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A REGULAR GRAPH OF GIRTH 6 AND VALENCY 11

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ABSTRACT. Let f(11,6) be the number of vertices of an (11,6)-cage. By giving a regular graph of girth 6 and valency 11, we show that $f(11,6) \leq 240$. This is the best known upper bound for f(11,6).

KEY WORDS AND PHRASES. Regular graph, cage, latin square.
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1. INTRODUCTION.

A graph is said to be <u>regular</u> of valency v if each of its vertices has valency v. A regular graph of valency v and girth g with the least possible number of vertices is called a <u>(v,g)-cage</u>. The number of vertices of a (v,g)-cage is denoted by f(v,g). The existence of (v,g)-cages was proved by Erdös and Sachs [1].

When g = 6 , it is easy to see that $f(v,6) \geq 2(v^2 - v + 1)$ $(v \geq 3)$. Also, it is known that, when v-1 is a prime power, $f(v,6) = 2(v^2 - v + 1)$ and the graph can be constructed by using a complete set of orthogonal latin squares (see [2, p.7]). When v=7, O'Keefe and Wong showed that f(7,6) = 90 [3]. Thus far, no other (v,6)-cages have been found.

The construction of an (11,6)-cage is of great interest and this would settle an open problem in projective geometry: can a projective plane of order 10 exist? We now give a regular graph of girth 6 and valency 11 with 240 vertices. Therefore, $f(11,6) \leq 240$. This is the best known upper bound for f(11,6). The graph is presented in the following way. Eighteen (18) "opposite" vertices A_n and B_n (n = 1,2, ...,9) are added to Figure 8 in [2, p.7] with k = 10. B_n is adjacent to A_n, 1n,2n, ...,10n (n=1,2,...,9). The array of numbers in Table I shows how the vertices of set Y are joined to those of set X, row N standing for set N of Y ; the number M appearing in row N means that vertex NM is adjacent to the vertex of set X in the same column (M, N = 1,2, ... , 10). We note that the sets 2,3, ... ,10 are obtained by permuting cyclically the rows of set 1. Because of this very unusual property of the graph, the author believes that this upper bound is not the best possible one.

Table I

(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(110)
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5

(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(210)
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6

(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)	(310)
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7

(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(410)
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8

(51)	(52)	(53)	(54)	(55)	(56)	(57)	(58)	(59)	(510)
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9

(61)	(62)	(63)	(64)	(65)	(66)	(67)	(68)	(69)	(610)
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10

(71)	(72)	(73)	(74)	(75)	(76)	(77)	(78)	(79)	(710)
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1

(81)	(82)	(83)	(84)	(85)	(86)	(87)	(88)	(89)	(810)
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2

(91)	(92)	(93)	(94)	(95)	(96)	(97)	(98)	(99)	(910)
5	9	2	7	3	1	8	10	A_9	4
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2

(101)	(102)	(103)	(104)	(105)	(106)	(107)	(108)	(109)	(1010)
7	6	10	3	8	4	2	9	1	5
A_1	8	7	1	4	9	5	3	10	6
3	A_2	9	8	2	5	10	6	4	7
2	4	A_3	10	9	3	6	1	7	8
6	3	5	A_4	1	10	4	7	2	9
9	7	4	6	A_5	2	1	5	8	10
4	10	8	5	7	A_6	3	2	6	1
10	5	1	9	6	8	A_7	4	3	2
8	1	6	2	10	7	9	A_8	5	3
5	9	2	7	3	1	8	10	A_9	4

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