340
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A REFOUNDATION OF MODAL LOGIC

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1 In this paper, we continue the work begun in [1], refounding modal logic. In [1] we constructed some systems of propositional modal logic and saw how the technique involved resolved many of the awkward problems surrounding such logics. The technique was then extended to modal predicate logics in a natural (but we now think philosophically insignificant) way. In this paper we extend the technique to modal predicate logics in a different way which, we think, resolves many of the problems associated with quantified modal notions.

Modal logic as presently conceived is ill-founded. And it is precisely for this reason that it seems to have run into a blind alley. When Lewis constructed the first modal logics, he constructed them as extensions of first-order languages (i.e., as first-order languages with one new connective $\Diamond$), since these were the only logics formalized at the time. No one has since questioned this assumption, which we think is wrong. As we argued in [1], necessary truth, like truth, is a semantic concept. It is hence impossible to formalize it properly within an extended first-order theory, and now that we have formalized semantic theories (since [9]), we are in a position to correct Lewis's mistake. Had, in fact, modal logic been invented in the 1960's instead of the 1920's, then it would have been originally formulated as a semantic theory. Now, reading necessity as a semantic operator is in accordance with Quine's first grade of modal involvement (see [4]), which he considers safe but uninteresting. We will show it to be far from uninteresting. Since quotation is referentially opague, he considers quantified modal logics impossible on this reading. We will show that this is not the case. Further, he regards all sentences of the third degree of modal involvement (i.e., quantified modal statements) as confused, meaningless, and leading to metaphysical commitment. We will show how we can make perfectly good sense of quantified modal statements without endangering such unpleasant ends. Finally, in the introduction, we note that one of the arguments we used in [1] to show that modality is a meta-concept, viz. the compulsive Liar paradox, is used by Prior in [6] to refute exactly this position. He assumes however, that modal logic must

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be a first-order Lewis-type system, which we used the compulsive Liar paradox to refute.

2 The language we construct in this paper is, in fact, an object-meta-language pair as in the system $\mathcal{G}_1$ of [1]. For the object language O, we take the two-valued predicate calculus. (Although we could carry out the same construction with any first-order, intuitionist, or many-valued theory with quantification. We note also that we could choose any of these for the logical basis of the meta-language $\mathcal{GP}$, and in this context, a free logic, i.e., one valid in the empty domain would be particularly appropriate.)

Object language, O The two-valued predicate calculus, with any of the usual axiomatizations.

Notation

Connectives:	$\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
Quantifiers:	$\forall, \exists$
Predicates:	$A, B, C, \dots$
Variables:	$x_0, y_0, z_0, \dots$

and perhaps some

Individual Constants: $a_0, b_0, c_0, \dots$

We write '$\vdash_O$' for 'is a thesis of O'.

Meta-Language $\mathcal{GP}$ We want the language $\mathcal{GP}$ to be able to talk about the formulas of O. We could supply a name for each symbol of O, e.g., '$\vee$' for '$\forall$', 'X_i' for 'x_i', '$\Rightarrow$' for '$\rightarrow$', 'θ' for 'A', '$-$' for '$\neg$', etc., and then stipulate that the name of any formula of O is its corresponding structural description, e.g., the name of $\forall x_0 A\, x_0 \rightarrow \neg A\, x_0$, would be $\vee X_0\, \theta\, X_0 \Rightarrow -\theta X_0$. However, because there is an obvious isomorphism between every formula and its structural description, we can use wffs of O autonomously, and dispense with the need for new symbols. Hence, we make the following convention:

Convention C If a formula of O occurs within the scope of a '$\vdash_O$', (or any other meta-language predicate we introduce) then the symbols do not have their normal meanings, but are autonomous, i.e., they name themselves; and their concatenation names the formula of O which is the concatenation of them.

We make this convention since, as we shall see, no confusion can arise as to whether the symbols are being used autonomously or not, this being determined unambiguously by the context. Further this makes the symbolism more easy to read, brings it in line with an informal mode of speech, and brings out the connection between $\mathcal{GP}$ and the more usual systems of modal logic.

We note also that according to Frege (see, e.g., [7]) something very similar happens in ordinary language, where quotation marks produce an oblique context, within which each expression denotes itself.

Notation of $\mathcal{GP}$

Individual Constants: Any symbol or linear concatenation of symbols of O is an *individual constant symbol* of $\mathcal{GP}$
Predicates: The only predicate is '$\vdash_0$'
Connectives and quantifiers: $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\forall$, $\exists$
Variables: $x_1, y_1, z_1, \ldots$

Now the variables of $\mathcal{GP}$ are to range over the individual constant symbols of O. This is a very important point; the objects quantified over in the meta-language are symbols, viz. a_0, b_0, c_0, etc. We could clearly have variables ranging over all the formulas of O. We would then, however, have to introduce additional machinery such as a predicate for 'is an individual constant symbol of O', for our purposes.

Formation Rules

We will use the following syntactic variables:

$p, q, \ldots$ will denote wffs of O.
$P, Q, \ldots$ will denote wffs of $\mathcal{GP}$.
$\alpha_0, \beta_0, \ldots$ will denote variables of O.
$\alpha_1, \beta_1, \ldots$ will denote variables of $\mathcal{GP}$.
$p(\alpha_0 \ldots \beta_0)$ will mean that the variables of p occur amongst $\alpha_0 \ldots \beta_0$.
$p(\alpha_1 \ldots \beta_1)$ will denote $p(\alpha_0 \ldots \beta_0)$ with $\alpha_1 \ldots \beta_1$ substituted for some occurrences of $\alpha_0 \ldots \beta_0$.

And

a) $\vdash_0 p$ is a wff of $\mathcal{GP}$.
b) If $P(\alpha_0 \ldots \beta_0)$ is a wff, then $P(\alpha_1 \ldots \beta_1)$ is.
c) If P, Q are wffs, so are $\neg P$, $P \rightarrow Q$, $(\exists\alpha_1)P$.

These clauses form a complete recursive definition of the wffs of $\mathcal{GP}$.

Definitions

D1 The usual definitions of $\wedge$, $\vee$, $\forall$, etc.
D2 $Lp =_{df} \vdash_0 p$.
D3 $Mp =_{df} \neg L \neg p$.

By *D2* we do not intend to imply that 'necessarily true' means 'provable' in some system or other. Axiomatization is usually a post-hoc characterization or a body of truths. We assume that the set of truths we are interested in has been axiomatized for us. So '*L*' is, in effect, a primitive constant. We note and accept Quine's thesis that necessity is not definable in terms of non-circular concepts. However, this does not mean that necessity is meaningless: many meaningful concepts can only be defined in circular terms (e.g., set, collection, class, etc.). '$\vdash_1$' will mean 'is a thesis of $\mathcal{GP}$'.

Axioms

M1-M5, as in [1]. These form a complete axiomatic basis for the two-valued predicate calculus.

M6 $Lp \rightarrow Mp$
M7 $L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq)$

M6 and *M7* state the consistency of O and the validity of *modus ponens* in O, respectively. (We note in passing that the converses of *M6* and *M7*, viz.

$$\neg L \neg p \rightarrow Lp \text{ and } (Lp \rightarrow Lq) \rightarrow (L(p \rightarrow q))$$

state forms of the completeness of O and the deduction theorem for O, respectively.) Plus some of *M8-M11*.

M8 $\vdash_1 (\exists\alpha_1)Lp(\alpha_1) \rightarrow L(\exists\alpha_0)p(\alpha_0)$
M9 $\vdash_1 L(\forall\alpha_0)p(\alpha_0) \rightarrow (\forall\alpha_1)Lp(\alpha_1)$
M10 $\vdash_1 (\forall\alpha_1)Lp(\alpha_1) \rightarrow L(\forall\alpha_0)p(\alpha_0)$
M11 $\vdash_1 L(\exists\alpha_0)p(\alpha_0) \rightarrow (\exists\alpha_1)Lp(\alpha_1)$

Exactly which of *M8-M11* one chooses is a matter for philosophical deliberation, which we will not enter into here. In this paper we will just say which combinations we use, and when. As a formalization of the concept of logical necessity, the order 8-11 represents in increasing order of dubiousness, the desirability of inclusion in the system. The line is normally drawn between 9 and 10. *M8* and *M9* state the rules of existential generalization, and instantiation for O, respectively. Alternatively, since *M8* asserts that inferences from $p(a)$ to $(\exists x)p(x)$, are valid in O, this can be taken as saying that all the individual constant symbols of O denote. If we were to choose for O a language in which this does not hold, we would have to modify *M8* and *M9* in a fairly obvious way. Further, if O had terms in it other than individual constants, e.g., if O had function symbols or a description operator, we may or may not choose to let the variables of $\mathcal{GP}$ range over all the terms of O. We will consider this point later in another context, but for the moment, we will just note that if we do allow this, and if these terms can fail to denote, then we shall need to make similar modifications to the axioms. *M10*, the Barcan formula in thin disguise, states in effect, the ω-completeness of O (or what would be ω-completeness if O were some formal arithmetic). *M11* states the ω-consistency of O similarly.

Rules of Deduction for $\mathcal{GP}$

a) Modus Ponens: From $\vdash_1 P$ and $\vdash_1 P \rightarrow Q$ infer $\vdash_1 Q$.
b) Universal Generalization: From $\vdash_1 P$ infer $\vdash_1 (\forall\alpha_1)P$.
c) Rule T: This translates between O and $\mathcal{GP}$.

From $\vdash_0 p$ infer $\vdash_1 Lp$ and vice-versa.

This completes the specification of $\mathcal{GP}$.

3 We will now give a few proofs in $\mathcal{GP}$. In the following proofs, theses of O and predicate calculus valid theses of $\mathcal{GP}$ will be taken for granted and usually omitted from the proof. $\mathcal{GP}$ contains a complete axiomatic basis for the system $\mathcal{G}_1$ of [1]. Hence, any thesis of $\mathcal{G}_1$ is derivable in $\mathcal{GP}$, and this forms the modal propositional logic basis of $\mathcal{GP}$. At the beginning of each proof we will note which of *M8-M11*, we use in the proof.

GP1 (*M8*) $\vdash_1 (\forall x_1) Lp(x_1) \rightarrow L(\exists x_0)\, p(x_0)$

Proof:

(i)	$\vdash_1 (\exists x_1) Lp(x_1) \rightarrow L(\exists x_0) p(x_0)$	(*M8*)
(ii)	$\vdash_1 (\forall x_1) Lp(x_1) \rightarrow (\exists x_1) Lp(x_1)$	(Pred. calc.)

so by (i), (ii), and a tautology, we have *GP1*.

GP2 (*M10*) $\vdash_1 (\forall x_1) Lp(x_1) \rightarrow \neg L(\exists x_0) \neg p(x_0)$

Proof:

(i)	$\vdash_1 (\forall x_1) Lp(x_1) \rightarrow L(\forall x_0) p(x_0)$	(*M10*)
(ii)	$\vdash_1 L(\forall x_0) p(x_0) \rightarrow \neg L \neg(\forall x_0) p(x_0)$	(*M6*)
(iii)	$\vdash_0 \neg(\forall x_0) p(x_0) \leftrightarrow (\exists x_0) \neg p(x_0)$	(Pred. calc.)
(iv)	$\vdash_1 L \neg(\forall x_0) p(x_0) \leftrightarrow L(\exists x_0) \neg p(x_0)$	((iii), **T**, *M7*)

so by (i), (ii), (iv), and a tautology we have *GP2*.

GP3 (*M9*) $\vdash_1 (\exists x_1) \neg Lp(x_1) \rightarrow \neg L(\forall x_0) p(x_0)$

Proof:

(i)	$\vdash_1 L(\forall x_0) p(x_0) \rightarrow (\forall x_1) Lp(x_1)$	(*M9*)
(ii)	$\vdash_1 \neg(\forall x_1) Lp(x_1) \rightarrow \neg L(\forall x_0) p(x_0)$	((i) and taut.)

so by (ii) and the predicate calculus, we have *GP3*.

GP4 (*M9*) $\vdash_1 (\exists x_1) \neg Lp(x_1) \rightarrow M(\exists x_0) \neg p(x_0)$

Proof:

(i)	$\vdash_1 \neg Lp \leftrightarrow M \neg p$	(*D3*)

so by i), *GP3*, and the predicate calculus, we have *GP4*.

GP5 (*M9*) $\vdash_1 L(\forall x_0) p(x_0) \rightarrow (\exists x_1) Lp(x_1)$

Proof:

(i)	$\vdash_1 L(\forall x_0) p(x_0) \rightarrow (\forall x_1) Lp(x_1)$	(*M9*)
(ii)	$\vdash_1 (\forall x_1) Lp(x_1) \rightarrow (\exists x_1) Lp(x_1)$	(Pred. calc.)

so by (i), (ii), and a tautology we have *GP5*.

Notice that since many things of the form $(\forall x_0) p(x_0)$ are provable in *O*, *GP5* implies that there are some individual constants in *O*. Since in our construction these were optional, this seems rather strange. However, a moment's reflection shows that this is a result of the fact that the predicate calculus is only valid in non-empty domains, and this is how this result slips in. So a free logic would be more appropriate for the logical basis of $\mathcal{GP}$. However, this is not of too great importance.

4 *Semantics* We now consider some semantics for $\mathcal{GP}$. From now on we shall only consider closed formulas of $\mathcal{GP}$. (An open sentence is deductively equivalent to its closure.) These have the property that all variables occurring in them have their semantic category determined unambiguously by their context. Thus the indices of variables may be dropped. We define P' to be the sentence P with subscripts omitted. Models, definitions of validity and completeness proofs for the above systems (i.e., with various of *M8-M11*) are easily found along the lines of Kripke semantics. The variations to the standard cases (S4 and S5) are straightforward. It is also easy to prove semantically that if P is any closed wff of $\mathcal{GP}$, then S4 $\vdash P'$ iff *M1-M9* $\vdash P$ and B + S4 $\vdash P'$ iff *M1-M10* $\vdash P$. A much more natural concept of model is, however, obtained as follows:

Since the vocabulary and theorems of a $\mathcal{GP}$ meta-language are in part determined by its object language, O, we define truth and validity for an O-$\mathcal{GP}$ pair. For the moment we consider the case where $\mathcal{GP}$ has axioms *M1-M9* and we will extend this later. So let O be any fixed but arbitary object language which is a) consistent and in which b) *modus ponens*, c) existential generalization, d) instantiation hold.

We define: O' is an *extension* of O if all the vocabulary of O is vocabulary of O', and every axiom or axiom scheme of O is an axiom or scheme of O'.

We define: an *O-$\mathcal{GP}$ model* is a consistent extension of O with at least one individual constant symbol.

Given any O-$\mathcal{GP}$ model O', the variables of $\mathcal{GP}$ range over the individual constant symbols of O', and the interpretation of L is the set of sentences

$$\{p;\ O' \vdash p\},\ \text{i.e., } Lp \text{ is true in } O' \text{ iff } O' \vdash p.$$

The definition of truth for wffs of $\mathcal{GP}$ is then in the usual way, by induction over the lengths of sentences.

We define: A wff is *valid* if it is true in every O-$\mathcal{GP}$ model. It is easy to check that every $\mathcal{GP}$ theorem of an O-$\mathcal{GP}$ pair is valid.

We have a very neat completeness proof for the above semantics. First we prove the Satisfaction Theorem.

Satisfaction Theorem *If Σ is a consistent set of sentences of an O-$\mathcal{GP}$ pair with respect to the rules and axioms of $\mathcal{GP}$ then Σ has an O-$\mathcal{GP}$ model.*

Proof: Given O-$\mathcal{GP}$, let O' be the extension of O whose vocabulary is precisely that of O plus the set of individual constants $\{a_{ij};\ i, j \in \omega\}$ (the case when O is uncountable is as usual) and whose axioms and schemes are precisely those of O. We can extend Σ to a maximally consistent (with respect to the rules and axioms of O'-$\mathcal{GP}$) set of sentences Σ^ω of O'-$\mathcal{GP}$, such that if $(\exists\alpha_1) P(\alpha_1) \in \Sigma^\omega$, then for some $i, j \in \omega, P(a_{ij}) \in \Sigma^\omega$ and if $(\exists\alpha_0) p(\alpha_0) \in \Sigma^\omega$, then for some $i, j \in \omega$, $p(a_{ij}) \in \Sigma^\omega$, in the usual way. Now let O'' be the language whose vocabulary is that of O' and whose axioms are all the sentences of O'' in Σ^ω. Let Δ be the set of axioms of O'' then

$$p \in \Delta \text{ iff } p \text{ is a wff of } O' \text{ and } p \in \Sigma^\omega.$$

Δ is clearly maximally consistent and O'' is trivially an extension of O' and O. Further, conditions a)-d) above hold in O'', since all instances of axiom schemes *M6-M9* are in Σ^ω. Hence O'' is an O-$\mathcal{GP}$ model. Hence all we have to do is prove that it is a model for Σ. Since Δ is maximal, if p is a wff of O'', then

$$\Delta \vdash p \text{ iff } p \in \Delta.$$

It is now easy to prove by the usual induction that for any wff P of $\mathcal{GP}$,

$$P \in \Sigma^{\omega} \text{ iff } P \text{ is true in } O''.$$

The proof is by induction over the length of P. All the induction steps are exactly as for the usual Henkin-type proof. We will prove the basis

$$\begin{aligned} Lp \in \Sigma^{\omega} &\text{ iff } p \in \Sigma^{\omega} \text{ (by rule } \mathbf{T}) \\ &\text{iff } p \in \Delta \\ &\text{iff } \Delta \vdash p \\ &\text{iff } Lp \text{ is true in } O''. \end{aligned}$$

Hence O'' is the required model for Σ^{ω} and thus Σ.

Completeness Theorem *If P is a valid O-$\mathcal{GP}$ sentence, then P is provable in O-$\mathcal{GP}$.*

Proof: This follows from the Satisfaction Theorem in the usual way.

To modify the above for the case where *M10* or *M11* is included, we simply specify that for models of *M10*, any O-$\mathcal{GP}$ model will satisfy the extra condition:

e) If $p(a)$ is provable for every individual constant a, then $(\forall x_0)\, p(x_0)$ is provable.

And for *M11*, the extra condition:

f) If $(\exists x_0)\, p(x_0)$ is provable, then there is some individual constant a, such that $p(a)$ is provable.

The above completeness proof now goes through verbatim. The only new point is checking whether O'' satisfies conditions e) and f), and this is straightforward.

Permeability We now consider an application of our semantics. We say that a wff P of O is *permeable* if it is not a thesis of O, and yet there is a proof of Lp (i.e., $\vdash_{\bar{O}} p$) in $\mathcal{GP}$. Are there any permeable wffs in $\mathcal{GP}$? This was a question which was answered negatively for $\mathcal{G}_1$ and related systems in [1]. We now determine the answer for $\mathcal{GP}$. Whereas before, however, the proof was quite long, our semantics and completeness proof just developed give us the answer for $\mathcal{GP}$ immediately.

Consider all the theses of $\mathcal{GP}$ of the form 'Lp'. Since the theses of $\mathcal{GP}$ are precisely the valid wffs, then Lp is provable in $\mathcal{GP}$ iff p is provable in every O-$\mathcal{GP}$ model. Now define the *minimal extension*, O^0, of O to be the extension of O with only one new individual constant* and no new axioms or

*Or none if there are already some in the vocabulary of O.

schemes (N.B. O^0 may not be an O-$\mathcal{GP}$ model if $\mathcal{GP}$ contains *M10* or *M11*). Then p is provable in O^0 iff p is provable in every O-$\mathcal{GP}$ model.

Proof: Since O^0 is contained in every extension of O, one way round is trivial. Conversely, if p is not provable in O^0, then $\{\neg p\} \cup O$ is consistent. Extend it in any possible way to satisfy conditions a)-f) above, and we have the result. Thus, we have the Permeability Theorem:

Permeability Theorem *O-$\mathcal{GP}$ has permeable wffs iff O^0 is not a conservative extension of O.*

Corollary *If O is a first-order language then O^0 is a conservative extension of O.*

So by the Completeness Theorem for first-order logic, we have that Lp is valid iff p is true in every model (in the usual sense) of O.

Comment 1 The only sort of theory that come to mind in which O^0 is not a conservative extension of O, are free logics in which nothing of the form $(\exists x)\, p(x)$ is provable unless something of the form $p(a)$ is provable.

Comment 2 Suppose we had used a free logic as the logical basis of $\mathcal{GP}$. (We noted earlier that this would be appropriate.) We would then have no need to stipulate the existence of an individual constant in the definition of extension and minimal extension. Hence O would always be the minimal extension of itself and there would be no permeable wffs; another reason why a free logic would be more appropriate.

5 We now consider some natural extensions of $\mathcal{GP}$ and their consequences.

Identity and description We can take for O, a first-order language with identity and description and this is straightforward, since description operators will always be within the scope of any L's. Identity and description can also be introduced into $\mathcal{GP}$, and this is more interesting. $\mathcal{GP}$ can have two sorts of identity, a) natural identity and b) necessary identity.

a) Natural identity is the true identity on the range of the variables of $\mathcal{GP}$, i.e., the individual constant symbols of O. So we introduce a new two-place predicate '=' into $\mathcal{GP}$ and $x_1 = y_1$ means that x_1 is the same individual constant symbol as y_1. Hence $a_0 = a_0$, $a_0 \neq b_0$, etc. Further, we need new axioms for '=', and we take the usual axioms for identity. With this sort of identity, we see that the quantifiers in $\mathcal{GP}$ have the meaning suggested by R. B. Marcus in [8]. She suggests that '$(\exists x)\, Lp(x)$' could mean 'Some substitution instance of $Lp(x)$ is true'. However, as an explication of what happens in ordinary discourse, this is unsatisfactory for a number of reasons, one of which we will consider in connection with descriptions in $\mathcal{GP}$.

b) Necessary identity is really an equivalence relation. We will use '$\simeq$' as the symbol for necessary identity, and we have the explicit definition:

D4 $x_1 \simeq y_1 =_{df} L(x_1 = y_1)$

where the second '=', being in the scope of an 'L', is the identity of O. The properties of necessary identity can be inferred from the properties of the identity of O, and hence we need no new axioms in $\mathcal{GP}$. For example, we prove:

a) $\vdash_1 (\forall x_1)\, x_1 \simeq x_1$

Proof:

(i)	$\vdash_0 (\forall x_0)\, x_0 = x_0$	(Axiom of O identity)
(ii)	$\vdash_1 L(\forall x_0)\, x_0 = x_0$	(Rule **T**)
(iii)	$\vdash_1 L(\forall x_0)\, x_0 = x_0 \to (\forall x_1)\, L(x_1 = x_1)$	(*M9*)
(iv)	$\vdash_1 (\forall x_1)\, L(x_1 = x_1)$	((ii), (iii), and *modus ponens*)
(v)	$\vdash_1 (\forall x_1)\, x_1 \simeq x_1$	((iv), *D4*)

b) $\vdash_1 x_1 \simeq y_1 \to (P(x_1) \leftrightarrow P(y_1))$

Proof: Let $P(x_1)$ be of the form $Lp(x_1)$, then

(i)	$\vdash_0 (\forall x_0)(\forall y_0)(x_0 = y_0 \to (p(x_0) \leftrightarrow p(y_0)))$	(Axiom of O identity)
(ii)	$\therefore\ \vdash_1 L(\forall x_0)(\forall y_0)(x_0 = y_0 \to (p(x_0) \leftrightarrow p(y_0)))$	(Rule **T**)
(iii)	$\therefore\ \vdash_1 (\forall x_1)(\forall y_1)\, L(x_1 = y_1 \to (p(x_1) \leftrightarrow p(y_1)))$	(*M9*)
(iv)	$\therefore\ \vdash_1 (\forall x_1)(\forall y_1)\, L(x_1 = y_1) \to (Lp(x_1) \leftrightarrow Lp(y_1))$	(*M7* and $L(p \wedge q) \to (Lp \wedge Lq)$)
(v)	$\vdash_1 (\forall x_1)(\forall y_1)\,[(x_1 \simeq y_1) \to (P(x_1) \leftrightarrow P(y_1))]$	((iv), *D4*)

This forms the basis for an induction over the length of P. The rest of the induction is as usual.

These are, of course, the normal axioms for identity.

Descriptions can also be introduced into $\mathcal{GP}$ in any of the usual ways. The question then arises, which form of identity it is best to use in connection with description. Considerations such as the following show that necessary identity is the appropraite one. Take as O, Peano arithmetic, and consider the sentence of $\mathcal{GP}$,

$$7 < (\iota x)\, L(8 < x \wedge x < 10) \tag{1}$$

i.e., seven is less than the number that is necessarily between eight and ten (which must clearly be nine). Now (1) is equivalent to

$(\exists !x)(L(8 < x \wedge x < 10) \wedge 7 = x)$, where $(\exists !x)A(x)$ means as usual (2)

$$(\exists x)(A(x) \wedge (\forall z)A(z) \to x = z) \tag{3}$$

If we interpret the equality in (3) as natural identity, then there is no such unique object as (2) asserts. For

$$L(8 < 9 \wedge 9 < 10)$$

and

$$L(8 < 3^2 \wedge 3^2 < 10)$$

but it is not the case that $9 = 3^2$ in the sense of natural identity, since they are not the same term. However, if we interpret the identity in (3) in the sense of necessary identity, then *D4* itself guarantees the existence of such a unique object, since

$$\text{if } L(8 < x \wedge x < 10), \text{ then } L(x = 9), \text{ i.e., } x \simeq 9.$$

Thus, the unique thing is nine or anything necessarily equivalent to it.

Intensional objects With necessary identity, the objects in our domain are, in effect, equivalence classes of terms of O under the equivalence relation $L(x_1 = y_1)$. We see that these objects have exactly the properties of the intensional objects invented by Church, Carnap, and others (see, e.g., [5]) in an attempt to make sense of quantified modal logics, viz. the law of substitutivity of equivalents holds for them in all contexts (see b) in the previous section). We see now what these strange objects really are. This brings us back to a point mentioned earlier. Namely, whether or not, if O has terms other than individual constants, to allow $\mathcal{GP}$ variables to range over all terms of O. Now, inferences of the form:

$$L(\text{Cicero is Cicero}) \tag{4}$$
$$(\exists x_1)\ L(x_1 \text{ is Cicero}) \tag{5}$$

are always valid in $\mathcal{GP}$. If we allow the variables to range over all terms of O the following inference is valid:

$$L(\text{The winner of the next game of chess will win the next game of chess}) \tag{6}$$
$$(\exists x_1)\ L(x_1 \text{ will win the next game of chess}) \tag{7}$$

Now this is an intuitively repugnant conclusion, and would seem a very good reason for having the quantifiers of $\mathcal{GP}$ ranging over only the individual constant symbols of O. The point to note here, however, is that someone may accept some intensional objects, namely individual concepts, and so believe the validity of (4)-(5), but may not accept them all, and believe that (6)-(7) is not valid. (E.g., the individual concept of Cicero satisfies (5), but there is no individual concept that satisfies (7).) Someone who accepts all intensional objects, however, would accept even the validity of (6)-(7), since the winner of the next game of chess *qua* intensional object satisfies (7). See, e.g., [2] for a discussion of this in connection with *M11*. We now see exactly what the difference is between these two positions, and why one understands that there is a sense in which (7) is true, although not the usual sense. It all depends on the range of the meta-language variables. While we are on the subject of intensional objects, we will clear up an objection to them. This objection is due to a misunderstanding of their nature. For a full account of the objection see, e.g., [3]. If we restrict the range of our variables to intensional objects (the argument runs), then it must be the case that if two things are the same then they must necessarily be the same, i.e.,

$$x = y \rightarrow L(x = y) \tag{8}$$

But, if p is an arbitrary true statement, then $x = (\iota y)(x = y \wedge p)$. Hence $L(x = (\iota y)(y = x \wedge p))$, so $L(x = x \wedge p)$, and hence Lp. So all modalities collapse.

The fallacy in this argument is in failing to realize that intensional objects are not actual objects, but merely façons de parler. If (8) is talking about intensional objects, then it must mean:

$$\vdash_1 x_1 \simeq y_1 \rightarrow L(x_1 = y_1)$$

where '=' is, of course, the identity of O. This is indeed true by the very definition of '$\simeq$'. It is also the case that

$$\vdash_0 p \rightarrow a = (\iota y)(y = a \wedge p).$$

However, it does not follows that

$$\vdash_1 a \simeq (\iota y)(y = a \wedge p)$$

without the premis $\vdash_0 p$, i.e., Lp, which was supposed to be the paradoxical conclusion. The whole thing is then basically a confusion over identities.

Iteration of modalities and translational meta-languages Finally in this section we consider extensions of $\mathcal{GP}$ that allow for iterated and different degrees of modality. In $\mathcal{GP}$ we have only formulas with one degree of modality. However, we can obtain iterated modalities by constructing a hierarchy of $\mathcal{GP}$ meta-languages. We can obtain formulas of mixed modality (e.g., $Lp \rightarrow p$), by making $\mathcal{GP}$ a translational meta-language, that is, making O a sub-language of $\mathcal{GP}$. We will have reason to refer to this theory in subsequent sections, and will call it 'Translational $\mathcal{GP}$'. For further comments on the above two extensions of $\mathcal{GP}$, see [1].

6 In his paper [3], Quine examines some of the problems of standard quantified modal logics, which lead to paradoxical results and metaphysical commitment. We will now consider three such problems and see how they are resolved in $\mathcal{GP}$.

A) Consider the inferences:

1) Necessarily, if there is life on the morning star then there is life on the morning star.
2) The morning star is the evening star.
3) Hence, necessarily if there is life on the morning star then there is life on the evening star.
4) Necessarily Cicero is Cicero.
5) Cicero is Tully.
6) Hence, necessarily Cicero is Tully.

To explain the prima facie paradoxical results 3) and 6), one must either find some reason to reject the principle of extensionality, and thus falsify the inferences, or one must interpret 3) and 6) in such a way that they are true. In which case it is odd that astronomers can discover truths or reason by looking down telescopes. Anyway, in $\mathcal{GP}$ the inferences to 3) and

6) are invalid; not because the principle of extensionality fails, but because 1) and 4) are expressions in an informal mode of speech of

1′) 'If there is life on the morning star then there is life on the morning star' is necessarily true,

and

4′) 'Cicero is Cicero' is necessarily true,

i.e., in 1) and 4), 'the morning star' and 'Cicero' occur autonomously, which they do not in 2) and 5).

B) The other side of the coin is as follows: In most modal predicate logics with identity, the following is provable,

7) $$(\forall x)(\forall y)(x = y \rightarrow L(x = y)).$$

Now this is a strange result which means that if I wished to know whether Cicero really was Tully, then I should sit in an armchair and deliberate on the meanings of 'Cicero' and 'Tully', or at worst do a few pencil calculations on my shirt cuff. To accept this result has distinct metaphysical implications. In $\mathcal{GP}$, however, 7) is not provable. Trivially because the intended interpretation of both '=' signs in 7) is identity between physical objects, i.e., the identity of O. So 7) is not even a wff of $\mathcal{GP}$. It is, however, a wff of Translational $\mathcal{GP}$ but it is still not provable in this. For further comments see [1].

C) A closely associated problem, and in fact the one at the root of the controversy over quantified modal logics, is the meaning of quantified modal statements. For consider, from 2) we get:

8) ($\exists x$) Necessarily if there is life on x then there is life on the morning star.

Now, what is this x? Presumably it is the morning star; that is, Venus; that is, the evening star; but this cannot be the case since 3) is false. It is exactly this sort of reasoning that leads to Quine's view that 'being the evening star is not a property that is actually possessed by the lump of rock that is the evening star, but of the way we refer to it'. Or to put it another way, a thing does not have necessary properties in itself, but only properties that necessarily follow from a certain specification of it. It is precisely this view of logical necessity that is embodied in the system $\mathcal{GP}$. Now the way out of the above dilemma taken by Church, Carnap, and others, is to postulate the existence of intensional objects. But the nature of such things is very unclear and their existence dubious, unless one accepts our explanation in section **5** of what they really are. Let us look at 8) in the terminology of $\mathcal{GP}$. Let 'e' be the evening star, 'm' the morning star, and $A(x)$ be 'there is life on x'. Then it is true that

$(\exists x_0)\, L(A(x_0) \rightarrow A(m))$ is meaningless, but not so

8′) $$(\exists x_1)\, L(A(x_1) \rightarrow A(m))$$

which is true, since it is obtained by existential generalization in $\mathcal{GP}$ from $L(A(\mathrm{m}) \rightarrow A(\mathrm{m}))$. We note also that $L(A(\mathrm{e}) \rightarrow A(\mathrm{m}))$ is false. It is clear then that we can attribute meanings to quantified modal statements without the usual paradoxical results and metaphysical obscurities, by simply considering them as expressions in an informal mode of speech of the corresponding statements in $\mathcal{GP}$, e.g., 8′) for 8).

There is a standard argument against this sort of explication, viz. Church's translation argument. Now of the many dubious points on which this argument rests, one is the assumption that there are things called meanings, a particular one of which is assigned to every meaningful sentence of every language. Such an argument can then convince only people who have this over-simplified view of meaning. A second point on which the argument rests is the assumption that a quotation is nothing more than a name, and that constituents of the quotation occur merely as an orthographic accident (c.f., 'cat' in 'cattle'). This is certainly false, however. Firstly, the object which a quotation names can always be recovered from the quotation, and this is not true in general of names. Secondly a quotation has a discernible inner structure isomorphic to the structure of that which it names, and this is certainly not true of single indissoluble entities. Hence, we will reject this assumption and with it the argument, and leave the point.

7 Finally, in this paper we consider a point which is often a problem for theories of this sort, namely, how we are to interpret sentences which appear to have quantifiers of mixed semantic category. Consider how, for example, we are to interpret

1) There is something that is greater than 9 and necessarily greater than 9,

i.e.,

2) $$(\exists x)(x > 9 \wedge L(x > 9)).$$

The problem is that as it stands 2) can be neither a sentence of O, nor $\mathcal{GP}$, nor Translational $\mathcal{GP}$, since we have a quantifier that binds variables both inside and outside an 'L'. We have three courses open to us.

a) We can declare all such sentences meaningless. However, this is not very satisfactory, since we would certainly like to infer 2) from

3) $$10 > 9 \wedge L(10 > 9).$$

b) As was suggested in [1] in another context, we could introduce another predicate 'T' into $\mathcal{GP}$, carry over convention C about autonomous usage, and axiomatize 'T' in such a way that it is possible to interpret 'Tp' as 'p is true'. E.g., we would have axioms such as:

$$Lp \rightarrow Tp.$$
$$Tp \rightarrow \neg T \neg p, \text{ etc.}$$

'Tp' is now a statement in $\mathcal{GP}$ which has exactly the same intuitive strength as p. We can hence, infer

4) $T(10 > 9) \wedge L(10 > 9)$ from 3)

and

5) $(\exists x_1)(T(x_1 > 9) \wedge L(x_1 > 9))$ from 4),

and interpret 2) as 5), which has the same meaning (and even rhetorical content) as 2).

c) One object may have many names, but it is an assumption of classical logic that all names name only one thing (or at least that all ambiguities are theoretically eliminable). Hence, if a is a term of O, we introduce a new function symbol $(\ )p$ into $\mathcal{GP}$ where $(a)p$ is to mean the object denoted by 'a'. $(\ )p$ is the projection of the terms of O into those objects of which O talks. We then add the new axiom:

$$(a)p = a \text{ for every closed term } a, \text{ of } O,$$

(where '=' is the identity of O, and hence, we must work in Translational $\mathcal{GP}$). We can now interpret 3) in Translational $\mathcal{GP}$ as:

$(\exists x_1)(L(x_1 > 9) \wedge (x_1)p > 9)$, or by an abuse of notation: $(\exists x_1)(L(x_1 > 9) \wedge x_1 > 9)$, since this can be interpreted in no other way. Similarly we interpret 'There is something that is necessarily greater than 8 and equal to the number of planets' as:

$$(\exists x_1)L(x_1 > 9) \wedge (x_1)p = \text{the numbers of planets.}$$

Thus we see that we can interpret sentences with quantifiers of mixed semantic category, in the way of either b) or c), quite satisfactorily.

8 We conclude then, this exposition of modal logic as a semantic theory. We claim that it is a much clearer, more natural way of formulating modal logic, and eliminates most standard problems. It allows us to make meaningful quantified modal statements without having to do a lot of ontological wriggling, explains intensional objects, and throws much light on the nature of logical necessity.

REFERENCES

[1] Priest, G., "Modality as meta-concept," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 401-414.

[2] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co., London (1968).

[3] Quine, W. V. O., "Reference and modality," in *From a Logical Point of View*, Harvard University Press, Cambridge, Massachusetts (1953), pp. 139-159.

[4] Quine, W. V. O., "Three grades of modal involvement," *Actes du XIème Congrès International de Philosophie*, Volume XIV, *Volume complêmentaire et communications du Colloque de Logique*, North-Holland, Amsterdam (1953), pp. 65-81.

[5] Carnap, R., *Meaning and Necessity. A study in semantics and modal logic*, University of Chicago Press, Chicago (1947).

[6] Prior, A. N., "Modal logic," in *The Encyclopedia of Philosophy*, ed. by P. Edwards, vol. 5, Macmillan, New York (1967), pp. 5-12.

[7] Frege, G., "On sense and reference," in *Translations from the Philosophical Writings of Gottlob Frege*, ed. by Peter Geach and Max Black. Basil Blackwell, Oxford (1952), pp. 56-78.

[8] Barcan Marcus, R. C., "Interpreting quantification," *Inquiry*, vol. 5 (1962), pp. 252-259.

[9] Tarski, A., "The concept of truth in formalized languages," in *Logic, Semantics, Metamathematics*, The Clarendon Press, Oxford (1956), pp. 152-278.

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