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A Reflection of Euler's Constant and Its Applications

Spyros Andreou[a*] and Jonathan Lambright[a]

[a] *Department of Engineering Technology and Mathematics, Savannah State University, Savannah, GA 31404, USA*

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ABSTRACT

One of the most fascinating and remarkable formulas existing in mathematics is the Euler Formula. It was formulated in 1740, constituting the main factor to reason why humankind can advance in science and mathematics. Accordingly, this research will continue investigating the potentiality of the Euler Formula or "the magical number e." The goal of the present study is to further assess the Euler formula and several of its applications such as the compound interest problem, complex numbers, trigonometry, signals (electrical engineering), and Ordinary Differential Equations. To accomplish this goal, the Euler Formula will be entered into the MATLAB software to obtain several plots representing the above applications. The importance of this study in mathematics and engineering will be discussed, and a case study on a polluted lake will be formulated.

1. Introduction –History

The purpose of this study is to explore the evolution of number *e*. It starts by introducing the history of the number, then it gives some series representations of the number along with its complex number application, its number approximation and its application to compound interest. The number came into mathematics during the 16th century when scientists were

*Corresponding author (Spyros S. Andreou). Tel/Fax: +1-(912)358-3276 E-mail address: andreous@savannahstate.edu. ©2012. International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies: Volume 3 No.4 ISSN 2228-9860 eISSN 1906-9642. Online Available at http://TuEngr.com/V03/371-380.pdf

working on logarithms and the rectangular hyperbola f(x) = 1/x by the Dutch mathematician Christiaan Huygens [1]. However, it was difficult for the mathematicians at that time to interpret the area under such curve from 1 to *e* which is equal to 1. It wasn't until 1683, [2], when Jacob Bernoulli looked at the problem of compound interest where he tried to find the limit of $(1 + 1/n)^n$ as n goes to infinity. He used the Binomial theorem to show that the limit had to lie between 2 and 3 so this is the first approximation for the number *e*. It took a while for further development of the number *e* due to the fact that mathematicians thought of the logarithm as a number instead of a function. Euler is credited for using the notation *e* in 1731 for the first time and historians say that he did not use it because his name starts with the letter e or even to mean 'exponential'. The historians [3] say that the letter *e* is the next vowel after 'a' which Euler was using already in his work. Euler demonstrated that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$, and using 20 such terms this is equal to 2.718281828459045235. Moreover the limit mentioned above for the compound interest is indeed *e*. Later Euler was the first to prove the number *e* is an irrational number and until today mathematicians cannot prove the nature of the number e^e. After Euler approximated the number *e* to 18 decimals, other people followed and approximated it with more decimals such as Williams Shanks with 205 in 1871 and with today's computers the decimals reached 100 billion in 2007 [4]. It is worth mentioning its function f(x) = e^x in calculus is studied comprehensively by mathematicians and scientists. The main result is that its derivative is equal to itself; that is, df(x)/dx = e^x. Furthermore, the exponential function e^x along with the complex exponential e^{ix} (in phasor theorem) are studied. A proof of the approximation of the number *e* is given. Next, a very important application is introduced in engineering by adding several sinusoids of the same frequency using the complex exponential. It was done mathematically as well as utilizing the MATLAB software. An excellent reference regarding MATLAB is provided in [5]. Finally, an application in Civil Engineering involving a solution of a first-order differential equation is presented.

2. Representing *e* Along with Complex Number Equation

The following series expressions are equal to the number *e* [6]

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}, \quad \left[\sum_{k=0}^{\infty} \frac{1-2k}{(2k)!}\right]^{-1}, \quad \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!}, \quad \frac{1}{2}\sum_{k=0}^{\infty} \frac{k+1}{k!}, \quad 2\sum_{k=0}^{\infty} \frac{k+1}{(2k+1)!} \tag{1}$$

and indeed many others can be found in the literature. The special case of the Euler formula $e^{ix} = cos(x) + i\,sin(x)$ with $x = \pi$ gives a beautiful formula $e^{i\pi} + 1 = 0$ an equation involving the fundamental numbers i, pi, e, 1, and 0 and involving the fundamental operations of equality, addition, multiplication and exponentiation.

3. Approximating the Number *e*

In this work, a proof of a numerical estimate [7], for e from its numerical representation of the natural logarithm function given by $\ln(x) = \int_1^x (1/t)dt$ for x > 0 is presented. Let n be a positive integer so that $\ln(1+1/n) = \int_1^{1+1/n} (1/t)dt$. Next is to show the following three steps which are:

Step 1: $1/(n+1) \le \ln(1 + 1/n) \le 1/n$

To demonstrate this, first is to show that $\ln(1 + 1/n) \le 1/n$. So, $\ln(1+1/n) = \int_1^{1+1/n} (1/t)dt \le \int_1^{1+1/n} 1dt = 1/n$ since $(1/t) \le 1$ throughout the interval of integration. Now,

$\ln(1+1/n) = \int_1^{1+1/n} (1/t)dt \ge \int_1^{1+1/n} dt/(1+1/n) = [1/(1+1/n)]1/n = 1/(n+1)$ since $1/t \ge 1/(1+1/n)$ throughout the interval of integration.

Step 2: $(1 + 1/n)^n \le e \le (1 + 1/n)^{n+1}$. This result is an elegant approximation of *e* but not a very efficient tool for evaluating *e*. To prove this, from step 1, it can be obtained that

$(1+1/n) \le e^{1/n} \Rightarrow (1+1/n)^n \le e$ and $e^{1/(n+1)} \le 1+1/n \Rightarrow e \le (1+1/n)^{n+1}$. Combining these two inequalities, we have $(1 + 1/n)^n \le e \le (1 + 1/n)^{n+1}$

Step 3: Using step 2 some evaluations for *e* are given for n = 1,000, n = 10,000, n = 100,000, n = 1,000,000, n = 10,000,000 and higher using the software MATLAB. The results are as follows:

For n = 1,000, $(1+ 1/1{,}000)^{1{,}000} = 2.716923932235594$,

$(1+ 1/1{,}000)^{1{,}001} = 2.719640856167829.$

For n = 10,000, $(1 + 1/10{,}000)^{10{,}000} = 2.718145926824926,$

$(1+ 1/10{,}000)^{10{,}001} = 2.718417741417608.$

For n = 100,000, $(1 + 1/100{,}000)^{100{,}000} = 2.718268237192298,$

$(1+ 1/100{,}000)^{100{,}001} = 2.718295419874670.$

For n=1,000,000, $(1+ 1/1{,}000{,}000)^{1{,}000{,}000} = 2.718280469095753,$

$(1+ 1/1{,}000{,}000)^{1{,}000{,}001} = 2.718283187376222.$

Finally for n = 10,000,000, $(1+ 1/10{,}000{,}000)^{10{,}000{,}000} = 2.718281694132082,$

$(1+ 1/10{,}000{,}000)^{10{,}000{,}001} = 2.718281965960251.$

Based on the above results it can be conjectured that $(1 + 1/x)^x \rightarrow e$ as $x \rightarrow \infty$. So, the proof is as follows [8]:

Let

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \tag{2}$$

Taking the natural logarithms on both sides, it can be obtained

$$\begin{aligned}
\ln y &= \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] \\
&= \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x}\right)^x\right] \\
&= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] \\
&= \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right]
\end{aligned} \tag{3}$$

At this point the form of the limit would be 0/0 and it is an indeterminate form. However, the L'Hôpital's Rule can be employed to attempt to find the limit by taking the derivative of the numerator and denominator and finding the limit of that ratio. Proceeding, in that manner it can be obtained that

$$\begin{aligned}
\ln y &= \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right] \\
&= \lim_{x\to\infty}\left[\left(\frac{-1/x^2}{1+\frac{1}{x}}\right)\Big/ -1/x^2\right] \\
&= \lim_{x\to\infty}\left[1+\frac{1}{x}\right] \\
&= 1
\end{aligned} \tag{4}$$

Thus, it can be observed that ln(y) = 1. This implies $y = e$ (by definitions of logarithms). Since it was let y = the original limit expression, then

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \tag{5}$$

This completes the proof.

Now plotting the above function using the following MATLAB code shown in figure 1, the graph of the function is obtained.

```
>> x = 0:0.001:50;
>> y = (1 + 1./x).^x;
>> plot(x,y)
>> xlabel('x')
>> ylabel('y')
>> title('Graph of function (1 + 1/x)^x')
```

Figure 1: MATLAB code to plot $f(x) = (1 + 1/x)^x$.

From the plot (Figure 2), it can be seen that the function converges between 2.6 and 2.8 and the middle. So, the initial approximation is confirmed by the software. Comparing 2.718281828459045 with the obtained results, it can be observed that the achieved accuracy of

*Corresponding author (Spyros S. Andreou). Tel/Fax: +1-(912)358-3276 E-mail address: andreous@savannahstate.edu.

five decimals is 2.71828.

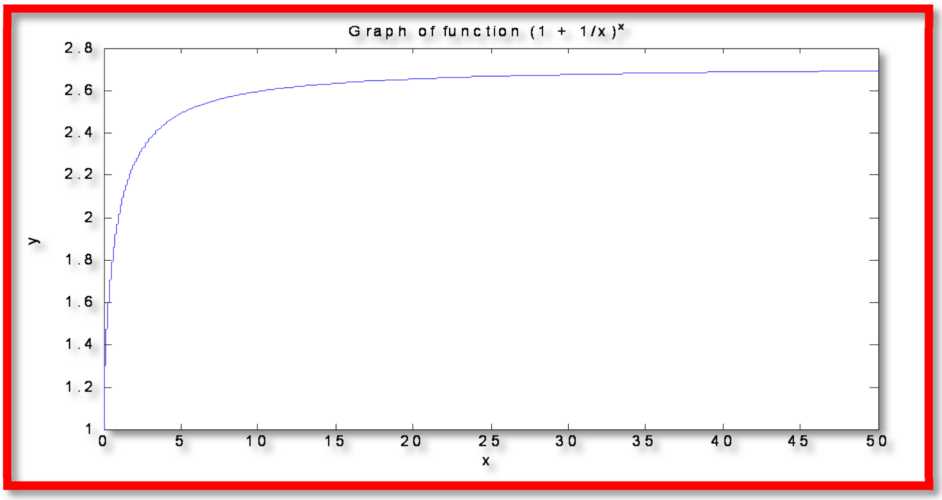

Figure 2: The exponential function as plotted in MATLAB.

4. Application 1 - The Compound-interest Problem

Jacob Bernoulli was working on the problem of compound interest and discovered the number *e*. For example, if an account is started with \$1.00 that pays 100% interest per year how much would it earn if it is compounded annually, semiannually, quarterly, monthly, weekly, daily, hourly or even every second. Simple MATLAB calculations reveal that the interest earned daily, hourly and secondly is not much different indicating convergence to number e. The calculations are as follows:

Let the account start with \$1 with 100% interest per year.

a) simple interest yields 1*(1 + 1)^1 = \$2.00 where 100% = 1 ➔ \$2.00

b) compounded semiannually yields 1*(1 + 1/2)^2 = \$2.25 ➔ \$2.25

c) compounded quarterly yields 1*(1 + 1/4)^4 = \$2.44140625 ➔ \$2.44

d) compounded monthly yields 1*(1 + 1/12)^12 = \$2.613035290224676 ➔ \$2.61

e) compounded weekly yields 1*(1 + 1/52)^52 = \$2.692596954437168 ➔ \$2.69

f) compounded daily yields 1*(1 + 1/365)^365 = \$2.714567482021973 ➔ \$2.71

g) compounded hourly yields

1*(1+1/(365*24))^ (365 *24) = \$2.718126691617908 ➔ \$2.72

h) compounded every second yields

1*(1 + 1/(365 *24*60)) ^(365*24*60) = \$2.718279242666355 ➔ \$2.72

So, if the amount of \$10,000.00 is deposited with 100% interest rate the maximum amount that can be earned is \$27,182.79 but the banks stop at daily so the amount that can be earned is \$27,145.67. However, typically the bank gives daily interest at 3% today (too high in 2012). So, the interest you receive to \$1.00 is 1*(1 + 0.03/365)^365 = \$1.030453263600551 where 3% = 0.03. So, if the amount of \$10,000.00 is deposited, the principal becomes \$10,304.53 at the end of the year compared to \$10,300.00 with simple interest.

5. Application 2 – Addition of Signals Using the Phasor Theorem

Many applications in engineering require adding two or more signals with the same frequency but with different amplitudes and time shifts (phases). The Phasor Theorem [9] is used in this situation. A phasor representation of a signal is $X_k = A_k e^{i\theta_k}$ where A is amplitude and θ is the phase (angle). The phase angle can be obtained from the time shift using the formula $\theta = -2\pi(t_m/T)$. Let the two signals be $x_1(t) = A_1\cos((2\pi/T)(t - t_{m1}))$ and $x_2(t) = A_2\cos((2\pi/T)(t - t_{m2}))$ where T is their period and t_{m1} and t_{m2} are their time shifts. So, their sum will be $x_3(t) = x_1(t) + x_2(t)$ with the same T and different time shift t_{m3}. The MATLAB code shown in Figure 3 is adding the two signals, $x_1(t)$ and $x_2(t)$. The graphs of the signals $x_1(t)$ and $x_2(t)$ along with their sum $x_3(t)$ are illustrated in Figure 4.

```
A1=24;%magnitude of the first signal
A2=1.2*A1;%magnitude of the second signal

T=1/6000;%period of both signals
time=[-T:T/50:T];
tm1=37.2*T;%time shifts tm1 and tm2
tm2=(41.3/12)*T;
x1t=A1*cos(2*pi*(1/T)*(time - tm1)); %first signal
x2t=A2*cos(2*pi*(1/T)*(time - tm2));%second signal
subplot(3,1,1)
plot(time,x1t), xlabel('time in seconds'), ylabel('signal                x1t'), title('Graph of signal x1t')
subplot(3,1,2)
plot(time, x2t),xlabel('time in seconds'),ylabel('signal    x2t'), title('Graph of signal x2t')
%now we would like to add them
x3t=x1t+x2t; %This addition is done by hand using the    complex exponential e^(jt)
subplot(3,1,3)
plot(time,x3t), xlabel ('time in seconds'), ylabel('signal x3t'), title('Graph of signal x3t=x1t+x2t')
```

Figure 3: MATLAB code to add two signals.

*Corresponding author (Spyros S. Andreou). Tel/Fax: +1-(912)358-3276 E-mail address: andreous@savannahstate.edu.

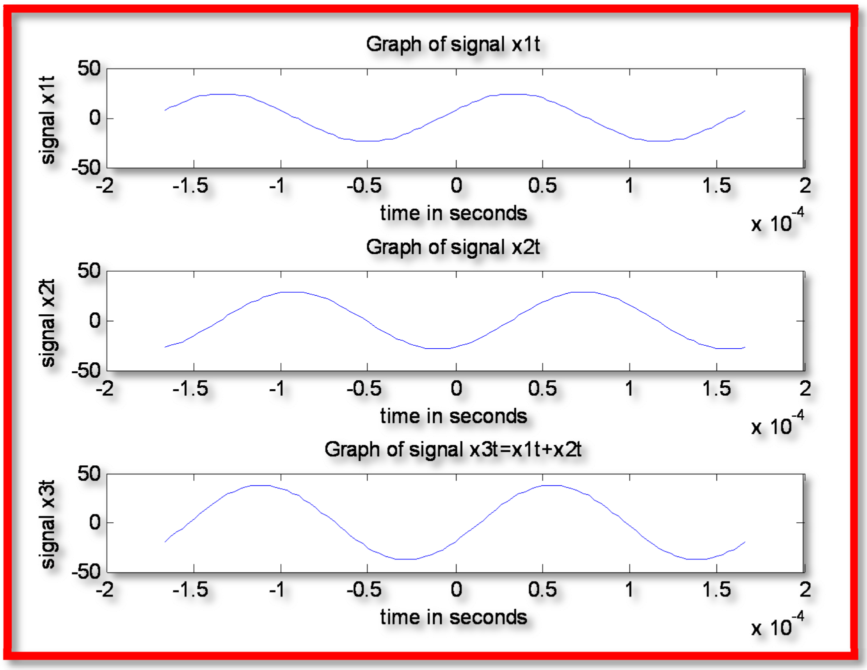

Figure 4: Signals $x_1(t)$, $x_2(t)$ and their sum $x_3(t)$.

6. First-Order Differential Equation Case Study in Environmental Engineering

Suppose that a polluted lake (see more details in [10]) has an initial concentration of bacteria of 10^7parts/m^3, whereas the acceptable level is only $5x10^6$ parts/m^3. When fresh water enters the lake the concentration of the bacteria will decrease. Let C be the concentration of the pollutant as a function of time (in weeks). The differential equation is given by $dC(t)/dt + 0.06C(t) = 0$, $C(0) = 10^7$. The question is to find the concentration of the pollutant after 7 weeks.

In order to solve the differential equation, concentration C and time t are separated and integrated. After some algebra, the exact solution is $C(t) = 10^7e^{(-0.06t)}$. Substituting t = 7 weeks it can be obtained that $C(7) = 6.5705x10^6$ parts/m^3. It can be observed that the solution involves the exponential function e^x and after 7 weeks the concentration of the pollutant is more than the acceptable level. Calculations show that it takes about 11.5 weeks for the lake to reach the acceptable level of pollutants.

7. Conclusion

The two irrational numbers e and π are so important in mathematics, sciences and engineering where most people knowing the latter. So, in this work the number e (more about this in [11]) and its function e^x are explored. The limit of the function $f(x) = (1 + 1/x)^x$ as $x \to \infty$ involving a lot of knowledge in mathematics is revisited and found to be the number e. Its main application to compound interest is investigated. Compound interest is in people's every- day life and the authors of this paper had this in mind. This number has significant applications in calculus where its function e^x plays a significant role and it is studied very thoroughly and carefully by the mathematicians. The importance of the exponential function is demonstrated in the second application using the phasor theorem to add two signals. Finally, a solution of a first – order differential equation is presented where it is always exponential. There are many further studies for undergraduates to get involved; for example, Linear Algebra and systems of differential equations (matrix exponential function).

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*Corresponding author (Spyros S. Andreou). Tel/Fax: +1-(912)358-3276 E-mail address: andreous@savannahstate.edu.

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Dr.Spyros Andreou received the B.S. degree and M.S. degree in Electrical Engineering from the University of Arizona, the Ph.D. degree in Electrical Engineering (control systems) and the M.S. degree in Applied Mathematics from the University of Arkansas. Dr. Andreou got his first academic job at New Mexico Highlands University as a Visiting Assistant Professor. In January 2003, Dr.Andreou joined Georgia Southern University as an Assistant Professor where he taught various engineering courses and worked in Matrix Difference Systems. In August 2006 joined the faculty at Savannah State University where he currently teaches Regents Engineering Transfer Program courses and involved in engineering education research via NSF grants. He is currently an Associate Professor within the Department of Engineering Technology & Mathematics at Savannah State University, Registered as a Professional Engineer (PE) in the State of New Mexico, a Senior Member of the Institute of Electrical and Electronics Engineers (SMIEEE) and a Member of the Institution of Engineering Technology (MIET) and Chartered Engineer (CEng) in the United Kingdom.

Dr.Jonathan Lambright received the B.S. degree in mechanical engineering from the North Carolina A&T State University and after working for 3 years as a Mechanical Engineer at the Department of Defense, Dr. Lambright returned to graduate school at North Carolina A & T and received his M.S. in Mechanical Engineering in 1990 with a focus in Computer Aided Design and Manufacturing. While at Georgia Tech Jonathan focused his studies and research on design methodology and manufacturing automation. During the period between 1992 and 1996 Jonathan worked for the Lockheed Martin Aeronautical Systems Co. in Marietta GA. At Lockheed he worked on various research and development projects within the Advanced Design department. The research projects at Lockheed consisted of Computer Assisted Manufacturing Tools, Design Tools using Knowledge Based Systems and Advance Database applications and he earned his Ph.D. He is currently an Associate Professor within the Department of Engineering Technology & Mathematics at Savannah State University.

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