\xi, \tau)\|_{L^2_\tau L^2_\xi}.$$

[1] Recall that in this case the initial value problem is locally well-posed in H^s for $s \geq 0$; see [7] and [33].

We often abbreviate $\|u\|_{s,b}$ for $\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R})}$. For any time interval I, we define the restricted spaces $X^{s,b}(I \times \mathbb{R})$ by

$$\|u\|_{X^{s,b}(I\times\mathbb{R})} := \inf\{\|U\|_{s,b} : U|_{I\times\mathbb{R}} = u\}.$$

We shall take advantage of the Strichartz estimate (see, e.g., [3])

$$\|u\|_{L_t^6 L_x^6} \lesssim \|u\|_{0,\frac{1}{2}+}, \tag{4}$$

which interpolates with the trivial estimate

$$\|u\|_{L_t^2 L_x^2} \lesssim \|u\|_{0,0}, \tag{5}$$

to give

$$\|u\|_{L_t^p L_x^p} \lesssim \|u\|_{0,\alpha(p)} \tag{6}$$

for any $p \in [2, 6]$ and $\alpha(p) = \frac{(3+)(p-2)}{4p}$. We also use

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{0,\frac{1}{2}+}, \tag{7}$$

which together with Sobolev embedding gives

$$\|u\|_{L_t^\infty L_x^\infty} \lesssim \|u\|_{\frac{1}{2}+,\frac{1}{2}+}. \tag{8}$$

The next lemma introduces two more estimates that are probably less known than the standard Strichartz estimates.

LEMMA 2.1. *For any $b > \frac{1}{2}$ and any function u for which the right-hand side is well defined, we have*

$$\|D_x^{\frac{1}{2}} u\|_{L_x^\infty L_t^2} \lesssim \|u\|_{X^{0,b}} \tag{9}$$

(smoothing effect estimate).

For any $s > \frac{1}{2}$ and $\rho \geq \frac{1}{4}$ we have

$$\|u\|_{L_x^2 L_t^\infty} \lesssim \|u\|_{X^{s,b}}, \tag{10}$$

$$\|u\|_{L_x^4 L_t^\infty} \lesssim \|u\|_{X^{\rho,b}} \tag{11}$$

(maximal function estimates).

Proof. The estimates (9), (10), and (11) come from estimating the solution $S(t)u_0$ of the linear one-dimensional Schrödinger IVP in the norm appearing in the left-hand side and from a standard argument of summation along parabolic curves; see, for example, the expository paper [13]. The smoothing effect and maximal function estimates for $S(t)u_0$ can be found, for example, in [24]. □

We also have the following improved Strichartz estimate (cf. Lemma 7.1 in [9]; see also [4, 28, 32]).

LEMMA 2.2. *For any Schwartz functions u, v with Fourier support in $|\xi| \sim R$, $|\xi| \ll R$, respectively, we have that*

$$\|uv\|_{L_t^2 L_x^2} = \|u\bar{v}\|_{L_t^2 L_x^2} \lesssim R^{-1/2} \|u\|_{0,1/2+} \|v\|_{0,1/2+}.$$

In our arguments we shall be using the trivial embedding

$$\|u\|_{s_1,b_1} \lesssim \|u\|_{s_2,b_2} \quad \text{whenever } s_1 \leq s_2, b_1 \leq b_2$$

so frequently that we will not mention this embedding explicitly.

We now give some useful notation for multilinear expressions. If $n \geq 2$ is an even integer, we define a *(spatial) n-multiplier* to be any function $M_n(\xi_1, \dots, \xi_n)$ on the hyperplane

$$\Gamma_n := \{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \xi_1 + \cdots + \xi_n = 0\},$$

which we endow with the standard measure $\delta(\xi_1 + \cdots + \xi_n)$, where δ is the Dirac delta.

If M_n is an n-multiplier and $f_1, \dots, f_n$ are functions on $\mathbb{R}$, we define the *n-linear functional* $\Lambda_n(M_n; f_1, \dots, f_n)$ by

$$\Lambda_n(M_n; f_1, \dots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \dots, \xi_n) \prod_{j=1}^{n} \hat{f}_j(\xi_j).$$

We adopt the notation

$$\Lambda_n(M_n; f) := \Lambda_n(M_n; f, \bar{f}, f, \bar{f}, \dots, f, \bar{f}).$$

Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even ξ_j indices or of the odd ξ_j indices.

If M_n is a multiplier of order n, $1 \leq j \leq n$ is an index, and $k \geq 1$ is an even integer, we define the *elongation* $\mathbf{X}_j^k(M_n)$ of M_n to be the multiplier of order $n+k$ given by

$$\mathbf{X}_j^k(M_n)(\xi_1, \dots, \xi_{n+k}) := M_n(\xi_1, \dots, \xi_{j-1}, \xi_j + \ldots + \xi_{j+k}, \xi_{j+k+1}, \dots, \xi_{n+k}).$$

In other words, $\mathbf{X}_j^k$ is the multiplier obtained by replacing ξ_j by $\xi_j + \cdots + \xi_{j+k}$ and advancing all the indices after ξ_j accordingly.

We shall often write ξ_{ij} for $\xi_i + \xi_j$, ξ_{ijk} for $\xi_i + \xi_j + \xi_k$, etc. We also write ξ_{i-j} for $\xi_i - \xi_j$, ξ_{ij-klm} for $\xi_{ij} - \xi_{klm}$, etc. Also, if $m(\xi)$ is a function defined in the frequency space, we use the notation $m(\xi_i) = m_i$, $m(\xi_{ij-k}) = m_{ij-k}$, etc.

In this paper we often use two very elementary tools: the mean value theorem (MVT) and the double mean value theorem (DMVT). While recalling the statement of the MVT will be an embarrassment, we think that doing so for the DMVT is a necessity to avoid later confusion.

LEMMA 2.3 (DMVT). *Assume $f \in C^2(\mathbb{R})$ and that $\max(|\eta|, |\lambda|) \ll |\xi|$; then*

$$|f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi)| \lesssim |f''(\theta)||\eta||\lambda|,$$

where $|\theta| \sim |\xi|$.

3. The gauge transformation, energy, and the almost conservation laws. In this section we summarize the main results presented in section 3 and 4 of [9]. Whatever is here simply stated and recalled is fully explained or proved in those sections.

We start by applying the gauge transform used in [27] in order to improve the derivative nonlinearity present in (1).

DEFINITION 3.1. *We define the nonlinear map $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by*

$$\mathcal{G}f(x) := e^{-i\int_{-\infty}^{x} |f(y)|^2 dy} f(x).$$

The inverse transform $\mathcal{G}^{-1}f$ *is then given by*

$$\mathcal{G}^{-1}f(x) := e^{i\int_{-\infty}^{x}|f(y)|^2 dy} f(x).$$

This transform is a bicontinuous map from H^s to itself for any $s \in [0,1]$.

Set $w_0 := \mathcal{G}u_0$, and $w(t) := \mathcal{G}u(t)$ for all times t. A straightforward calculation shows that the IVP (1) transforms into

$$\begin{cases} i\partial_t w + \partial_x^2 w = -iw^2\partial_x \bar{w} - \frac{1}{2}|w|^4 w, \\ w(x,0) = w_0(x), \qquad x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \tag{12}$$

Also, the smallness condition (2) becomes

$$\|w_0\|_{L^2} < \sqrt{2\pi}. \tag{13}$$

By the bicontinuity we thus see that global well-posedness of (1) in H^s is equivalent to that of (12). From [27, 30, 31], we know that both Cauchy problems are locally well-posed in $H^s, s \geq \frac{1}{2}$, and globally well-posed in H^1 assuming (13). By standard limiting arguments, we thus see that Theorem 1.1 will follow if we can show the following.

Proposition 3.2. *Let w be a global H^1 solution to* (12) *obeying* (13). *Then for any $T > 0$ and $s > \frac{1}{2}$ we have*

$$\sup_{0\leq t\leq T} \|w(t)\|_{H^s} \lesssim C_{(\|w_0\|_{H^s},T)},$$

where the right-hand side does not depend on the H^1 norm of w.

We now pass to the considerations on the energy associated with solutions of (12).

Definition 3.3. *If $f \in H^1(\mathbb{R})$, we define the energy $E(f)$ by*

$$E(f) := \int \partial_x f \partial_x \overline{f}\, dx - \frac{1}{2}\mathrm{Im}\int f\overline{f} f \partial_x \overline{f}\, dx.$$

By the Gagliardo–Nirenberg inequality we have

$$\|\partial_x f\|_2 \leq C_{\|f\|_2} E(f)^{1/2} \tag{14}$$

for any $f \in H^1$ such that $\|f\|_2 < \sqrt{2\pi}$.

By Plancherel, we write $E(f)$ using the Λ notation and Fourier transform properties as

$$E(f) = -\Lambda_2(\xi_1\xi_2; f) - \frac{1}{2}\mathrm{Im}\Lambda_4(i\xi_4; f). \tag{15}$$

Expanding out the second term using $\mathrm{Im}(z) = (z - \bar{z})/2i$, and using symmetry, we may rewrite this as

$$E(f) = -\Lambda_2(\xi_1\xi_2; f) + \frac{1}{8}\Lambda_4(\xi_{13-24}; f). \tag{16}$$

One can use the same notation to rewrite the L^2 norm as

$$\|w(t)\|_2^2 = \Lambda_2(1; w(t)).$$

Lemma 3.4 (see [27]). *If w is an H^1 solution to* (12) *for $t \in [0,T]$, then we have*

$$\|w(t)\|_2 = \|w_0\|_2$$

and

$$E(w(t)) = E(w_0)$$

for all $t \in [0,T]$.

In [9] this lemma was proved using the following general proposition (cf. [9]).

PROPOSITION 3.5. *Let $n \geq 2$ be an even integer, let M_n be a multiplier of order n, and let w be a solution of* (12). *Then*

$$\begin{aligned}\partial_t \Lambda_n(M_n; w(t)) = {} & i\Lambda_n \left(M_n \sum_{j=1}^{n} (-1)^j \xi_j^2 ; w(t) \right) \\ & - i\Lambda_{n+2} \left(\sum_{j=1}^{n} \mathbf{X}_j^2(M_n)\xi_{j+1}; w(t) \right) \\ & + \frac{i}{2}\Lambda_{n+4} \left(\sum_{j=1}^{n} (-1)^{j-1} \mathbf{X}_j^4(M_n); w(t) \right).\end{aligned} \tag{17}$$

We summarize below the idea we used to prove Proposition 3.2 for $s > \frac{2}{3}$ in [9]. Because we do not want to use the H^1 norm of w, we cannot directly use the energy $E(w(t))$ defined above. So we introduced a substitute notion of "energy" that could be defined for a less regular solution and that had a very slow increment in time. In frequency space consider an even C^∞ monotone multiplier $m(\xi)$ taking values in $[0, 1]$ such that

$$m(\xi) := \begin{cases} 1 & \text{if } |\xi| < N, \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{if } |\xi| > 2N. \end{cases} \tag{18}$$

Define the multiplier operator $I : H^s \longrightarrow H^1$ such that $\widehat{Iw}(\xi) := m(\xi)\widehat{w}(\xi)$. This operator is smoothing of order $1 - s$; indeed one has

$$\|u\|_{s_0,b_0} \lesssim \|Iu\|_{s_0+1-s,b_0} \lesssim N^{1-s}\|u\|_{s_0,b_0} \tag{19}$$

for any $s_0, b_0 \in \mathbb{R}$. Our substitute energy was defined by

$$E_N(w) := E(Iw).$$

Note that this energy makes sense even if w is only in H^s. In general, the energy $E_N(w(t))$ is not conserved in time, but we showed that the increment was very small in terms of N.

To proceed with the improvement of the "I-method," let us consider a symmetric multiplier $m(\xi)^2$ and let I be the multiplier operator associated with it. Then we write

$$E^1(w) := E(Iw).$$

Clearly, if m is the multiplier in (18), then

$$E^1(w) = E_N(w),$$

so we can think about $E^1(w)$ as the first generation of a family of modified energies. In this paper we introduce the second generation in detail, but formally the method can be used to define an infinite family of modified energies. We write

$$E^2(w) = -\Lambda_2(m_1\xi_1 m_2\xi_2, w) + \frac{1}{2}\Lambda_4\left(M_4(\xi_1, \xi_2, \xi_3, \xi_4), w\right), \tag{20}$$

[2] This eventually will be taken to be exactly the multiplier in (18).

where M_4 will be determined later. Assume now that w is a solution of (12). Because w is fixed we drop it from the definition of E^2. We are interested in the increment of this second generation of energies, and hence we compute $\frac{d}{dt}E^2$. Differentiating $\Lambda_2(m_1\xi_1 m_2\xi_2)$ using Proposition 3.5, using the identity $\xi_1 + \cdots + \xi_n = 0$ and symmetrizing, we have

$$\begin{aligned}\frac{d}{dt}\Lambda_2(m_1\xi_1 m_2\xi_2) =& - i\Lambda_2(m_1\xi_1 m_2\xi_2(\xi_1^2 - \xi_2^2)) - i\Lambda_4(m_{123}\xi_{123}m_4\xi_4\xi_2 + m_1\xi_1 m_{234}\xi_{234}\xi_3)\\ &+ \frac{i}{2}\Lambda_6(m_{12345}\xi_{12345}m_6\xi_6 - m_1\xi_1 m_{23456}\xi_{23456})\\ =& \frac{i}{2}\Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4)) + \frac{i}{6}\Lambda_6(\sigma_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)),\end{aligned}$$

where

$$\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4) = m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2 \tag{21}$$

and

$$\sigma_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \sum_{j=1}^{6}(-1)^{j-1}m_j^2\xi_j^2. \tag{22}$$

Notice that the contribution of Λ_2 is zero because the factor $(\xi_1^2 - \xi_2^2)$ is zero over the set of integration $\xi_1 + \xi_2 = 0$.

Differentiating $\Lambda_4(M_4)$, we have

$$\begin{aligned}&\frac{d}{dt}\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4))\\ =& i\Lambda_4\left(M_4\sum_{j=1}^{4}(-1)^j\xi_j^2\right)\\ &- i\Lambda_6(M_4(\xi_{123},\xi_4,\xi_5,\xi_6)\xi_2 + M_4(\xi_1,\xi_{234},\xi_5,\xi_6)\xi_3\\ &+ M_4(\xi_1,\xi_2,\xi_{345},\xi_6)\xi_4 + M_4(\xi_1,\xi_2,\xi_3,\xi_{456})\xi_5)\\ &+ \frac{i}{2}\Lambda_8(M_4(\xi_{12345},\xi_6,\xi_7,\xi_8) - M_4(\xi_1,\xi_{23456},\xi_7,\xi_8)\\ &+ M_4(\xi_1,\xi_2,\xi_{34567},\xi_8) - M_4(\xi_1,\xi_2,\xi_3,\xi_{45678}))\\ =& i\Lambda_4\left(M_4\sum_{j=1}^{4}(-1)^j\xi_j^2\right)\\ &- \frac{i}{36}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}}\Lambda_6(M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b + M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c\\ &+ M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d + M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e)\\ &+ C\sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}}\Lambda_8(M_4(\xi_{abcde},\xi_f,\xi_g,\xi_h) + M_4(\xi_a,\xi_b,\xi_{cdefg},\xi_h)\\ &- M_4(\xi_a,\xi_{bcdef},\xi_g,\xi_h) - M_4(\xi_a,\xi_b,\xi_c,\xi_{defgh})).\end{aligned}$$

Then

$$
\begin{aligned}
\frac{d}{dt}E^2(w) = &-\frac{i}{2}\Lambda_4(\sigma_4(\xi_1,\xi_2,\xi_3,\xi_4)) + \frac{i}{2}\Lambda_4\left(M_4\sum_{j=1}^{4}(-1)^j\xi_j^2\right)\\
&-\frac{i}{6}\Lambda_6(\sigma_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6))\\
&-\frac{i}{72}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}}\Lambda_6(M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b + M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c\\
&+ M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d + M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e)\\
&+ C_1\sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}}\Lambda_8(M_4(\xi_{abcde},\xi_f,\xi_g,\xi_h) + M_4(\xi_a,\xi_b,\xi_{cdefg},\xi_h)\\
&- M_4(\xi_a,\xi_{bcdef},\xi_g,\xi_h) - M_4(\xi_a,\xi_b,\xi_c,\xi_{defgh})).
\end{aligned}
$$

We abbreviate the 6-linear and the 8-linear expressions as $\Lambda_6(M_6(\xi_1,\xi_2,\dots,\xi_6))$ and $\Lambda_8(M_8(\xi_1,\xi_2,\dots,\xi_8))$. We are now ready to make our choice for M_4. From our calculations in [9], we realized that the estimates for the different pieces of Λ_n appearing in the right-hand side of $\frac{d}{dt}E_N(w)$ are easier for n larger.[3] We decided to use the freedom of choosing M_4 to cancel the Λ_4 contribution obtained above. Hence, using (21), we set

$$
M_4(\xi_1,\xi_2,\xi_3,\xi_4) = -\frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2}{\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2}, \tag{23}
$$

which in the set of integration $\xi_1+\xi_2+\xi_3+\xi_4=0$ can also be written as

$$
M_4(\xi_1,\xi_2,\xi_3,\xi_4) = -\frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2}{2\xi_{12}\xi_{14}}.
$$

Remark 3.6. If we assume that $m(\xi)=1$, then $E^2(w)=E(w)$. In fact, on the set $\xi_1+\xi_2+\xi_3+\xi_4=0$ we have

$$
\begin{aligned}
&m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2\\
&= \xi_1^2\xi_3 + \xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2\\
&= (\xi_1+\xi_3)(\xi_1\xi_3 - \xi_2\xi_4)\\
&= (\xi_1+\xi_3)(\xi_1\xi_3 + (\xi_1+\xi_3+\xi_4)\xi_4)\\
&= -(\xi_1+\xi_3)(\xi_1+\xi_4)(\xi_1+\xi_2);
\end{aligned}
$$

hence

$$
M_4(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{1}{2}(\xi_1+\xi_3) \tag{24}
$$

and

$$
E^2(w) = -\Lambda_2(\xi_1\xi_2) + \frac{1}{4}\Lambda_4(\xi_{13}),
$$

[3]Compare, for example, sections 8, 9, and 10 in [9].

which is exactly the value of $E(w)$ in (15).

Once again we recall that we assume throughout the paper that $s \in (\frac{1}{2}, \frac{2}{3}]$ and that the multiplier m is defined as in (18). To stress the fact that with this choice the energy $E^2(w)$ depends on the parameter N, we write $E^2(w) = E^2_N$. We now summarize some of the above observations in the following.

PROPOSITION 3.7. *Let w be an H^1 global solution to* (12). *Then for any $T \in \mathbb{R}$ and $\delta > 0$ we have*

$$E^2_N(w(T+\delta)) - E^2_N(w(T)) = \int_T^{T+\delta} [\Lambda_6(M_6; w(t)) + \Lambda_8(M_8; w(t))]\, dt,$$

where the multipliers M_6 and M_8 are given by

$$\begin{aligned} M_6 :=& -\frac{i}{6}\sigma_6(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) \\ & - \frac{i}{72}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}} (M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b + M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c \\ & + M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d + M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e), \\ M_8 :=& C_2 \sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} (M_4(\xi_{abcde},\xi_f,\xi_g,\xi_h) + M_4(\xi_a,\xi_b,\xi_{cdefg},\xi_h) \\ & - M_4(\xi_a,\xi_{bcdef},\xi_g,\xi_h) - M_4(\xi_a,\xi_b,\xi_c,\xi_{defgh})), \end{aligned}$$

where C_2 is an absolute constant. Furthermore, if $|\xi_j| \ll N$ for all j, then the multipliers M_6 and M_8 vanish.

We end this section with a lemma that shows the energy $E^2_N(w)$ has the same strength as $\|Iw\|_{H^1}$.

LEMMA 3.8. *Assume that w satisfies $\|w\|_{L^2} < \sqrt{2\pi}$, $\|Iw\|_{H^1} = O(1)$. Then, for $N \gg 1$,*

$$\|\partial_x Iw\|^2_{L^2} \lesssim E^2_N(w). \tag{25}$$

The proof of this lemma relies strongly on the estimate of the multiplier M_4, and it can be found in the next section.

4. Estimates for M_4 and proof of Lemma 3.8. Before we start with our estimates we recall some notation that we used in [9]. Let $n = 4$, 6, or 8 and let $\xi_1, \dots, \xi_n$ be frequencies such that $\xi_1 + \cdots + \xi_n = 0$. Define $N_i := |\xi_i|$, and $N_{ij} := |\xi_{ij}|$. We adopt the notation that

$$1 \le soprano, alto, tenor, baritone \le n$$

are the distinct indices such that

$$N_{soprano} \ge N_{alto} \ge N_{tenor} \ge N_{baritone}$$

are the highest, second highest, third highest, and fourth highest values of the frequencies $N_1, \dots, N_n$, respectively. (If there is a tie in frequencies, we break the tie arbitrarily.) Since $\xi_1 + \cdots + \xi_n = 0$, we must have $N_{soprano} \sim N_{alto}$. Also, from Proposition 3.7 we see that M_n vanishes unless $N_{soprano} \gtrsim N$.

In this section whenever we write $\max|f(\theta)|$ for a function f we understand that the maximum is taken for $|\theta| \sim N_{soprano}$.

LEMMA 4.1. *Assume M_4 is the multiplier defined in* (23) *and $m(\xi)$ is as in* (18). *Then*

$$|M_4(\xi_1, \dots, \xi_4)| \lesssim m^2(N_{soprano})N_{soprano}. \tag{26}$$

Proof. We observe that to prove (26) it suffices to prove

$$|\sigma_4(\xi_1, \dots, \xi_4)| \lesssim |\xi_{12}||\xi_{12}|m^2(N_{soprano})N_{soprano}.$$

Without loss of generality we may assume that $N_{soprano} = N_1$. By symmetry we can assume that $|\xi_{12}| \leq |\xi_{14}|$. We divide the analysis into two cases: Case (a) when $N_1 \lesssim |\xi_{14}|$ and Case (b) when $|\xi_{14}| \ll N_1$.

Case (a). We write

$$\begin{aligned} |\sigma_4(\xi_1, \dots, \xi_4)| &= |m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2(-\xi_{12} - \xi_3) + m_3^2\xi_3^2\xi_1 + m_{12+3}^2\xi_{12+3}^2\xi_2| \\ &= |\xi_3(m_1^2\xi_1^2 - m_{1-12}^2\xi_{1-12}^2) + \xi_1(m_3^2\xi_3 - m_{3+12}^2\xi_{3+12}^2) \\ &\quad - \xi_{12}(m_2^2\xi_2^2 - m_{12+3}^2\xi_{12+3}^2)|. \end{aligned} \tag{27}$$

Then the MVT shows that

$$|\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim |\xi_{12}|N_1 \max|(m(\xi)^2\xi^2)'|, \tag{28}$$

where $|\xi| \lesssim N_1$. Now it is easy to see that for m defined in (18)

$$(m^2(\xi)\xi^2)' \sim m^2(\xi)\xi$$

and that the function $m^2(\xi)\xi$ is nondecreasing. Then (28) immediately gives (26).

Case (b). We first write σ_4 so that the DMVT in Lemma 2.3 can be applied. For simplicity we write $m^2(\xi)\xi^2 = f(\xi)$. Then in the set $\xi_1 + \cdots + \xi_4 = 0$ we have

$$\begin{aligned} \sigma_4(\xi_1, \dots, \xi_4) &= f(\xi_1)\xi_3 + f(\xi_2)\xi_4 + f(\xi_3)\xi_1 f(\xi_4)\xi_2 \\ &= \xi_3[f(\xi_1) - f(\xi_2)] + \xi_1[f(\xi_3) - f(-\xi_4)] - \xi_{12}[f(\xi_2) - f(-\xi_4)] \\ &= \xi_3[f(\xi_1) - f(\xi_2) + f(\xi_3) - f(-\xi_4)] \\ &\quad + (\xi_1 - \xi_3)[f(\xi_3) - f(\xi_3 - \xi_{12})] - \xi_{12}[f(\xi_2) - f(-\xi_4)] \\ &= \xi_3[f(\xi_1 - \xi_{12} - \xi_{14}) - f(\xi_1 - \xi_{12}) - f(\xi_1 - \xi_{14}) + f(\xi_1)] \\ &\quad + (-\xi_3 + \xi_1)[f(\xi_3) - f(\xi_3 - \xi_{12})] - \xi_{12}[f(\xi_2) - f(\xi_2 + \xi_{14} - \xi_{12})], \end{aligned}$$

where we often used the fact that $f(\xi)$ is an even function. Using the DMVT in the first term of the right-hand side of the inequality and the MVT in the remaining two terms we obtain

$$\sigma_4(\xi_1, \dots, \xi_4) \lesssim |\xi_1||f''(\theta)||\xi_{12}||\xi_{14}| + |\xi_{12}| \max|f'|(|\xi_{3-1}| + |\xi_{14}| + |\xi_{12}|), \tag{29}$$

where $|\theta| \sim N_1$. Now observe that

$$|\xi_{3-1}| = |\xi_{12} + \xi_{14}| \lesssim |\xi_{14}|$$

and that $|f''(\theta)| \lesssim m(N_1)^2$, so inserting (29) in the definition of M_4 we obtain (26). $\square$

We need two more local estimates for M_4.

LEMMA 4.2.

- *Assume that* $|\xi_1| \sim |\xi_3| \gtrsim N \gg |\xi_2|, |\xi_4|$*; then*

$$|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \lesssim m(N_{soprano})^2 N_{tenor}. \tag{30}$$

- *Assume that* $|\xi_1| \sim |\xi_2| \gtrsim N \gg |\xi_3|, |\xi_4|$*; then*

$$M_4(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{m_1^2\xi_2^2}{2\xi_1} + R(\xi_1,\dots,\xi_4), \tag{31}$$

where

$$|R(\xi_1,\dots,\xi_4)| \lesssim N_{tenor}.$$

Proof. The first part of the lemma follows from the MVT. In fact,

$$\begin{aligned}\left|\frac{m_1^2\xi_1^2\xi_3 + \xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + \xi_4^2\xi_2}{\xi_{12}\xi_{14}}\right| &\lesssim \frac{|\xi_1\xi_3\xi_{13}| \max|(m(\xi)^2\xi)'| + |\xi_{24}\xi_2\xi_4|}{|\xi_1|^2}\\ &\lesssim m(N_{soprano})^2 N_{tenor},\end{aligned}$$

where again we used that $|(m(\xi)^2\xi)'| \sim |m(\xi)\xi|$.

To prove the second part of the lemma we use the identity

$$\frac{1}{\xi_{14}} = \frac{1}{\xi_1} - \frac{\xi_4}{\xi_{14}}\frac{1}{\xi_1},$$

and we write

$$-2M_4(\xi_1,\xi_2,\xi_3,\xi_4) + \frac{m_1^2\xi_2^2}{\xi_1} = R_1(\xi_1,\dots,\xi_4) + R_2(\xi_1,\dots,\xi_4),$$

where

$$\begin{aligned}R_1(\xi_1,\dots,\xi_4) &= \frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2 + m_1^2\xi_2^2\xi_{12}}{\xi_{12}\xi_1},\\ R_2(\xi_1,\dots,\xi_4) &= -\frac{\xi_4}{\xi_{14}}\frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2}{\xi_{12}\xi_1}.\end{aligned}$$

We first estimate R_1:

$$\begin{aligned}R_1(\xi_1,\dots,\xi_4) &= \frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + \xi_3^2\xi_1 + \xi_4^2\xi_2 - m_1^2\xi_2^2\xi_{34}}{\xi_{12}\xi_1}\\ &= \frac{m_1^2\xi_3(\xi_1^2 - \xi_2^2) + \xi_2^2\xi_4(m_2^2 - m_1^2) + \xi_3^2(\xi_1 + \xi_2) + \xi_2(\xi_4^2 - \xi_3^2)}{\xi_{12}\xi_1}, \text{ and}\end{aligned}$$

hence, by the MVT,

$$|R_1(\xi_1,\dots,\xi_4)| \lesssim N_{tenor}.$$

On the other hand,

$$R_2(\xi_1,\dots,\xi_4) = -\frac{\xi_4}{\xi_{14}}\frac{m_1^2\xi_1^2(\xi_3+\xi_4) + (m_2^2\xi_2^2 - m_1^2\xi_1^2)\xi_4 + \xi_3^2\xi_{12} + \xi_2\xi_{34}\xi_{3-4}}{\xi_{12}\xi_1}, \text{ and}$$

hence, again by the MVT,

$$|R_2(\xi_1,\dots,\xi_4)| \lesssim N_{tenor}. \qquad \square$$

Proof of Lemma 3.8.

Proof. We rewrite $E_N^2(w)$ as

$$E_N^2(w) = -\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{8}\Lambda_4(\xi_{13-24}m_1m_2m_3m_4) + \frac{1}{8}\Lambda_4(4M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \xi_{13-24}m_1m_2m_3m_4).$$

In Lemma 3.6 of [9] we proved the estimate

$$\|\partial_x Iw\|_{L^2}^2 \lesssim -\Lambda_2(m_1\xi_1m_2\xi_2) + \frac{1}{8}\Lambda_4(\xi_{13-24}m_1m_2m_3m_4)$$

for $\|Iw\|_{L^2} < \sqrt{2\pi}$. Hence we have only to show that

$$|\Lambda_4(4M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \xi_{13-24}m_1m_2m_3m_4)| \lesssim O\left(\frac{1}{N^\alpha}\right)\|Iw\|_{H^1}^4 \tag{32}$$

for some $\alpha > 0$.

We first perform a Littlewood–Paley decomposition of the four factors w so that the ξ_i are essentially the constants N_i, $i = 1, \dots, 4$. To recover the sum at the end we borrow a $N_{soprano}^{-\epsilon}$ from the large denominator $N_{soprano}$ and often this will not be mentioned.

If all $|\xi_j|$ are less than $\frac{N}{100}$, the left-hand side of (32) vanishes thanks to (23). Therefore, we may assume $N_{soprano} \gtrsim N$. Also note $N_{alto} \gtrsim N$ on the set $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$. Then it is obvious that

$$|\Lambda_4(\xi_{13-24}m_1m_2m_3m_4)| \lesssim \frac{1}{N}\|Iw\|_{H^1}^2\|Iw\|_{L^\infty}^2 \lesssim \frac{1}{N}\|Iw\|_{H^1}^4.$$

Next we control the contribution of $\Lambda_4(M_4)$ in (32). By (26), we have

$$|\Lambda_4(M_4(\xi_1,\xi_2,\xi_3,\xi_4))| \lesssim \frac{1}{N_{soprano}^{1-}m(N_{baritone})^2N_{baritone}}\|Iw\|_{H^1}^4 \lesssim \frac{1}{N^{1-}}\|Iw\|_{H^1}^4,$$

where again we used the fact that $m^2(\xi)\xi$ is nondecreasing. □

5. Local estimates. This section contains a refinement of the results presented in section 5 of [9]. We start with the main result.

THEOREM 5.1. *Let w be a H^1 global solution to* (12) *and let $T \in \mathbb{R}$ be such that*

$$\|Iw(T)\|_{H^1} \le C_0$$

for some $C_0 > 0$. Then we have

$$\|Iw\|_{X^{1,b}([T,T+\delta]\times\mathbb{R})} \lesssim 1$$

for any $\frac{1}{2} < b < \frac{3}{4}$ and for some $\delta > 0$ depending on C_0.

Remark 5.2. This theorem is stronger than the corresponding Theorem 5.1 in [9] because b can be arbitrarily close to $\frac{3}{4}$, and this is essential to obtain our sharp global well-posedness result.

As explained in [9] the proof of Theorem 5.1 is a consequence of the following multilinear estimates.

LEMMA 5.3. *For the Schwartz function w and $\frac{1}{2} < b < \frac{3}{4}$, $b' < \frac{3}{4}$, we have*

$$\|I(w\partial_x \overline{w} w)\|_{1,b'-1} \lesssim \|Iw\|^2_{1,\frac{1}{2}+}\|Iw\|_{1,b}, \tag{33}$$

$$\|I(w\overline{w}w\overline{w}w)\|_{1,b'-1} \lesssim \|Iw\|^5_{1,\frac{1}{2}+}. \tag{34}$$

Proof. The proof of (34) follows from the same arguments used to prove (17) in [9], and we do not present it here again. The proof of (33) on the other hand is more delicate than the one given in [9] for (16), so we decided to give all the details. By standard duality arguments in L^2 and renormalization, it is easy to see that (33) is equivalent to

$$\int_* \frac{m_4\langle\xi_4\rangle|\xi_2|\langle\tau_4+\xi_4^2\rangle^{b'-1}}{\sum_{i=1}^3\langle\tau_i+(-1)^i\xi_i^2\rangle^{b-\frac{1}{2}-}\prod_{j=1}^3 m_j\langle\xi_j\rangle\langle\tau_j+(-1)^j\xi_j^2\rangle^{\frac{1}{2}+}}\prod_{j=1}^4 F_j(\tau_j,\xi_j) \lesssim \prod_{j=1}^4\|F_j\|_{L^2}, \tag{35}$$

where all functions F_j are real-valued and nonnegative. If

$$\frac{m_4\langle\xi_4\rangle|\xi_2|}{\prod_{j=1}^3 m_j\langle\xi_j\rangle} \lesssim 1, \tag{36}$$

then the L^2 estimate (5) for F_4 and the Strichartz estimate (6) with $p=6$ for F_1, F_2, F_3 automatically shows (35) for $b > \frac{1}{2}$, $b' \leq 1$. Then we may assume

$$\frac{m_4\langle\xi_4\rangle|\xi_2|}{\prod_{j=1}^3 m_j\langle\xi_j\rangle} \gg 1,$$

which, one can easily check, can happen only when

$$|\xi_2| \gg 1, \quad |\xi_{12}| \gg 1, \quad |\xi_{14}| \gg 1.$$

We recall (cf. [3] and [9]) the fundamental inequality

$$|\xi_{12}\xi_{14}| \lesssim \max_{j=1,2,3,4}\{\langle\tau_j+(-1)^j\xi_j^2\rangle\}. \tag{37}$$

Then we proceed with a case by case analysis: Case (a) if $\max_{j=1,2,3}\{\langle\tau_4+\xi_4^2\rangle, \langle\tau_j+(-1)^j\xi_j^2\rangle\} = \langle\tau_4+\xi_4^2\rangle$ and Case (b) if $\max_{j=1,2,3}\{\langle\tau_4+\xi_4^2\rangle, \langle\tau_j+(-1)^j\xi_j^2\rangle\} = \langle\tau_i+(-1)^j\xi_i^2\rangle$ for some $i=1,2,3$.

- *Case* (a). In this case we replace in the denominator $\langle\tau_4+\xi_4^2\rangle^{1-b'}$ with $(\langle\xi_{12}\rangle\langle\xi_{14}\rangle)^{1-b'}$. Then, using the same argument that in [9] led us from (16) to (18), we can show that (35) is equivalent to

$$\int_* \frac{\langle\xi_4\rangle^s\langle\xi_2\rangle^{1-s}}{(\langle\xi_{12}\rangle\langle\xi_{14}\rangle)^{1-b'}\langle\xi_1\rangle^s\langle\xi_3\rangle^s\prod_{j=1}^3\langle\tau_j+(-1)^j\xi_j^2\rangle^{\frac{1}{2}+}}\prod_{j=1}^4 F_j(\tau_j,\xi_j) \lesssim \prod_{j=1}^4\|F_j\|_{L^2}. \tag{38}$$

To have an idea of the "numerics" involved while proceeding with the proof, the reader should keep in mind that the interesting case is when $s = \frac{1}{2}+$ and $1 - b' = \frac{1}{4}+$. Since $\xi_{14} = -\xi_{32}$, by symmetry, we may assume that $|\xi_1| \geq |\xi_3|$. Then, using the fact that $\xi_4 = -\xi_3 - \xi_{12}$, we can write

$$\frac{\langle \xi_4 \rangle^s \langle \xi_2 \rangle^{1-s}}{(\langle \xi_{12} \rangle \langle \xi_{14} \rangle)^{1-b'} \langle \xi_1 \rangle^s \langle \xi_3 \rangle^s} = A_1 + A_2, \tag{39}$$

where

$$A_1 \lesssim \frac{\langle \xi_2 \rangle^{1-s}}{(\langle \xi_{12} \rangle \langle \xi_{14} \rangle)^{1-b'} \langle \xi_1 \rangle^s},$$
$$A_2 \lesssim \frac{\langle \xi_{12} \rangle^{s-1+b'} \langle \xi_2 \rangle^{1-s}}{\langle \xi_{14} \rangle^{1-b'} \langle \xi_1 \rangle^s \langle \xi_3 \rangle^s}.$$

We now write $\xi_{12} = -\xi_{14} - \xi_3 + \xi_1$, and we write

$$A_2 = A_2^1 + A_2^2 + A_2^3,$$

where

$$A_2^1 \lesssim \frac{\langle \xi_2 \rangle^{1-s}}{\langle \xi_{14} \rangle^{2(1-b')-s} \langle \xi_1 \rangle^s \langle \xi_3 \rangle^s},$$
$$A_2^2 \lesssim \frac{\langle \xi_2 \rangle^{1-s}}{\langle \xi_{14} \rangle^{1-b'} \langle \xi_3 \rangle^{1-b'} \langle \xi_1 \rangle^s},$$
$$A_2^3 \lesssim \frac{\langle \xi_2 \rangle^{1-s}}{\langle \xi_{14} \rangle^{1-b'} \langle \xi_1 \rangle^{1-b'} \langle \xi_3 \rangle^s}.$$

It is now easy to see that, for $1 - b' \geq \frac{s}{2}$,

$$A_1, A_2^i(\xi_1, \xi_2, \xi_3) \lesssim \frac{\langle \xi_2 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^{\frac{s}{2}} \langle \xi_3 \rangle^{\frac{s}{2}}} \quad \text{for all } i = 1, 2, 3.$$

Then by (9) and (11) we obtain

$$\int_* \frac{\langle \xi_4 \rangle^s \langle \xi_2 \rangle^{1-s}}{(\langle \xi_{12} \rangle \langle \xi_{14} \rangle)^{1-b'} \langle \xi_1 \rangle^s \langle \xi_3 \rangle^s \prod_{j=1}^3 \langle \tau_j + (-1)^j \xi_j^2 \rangle^{\frac{1}{2}+}} \prod_{j=1}^4 F_j(\tau_j, \xi_j)$$
$$\lesssim \|\widetilde{\mathcal{F}}^{-1}(F_4)\|_{L^2_{xt}} \left\| \widetilde{\mathcal{F}}^{-1} \left(\frac{\langle \xi \rangle^{\frac{1}{2}}}{\langle \tau + \xi^2 \rangle^{\frac{1}{2}+}} F_2 \right) \right\|_{L^\infty_x L^2_t} \|\widetilde{\mathcal{F}}^{-1} \left(\frac{\langle \xi \rangle^{-\frac{s}{2}}}{\langle \tau - \xi^2 \rangle^{\frac{1}{2}+}} F_3 \right) \|_{L^4_x L^\infty_t}$$
$$\times \|\widetilde{\mathcal{F}}^{-1} \left(\frac{\langle \xi \rangle^{-\frac{s}{2}}}{\langle \tau - \xi^2 \rangle^{\frac{1}{2}+}} F_1 \right) \|_{L^4_x L^\infty_t} \lesssim \prod_{j=1}^4 \|F_j\|_{L^2}.$$

- *Case* (b). In this case we borrow a power $\alpha = b' - \frac{1}{2}+$ from the large denominator, and we reduce our estimate to

$$\int_* \frac{\langle \xi_4 \rangle^s \langle \xi_2 \rangle^{1-s}}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s \prod_{j=1}^4 \langle \tau_j + (-1)^j \xi_j^2 \rangle^{\frac{1}{2}+}} \prod_{j=1}^4 F_j(\tau_j, \xi_j) \lesssim \prod_{j=1}^4 \|F_j\|_{L^2}.$$

Again by symmetry we can assume that $|\xi_1| \geq |\xi_3|$. We first observe that if the exponent of $\langle \xi_4 \rangle$ were $\frac{1}{2}$, then we could simply use (9) for the function F_2 and (10) for the function F_4 to obtain the estimate as we did above. However, in our case $s > \frac{1}{2}$, so we have to do a bit more work. We subdivide the analysis into subcases.

– *Subcase* (1). $|\xi_4| \lesssim |\xi_2|$. In this case we can write

$$\langle \xi_4 \rangle^s \langle \xi_2 \rangle^{1-s} \lesssim \langle \xi_4 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}},$$

and we can indeed use (9) and (10).

– *Subcase* (2). $|\xi_2| \ll |\xi_4|$. Because we assumed that $|\xi_3| \leq |\xi_1|$ and we are on the set $\xi_1 + \cdots + \xi_4 = 0$, it follows that $|\xi_4| \lesssim |\xi_1|$. Then the estimate becomes

$$\begin{aligned} &\int_* \frac{\langle \xi_2 \rangle^{1-s}}{\langle \xi_3 \rangle^s \prod_{j=1}^4 \langle \tau_j + (-1)^j \xi_j^2 \rangle^{\frac{1}{2}+}} \prod_{j=1}^4 F_j(\tau_j, \xi_j) \\ &\lesssim \left\| \widetilde{\mathfrak{F}}^{-1} \left(\frac{1}{\langle \tau + \xi^2 \rangle^{\frac{1}{2}+}} F_4 \right) \right\|_{L^4_{xt}} \left\| \widetilde{\mathfrak{F}}^{-1} \left(\frac{1}{\langle \tau - \xi^2 \rangle^{\frac{1}{2}+}} F_1 \right) \right\|_{L^4_{xt}} \\ &\times \left\| \widetilde{\mathfrak{F}}^{-1} \left(\frac{\langle \xi \rangle^{1-s}}{\langle \tau + \xi^2 \rangle^{\frac{1}{2}+}} F_2 \right) \right\|_{L^\infty_x L^2_t} \\ &\times \left\| \widetilde{\mathfrak{F}}^{-1} \left(\frac{\langle \xi \rangle^{-s}}{\langle \tau - \xi^2 \rangle^{\frac{1}{2}+}} F_3 \right) \right\|_{L^2_x L^\infty_t} \lesssim \prod_{j=1}^4 \|F_j\|_{L^2}, \end{aligned}$$

thanks to (6) for $p = 2$, (9), and (10). □

6. Proof of Proposition 3.2. Based on Lemma 3.8, Theorem 5.1, and the arguments presented in [9, section 6] (see also the comments in [9, section 7]), the only result that one needs to obtain is the following.

LEMMA 6.1. *For any Schwartz function w, we have*

$$\left| \int_T^{T+\delta} \Lambda_n(M_n; w(t))\, dt \right| \lesssim \frac{1}{N^{2-}} \|Iw\|^n_{X^{1,3/4-}([T,T+\delta]\times\mathbb{R})} \tag{40}$$

for $n = 6, 8$, where M_6, M_8 are defined in Proposition 3.7.

In [9] we were only able to obtain a decay of N^{-1+}, which is why we could only prove global well-posedness for $s > \frac{2}{3}$.

The proof of this lemma is a corollary of the four lemmas that follow in this section.

LEMMA 6.2 ($n = 8$).

$$|M_8(\xi_1, \xi_2, \ldots, \xi_8)| \lesssim N_{soprano} m^2(N_{soprano}).$$

This is a simple consequence of Lemma 4.1. We now turn to the estimate of $\frac{d}{dt}E^2(Iw)$ involving Λ_8.

LEMMA 6.3.

$$\left| \int_T^{T+\delta} \int \Lambda_8(M_8(\xi_1, \xi_2, \ldots, \xi_8))\, dt \right| \lesssim \frac{1}{N^{2-}} \|Iw\|^8_{1,\frac{1}{2}+}.$$

Proof. As in the proof of Lemma 3.8, also in this case we first perform a Littlewood–Paley decomposition of the eight factors w so that the ξ_i essentially are the constants N_i, $i = 1, \dots, 8$. To recover the sum at the end we borrow a $N_{soprano}^{-\epsilon}$ from the large denominator $N_{soprano}$. Often this will not be mentioned, and it will only be recorded at the end by paying a price equivalent to N^{0+}. Below we often use the set of indices $R = \{soprano, alto, tenor\}$. Again we proceed by analyzing different cases.

- *Case* (a). $N_{soprano} \sim N_{tenor}$. By Lemma 6.2 and the fact that $m(\xi)\langle\xi\rangle^{\frac{1}{2}}$ is increasing, we have

$$\left|\int_T^{T+\delta}\int \Lambda_8(M_8(\xi_1,\xi_2,\dots,\xi_8))\,dt\right|$$
$$\lesssim \sum_{R,j}\frac{N_{soprano}}{m(N_{tenor})}\|D_xIw_{soprano}\|_{L^6}\|D_xIw_{alto}\|_{L^6}\|D_xIw_{tenor}\|_{L^6}$$
$$\prod_{j,k\notin R}\|D_xIw_j\|_{L^6}\|D_x^{1/2-}Iw_k\|^2_{L^\infty} \lesssim \frac{1}{N^{2-}}\|Iw\|^8_{1,\frac{1}{2}+}.$$

- *Case* (b). $N_{soprano} \gg N_{tenor}$. By Lemma 2.2, and again the monotonicity of $m(\xi)\langle\xi\rangle^{1/2}$, we have

$$\left|\int_T^{T+\delta}\int \Lambda_8(M_8(\xi_1,\xi_2,\dots,\xi_8))\,dt\right|$$
$$\lesssim N_{soprano}\|Iw_{soprano}w_{tenor}\|_{L^2}\|Iw_{alto}w_{baritone}\|_{L^2}$$
$$\times\ \|w\|^4_{L^\infty} \lesssim \frac{1}{N^{2-}}\|Iw\|^8_{1,\frac{1}{2}+}. \qquad \square$$

LEMMA 6.4 ($n = 6$).

- *If $N_{tenor} \gtrsim N$, we have*

$$|M_6(\xi_1,\xi_2,\dots,\xi_6)| \lesssim m(N_{soprano})^2N^2_{soprano}. \tag{41}$$

- *If $N_{tenor} \ll N$, we have*

$$|M_6(\xi_1,\xi_2,\dots,\xi_6)| \lesssim N_{soprano}N_{tenor}. \tag{42}$$

Proof. If $N_{soprano} \ll N$, M_6 vanishes. Then we may assume $N_{soprano} \gtrsim N$. Also in the set $\xi_1 + \cdots + \xi_6 = 0$ we have $N_{alto} \sim N_{soprano}$.

The proof of (41) follows from (26). The proof of (42) is more delicate. By symmetry we assume $soprano = 1$, $N_1 \geq N_3 \geq N_5$, $N_2 \geq N_4 \geq N_6$. Again we analyze different cases.

- *Case* (a). $alto = 2$. The MVT shows

$$|\sigma_6(\xi_1,\xi_2,\dots,\xi_6)| \lesssim m(N_1)^2N_1N_{12} + m(N_{tenor})^2N^2_{tenor}$$
$$\lesssim m(N_{soprano})^2N_{soprano}N_{tenor}.$$

Next we estimate the second term in M_6:

$$\sum(M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b + M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c + M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d + M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e).$$

Again by (26) one has that

$$|M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_g| \lesssim m(N_{soprano})^2 N_{soprano} N_{tenor} \tag{43}$$

for every $a,\dots,g \in \{1,\dots,6\}$ and $g \neq soprano, alto$. Thus we have only to consider the contributions

$$\begin{aligned}
&\left|\sum_{(a,e)\in\{3,5\}}\sum_{(d,f)\in\{4,6\}} M_4(\xi_{a21},\xi_d,\xi_e,\xi_f)\xi_2 + M_4(\xi_a,\xi_{21d},\xi_e,\xi_f)\xi_1\right| \\
&+\left|\sum_{(a,c)\in\{3,5\}}\sum_{(d,f)\in\{4,6\}} M_4(\xi_a,\xi_{12b},\xi_e,\xi_f)\xi_1 + M_4(\xi_a,\xi_b,\xi_{12e},\xi_f)\xi_2\right| \\
&+\left|\sum_{(a,c)\in\{3,5\}}\sum_{(d,f)\in\{4,6\}} M_4(\xi_a,\xi_b,\xi_{12c},\xi_f)\xi_2 + M_4(\xi_a,\xi_b,\xi_c,\xi_{12f})\xi_1\right| \\
&+\left|\sum_{(a,e)\in\{3,5\}}\sum_{(d,f)\in\{4,6\}} M_4(\xi_{a2c},\xi_d,\xi_1,\xi_f)\xi_2 + M_4(\xi_a,\xi_2,\xi_c,\xi_{d1f})\xi_1\right| = \sum_{i=1}^{4} I_i.
\end{aligned}$$

Observe first that all the variables appearing in the function M_4 in $\sum_{i=1}^{3} I_i$ are strictly smaller that $\frac{N}{2}$, and hence by (24) it follows that

$$\sum_{i=1}^{3} I_i \lesssim N_{soprano} N_{tenor}.$$

To estimate I_4 we use (30) and the symmetry of M_4. Then also in this case we obtain

$$I_4 \lesssim N_{soprano} N_{tenor}.$$

- *Case* (b). $alto = 3$. In this case we need some cancellation between the large terms coming from $\sigma_6(\xi_1,\dots,\xi_6)$ and the large terms of the sum of the M_4. From (43) it is easy to see that one needs to estimate only

$$\begin{aligned}
\widetilde{M}_6(\xi_1,\dots,\xi_6) = &-\frac{1}{6}(m_1^2\xi_1^2 + m_3^2\xi_3^2) \\
&-\frac{\xi_1}{36}\left(\sum_{(b,d,f)\in\{2,4,6\}} M_4(\xi_a,\xi_{b1d},\xi_3,\xi_f) + M_4(\xi_a,\xi_b,\xi_3,\xi_{d1f})\right) \\
&-\frac{\xi_3}{36}\left(\sum_{(b,d,f)\in\{2,4,6\}} M_4(\xi_a,\xi_b,\xi_1,\xi_{d3f}) + M_4(\xi_a,\xi_{b3d},\xi_1,\xi_f)\right).
\end{aligned}$$

We now use (31) and the symmetries of M_4 to write

$$\begin{aligned}
\widetilde{M_6}(\xi_1,\dots,\xi_6) = &-\frac{1}{6}(m_1^2\xi_1^2+m_3^2\xi_3^2)\\
&-\frac{\xi_1}{72}\left(\sum_{(b,d,f)\in\{2,4,6\}}\frac{m_3^2(\xi_{b1d}^2+\xi_{b1f}^2)}{\xi_3}\right)+O(N_{soprano}N_{tenor})\\
&-\frac{\xi_3}{72}\left(\sum_{(b,d,f)\in\{2,4,6\}}\frac{m_1^2(\xi_{d3f}^2+\xi_{b3d}^2)}{\xi_1}\right)+O(N_{soprano}N_{tenor})\\
=&-\frac{1}{6}(m_1^2\xi_1^2+m_3^2\xi_3^2)\\
&+\frac{1}{72}\left(\sum_{(b,d,f)\in\{2,4,6\}}m_3^2(\xi_{b1d}^2+\xi_{b1f}^2)\right)+O(N_{soprano}N_{tenor})\\
&+\frac{1}{72}\left(\sum_{(b,d,f)\in\{2,4,6\}}m_1^2(\xi_{d3f}^2+\xi_{b3d}^2)\right)+O(N_{soprano}N_{tenor})\\
=&-\frac{1}{72}m_3^2\sum_{(b,d,f)\in\{2,4,6\}}(\xi_3^2-\xi_{1bd}^2)+(\xi_3^2-\xi_{1fb}^2)\\
&-\frac{1}{72}m_1^2\sum_{(b,d,f)\in\{2,4,6\}}(\xi_1^2-\xi_{3bf}^2)+(\xi_1^2-\xi_{b3d}^2)\\
&+O(N_{soprano}N_{tenor}),
\end{aligned}$$

and now it is clear that also in this case

$$|\widetilde{M_6}(\xi_1,\dots,\xi_6)|\lesssim N_{soprano}N_{tenor}. \qquad \square$$

Lemma 6.5.

$$\left|\int_T^{T+\delta}\int\Lambda_6(M_6(\xi_1,\xi_2,\dots,\xi_6))\,dt\right|\lesssim\frac{1}{N^{2-}}\|Iw\|^6_{1,\frac{3}{4}-}. \tag{44}$$

Proof. Also in this case one uses a Littlewood–Paley decomposition to start. We divide the proof into three different cases: Case (a) when $N_{baritone}\gtrsim N$, Case (b) when $N_{soprano}\geq N_{tenor}\gtrsim N\gg N_{baritone}$, and Case (c) when $N_{soprano}\sim N_{alto}\gtrsim N\gg N_{tenor}$. Below we often use the two sets of indices $S=\{soprano, alto, tenor, baritone\}$ and $R=\{soprano, alto, tenor\}$. We also recall that thanks to the fact that $m(\xi)|\xi|^{\frac{1}{2}}$ is not decreasing,

$$m(\xi)(1+|\xi|)\gtrsim\begin{cases}N & \text{if } |\xi|>\frac{N}{2},\\ 1 & \text{if } |\xi|\leq\frac{N}{2}.\end{cases} \tag{45}$$

- *Case* (a). $N_{baritone} \gtrsim N$. By Lemma 6.4, (45), and the Strichartz estimate (4), we have

$$\left| \int_T^{T+\delta} \int \Lambda_6(M_6(\xi_1, \xi_2, \dots, \xi_6))\, dt \right|$$
$$\lesssim \sum_{S,j} \frac{1}{N_{soprano} N} m(N_{soprano}) N_{soprano} \|w_{soprano}\|_{L^6}$$
$$\times\, m(N_{alto}) N_{alto} \|w_{alto}\|_{L^6} m(N_{tenor}) N_{tenor} \|w_{tenor}\|_{L^6}$$
$$\times\, m(N_{baritone}) N_{baritone} \|w_{baritone}\|_{L^6}$$
$$\times \prod_{j \notin S} \|Iw_j\|_{L^6} \lesssim \frac{1}{N^{2-}} \|Iw\|^6_{1,\frac{1}{2}+}.$$

- *Case* (b). $N_{soprano} \geq N_{tenor} \gtrsim N \gg N_{baritone}$. This is the only part in which we need to use the space $X^{1,b}$ with $b \sim \frac{3}{4}-$. By Lemma 6.4 and (45) we have

$$\left| \int_T^{T+\delta} \int \Lambda_6(M_6(\xi_1, \xi_2, \dots, \xi_6))\, dt \right|$$
$$\lesssim \sum_{R,j} \frac{1}{N_{soprano}} m(N_{soprano}) N_{soprano} \|w_{soprano} w_{baritone}\|_{L^2}$$
$$\times\, m(N_{alto}) N_{alto} \|w_{alto}\|_{L^6} m(N_{tenor}) N_{tenor} \|w_{tenor}\|_{L^6} \prod_{j \notin R} \|D_x^{\frac{1}{2}} I w_j\|_{L^{12}}.$$

Using Lemma 2.2 and (45), it is easy to see that

$$m(N_{soprano}) N_{soprano} \|w_{soprano} w_{baritone}\|_{L^2}$$
$$\lesssim N^{-1/2} \|Iw_{soprano}\|_{X^{1,\frac{1}{2}+}} \|Iw_{baritone}\|_{X^{1,\frac{1}{2}+}}.$$

Also by the Sobolev inequalities and again (45)

$$\prod_{j \notin R} \|D_x^{\frac{1}{2}} I w_j\|_{L^{12}} \lesssim \prod_{j \notin R} \|Iw_j\|_{X^{1,\frac{1}{2}+}}.$$

Collecting the above estimates one obtains

$$\left| \int_T^{T+\delta} \int \Lambda_6(M_6(\xi_1, \xi_2, \dots, \xi_6))\, dt \right| \lesssim \frac{1}{N^{\frac{3}{2}-}} \|Iw\|^6_{1,\frac{1}{2}+}.$$

Unfortunately, the decay $N^{-\frac{3}{2}+}$ is not enough for our purposes. Because the local estimate allow us to handle terms of type $\|Iw\|_{1,\frac{3}{4}-}$ (see section 5), we take advantage of the extra denominators. To see this we use the identity

$$\xi_1 + \cdots + \xi_4 = 0 \implies \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = 2\xi_{12}\xi_{14},$$

proved in [9]. We consider only the case $N_1 = N_{soprano}, N_2 = N_{alto}$, and $N_3 = N_{tenor}$. Indeed if $N_5 = N_{tenor}$ the argument is easier. Then in the set $\xi_1 + \cdots + \xi_6 = 0$ we write

$$\begin{aligned}\sum_{i=1}^{6}(-1)^{i-1}\xi_i^2 &= \xi_1^2 - \xi_2^2 + \xi_3^2 - (\xi_4+\xi_5+\xi_6)^2 \\ &\quad +(\xi_4+\xi_5+\xi_6)^2 - \xi_4^2 + \xi_5^2 - \xi_6^2 \\ &= 2\xi_{12}\xi_{1456} + (\xi_4+\xi_5+\xi_6)^2 - \xi_4^2 + \xi_5^2 - \xi_6^2,\end{aligned}$$

which implies that

$$\left|\sum_{i=1}^{6}(-1)^{i-1}\xi_i^2\right| \gtrsim N^2,$$

and for $\lambda_1 + \cdots + \lambda_6 = 0$

$$N^2 \lesssim \max_{i=1,\dots,6} |\lambda_i + (-1)^i \xi_i^2|. \tag{46}$$

If the integral in time were performed on the whole real line instead of $[T, T+\delta]$, then, after paying the price of the extra factor $\max_{i=1,\dots,6}|\lambda_i + (-1)^i\xi_i^2|^{\frac{1}{4}}$, one would obtain

$$\left|\int_T^{T+\delta}\int \Lambda_6(M_6(\xi_1,\xi_2,\dots,\xi_6))\,dt\right| \lesssim \frac{1}{N^{2-}}\|Iw\|^6_{1,\frac{3}{4}-}.$$

This argument has to be modified when the time integral is performed on a finite interval $[T, T+\delta]$, due to the fact that $\chi_{[T,T+\delta]}$, the characteristic function of the interval $[T, T+\delta]$, is not smooth enough. A similar difficulty was encountered also in [9]. We split

$$\chi_{[T,T+\delta]}(t) = a(t) + b(t),$$

where

$$\hat{a}(\tau) = \widehat{\chi_{[T,T+\delta]}}(\tau)\eta(\tau/N^2),$$

and η is supported on a small interval of 0 and equals 1 near 0, so a is smoothing out $\chi_{[T,T+\delta]}$ at scale N^{-2}. If one replaces $\chi_{[T,T+\delta]}(t)$ with $a(t)$, then the argument above works because the Fourier transform of $a(t)$ is supported on $|\tau| \ll N^2$, and one can still obtain the crucial inequality (46). We now have to deal with $b(t)$. It is easy to check that

$$\|b(t)\|_{L^1_t} \lesssim N^{-2}.$$

So we just have to show that

$$\sup_t |\Lambda_6(M_6; w_1(t),\dots,w_6(t))| \lesssim \prod_{j=1}^{6}\|Iw_j\|_{X^{1,\frac{3}{4}-}}. \tag{47}$$

We can crudely use Lemma 6.4 and obtain

$$|\Lambda_6(M_6; w_1(t), \dots, w_6(t))| \lesssim m_{soprano}^2 N_{soprano}^2 \|w_{soprano}\|_{L_t^\infty L_x^2} \|w_{alto}\|_{L_t^\infty L_x^2} \times \|w_{tenor}\|_{L_t^\infty L_x^\infty} \|w_{baritone}\|_{L_t^\infty L_x^\infty} \prod_{j \notin S} \|Iw_j\|_{L_t^\infty L_x^\infty},$$

which gives (47) by the Sobolev embedding theorem.

- *Case* (c). $N_{soprano} \sim N_{alto} \gtrsim N \gg N_{tenor}$. By Lemma 6.4, Lemma 2.2, Sobolev inequality, and (45), we have

$$\left| \iint \Lambda_6(M_6(\xi_1, \xi_2, \dots, \xi_6)) \right| \lesssim \sum_{S,j} \frac{1}{m_{alto}^2 N_{alto}} N_{soprano} N_{tenor} \times \|Iw_{soprano} Iw_{tenor}\|_{L^2} \times N_{alto} \|Iw_{alto} Iw_{baritone}\|_{L^2} \prod_{j \notin S} \|w_j\|_{L^\infty} \lesssim \frac{1}{N^{2-}} \|Iw\|_{1,\frac{1}{2}+}.$$

This concludes the proof of the lemma. □

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