A Regularity Theorem for Minimizers of Quasiconvex Integrals

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Summary

We prove $C^{1,\alpha}$ partial regularity for minimizers of functionals with quasiconvex integrand $f(x, u, Du)$ depending on vector-valued functions u. The integrand is required to be twice continuously differentiable in Du, and no assumption on the growth of the derivatives of f is made: a polynomial growth is required only on f itself.

Introduction

Consider the functional $I(u) = \int_\Omega f(Du(x))\, dx$, where Ω is an open subset of $\mathbb{R}^n$,

$$u : \Omega \to \mathbb{R}^N$$

and $f: \mathbb{R}^{nN} \to \mathbb{R}$.

The regularity of minimizers of I has been widely investigated (see [8] and its extensive bibliography), but until recently the function f was required to be convex, which rules out many interesting physical examples (see [2]) and is far from quasiconvexity (this condition is necessary and sufficient for the semicontinuity of I on appropriate Sobolev spaces, see [1], and so it is a fundamental assumption for the existence of such minimizers).

EVANS [5] proved in 1984 the $C^{1,\alpha}$ partial regularity of minimizers of I under the assumptions that f is of class C^2,

$$|D^2 f(\xi)| \leqq c(1 + |\xi|^{p-2}) \tag{1.1}$$

for some $p \geqq 2$, and f is uniformly strictly quasiconvex, *i.e.*

$$\int_\Omega f(\xi + D\varphi(x))\, dx \geqq \int_\Omega [f(\xi) + \gamma(|D\varphi(x)|^2 + |D\varphi(x)|^p)]\, dx \tag{1.2}$$

for some $\gamma > 0$ and all $\varphi \in C_0^1(\Omega; \mathbb{R}^N)$. This conclusion may be generalized ([7], [9], [10]) to the case when f depends also on (x, u).

It is clear that assumption (1.2) considerably enlarges the class of functions to which the theory applies: see [5], section 8. However, while condition (1.1) is natural when f is a convex function with polynomial growth, it seems too strong when f is quasiconvex: for instance, the function $(n = N = p)$

$$f(\xi) = |\xi|^2 + |\xi|^n + \sqrt{1 + |\det \xi|^2}$$

is of class C^2 and satisfies (1.2), but not (1.1). More generally, let $1 < \alpha < 2$, $p = n\alpha$ and let $\beta: \mathbb{R} \to \mathbb{R}$ be a strictly convex function of class C^2 with $|\beta(t)| \leqq c(1 + |t|^\alpha)$: then again

$$f(\xi) = |\xi|^2 + |\xi|^p + \beta(\det \xi)$$

satisfies (1.2) and not (1.1).

In this paper we prove $C^{1,\alpha}$ partial regularity (theorem [II.1]) for minimizers of I under the assumptions that f satisfies (1.2) and is of class C^2; while there are no restrictions on its second derivatives, instead it satisfies the inequality

$$|f(\xi)| \leqq c(1 + |\xi|^p).$$

The examples above satisfy these assumptions.

A similar conclusion (theorem [II.2]) is proved when f depends also on (x, u).

The proofs use essentially two main tools: the blow-up method (as used in [6], where it is shown that it is not necessary to pass through a Caccioppoli inequality, which would require restrictions on the second derivatives of f), and the approximation lemma [II.6] combined with a higher integrability result for minima of certain non-coercive functionals.

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Statements and Preliminary Lemmas

We now lay down the definitions we shall use to state our main results. Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $p \geqq 2$. We begin with the particular case in which f is independent of (x, u): let $f: \mathbb{R}^{nN} \to \mathbb{R}$ satisfy

$$f \text{ is of class } C^2 \tag{2.1}$$

$$|f(\xi)| \leqq L(1 + |\xi|^p) \tag{2.2}$$

$$\int_\Omega f(\xi + D\varphi(x))\, dx \geqq \int_\Omega [f(\xi) + \gamma(|D\varphi(x)|^2 + |D\varphi(x)|^p)]\, dx \quad \text{for every } \xi \in \mathbb{R}^{nN} \text{ and } \varphi \in C_0^1(\Omega; \mathbb{R}^N) \tag{2.3}$$

for suitable positive constants L, γ.

By (2.3), the function f is quasiconvex; therefore step 2 of [11], page 6, applies and we may assume

$$|Df(\xi)| \leqq L(1 + |\xi|^{p-1}). \tag{2.4}$$

For every $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ we set

$$I(u) = \int_\Omega f(Du(x))\, dx.$$

We say that u is a minimizer of I if

$$I(u) \leqq I(u + \varphi) \quad \text{for every } \varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N).$$

Then we have:

Theorem [II.1]. *Let f be as above, and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a minimizer of I. Then there is an open subset Ω_0 of Ω such that*

$$\operatorname{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\mu}(\Omega_0; \mathbb{R}^N) \text{ for every } \mu < 1.$$

If f depends also on (x, u), we assume that $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfies

$$f_{\xi\xi}(x, u, \xi) \quad \text{is continuous;} \tag{2.5}$$

$$|f(x, u, \xi)| \leqq L(1 + |\xi|^p); \tag{2.6}$$

$$|f(x, u, \xi) - f(y, v, \xi)| \leqq L(1 + |\xi|^p)\, \omega(|x - y|^p + |u - v|^p), \tag{2.7}$$

where $\omega(t) \leqq t^\sigma$, $0 < \sigma < 1/p$ and ω is bounded, concave, non-negative and increasing;

$$\int_\Omega f(x, u, \xi + D\varphi(y))\, dy \geqq \int_\Omega [f(x, u, \xi) + \gamma(|D\varphi(y)|^2 + |D\varphi(y)|^p)]\, dy$$
$$\text{for every } (x, u, \xi) \text{ and every } \varphi \in C_0^1(\Omega; \mathbb{R}^N); \tag{2.8}$$

there is a continuous function $\psi\colon \mathbb{R}^{nN} \to \mathbb{R}$ satisfying

$$f(x, u, \xi) \geqq \psi(\xi) \tag{2.9}$$

and

$$\int_\Omega \psi(D\varphi(y))\, dy \geqq \int_\Omega [\psi(0) + \gamma\, |D\varphi(y)|^p]\, dy \quad \text{for every } \varphi \in C_0^1(\Omega; \mathbb{R}^N),$$

with $L, \gamma > 0$.

As before, (2.6) and (2.8) imply

$$|f_\xi(x, u, \xi)| \leqq L(1 + |\xi|^{p-1}). \tag{2.10}$$

We remark that (2.9) is obviously satisfied if $f(x, u, \xi) \geqq |\xi|^p$, and that (2.9) allows also integrands f with variable sign. Set $I(u) = \int_\Omega f(x, u(x), Du(x))\, dx$; then we have

Theorem [II.2]. *Let f satisfy* (2.5), ..., (2.9), *and let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be a minimizer of I. Then there is an open subset* Ω_0 *of* Ω *such that*

$$\operatorname{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\mu}(\Omega_0; \mathbb{R}^N) \quad \text{for some } \mu < 1.$$

We remark here that assumptions (2.6), ..., (2.9) may be slightly weakened (see for instance [7], Remark 2).

It is worth noting that if the minimizer u happens to be continuous (for instance if $p > n$), then assumption (2.9) (first used in [10]), which is employed only in Lemma [IV.3] and Remark [IV.4], may be dropped. The same is true also when f depends only on (x, ξ).

In the sequel we denote by the same letter c any positive constant, which may vary from line to line.

If g is any vector-valued function, we denote by $(g)_{x_0,r}$ the mean value of g on $B_r(x_0)$; if no confusion is possible, we will simply write $(g)_r$ and B_r instead of $(g)_{x_0,r}$ and $B_r(x_0)$. We shall use in the proofs of Theorems [II.1], [II.2] the following lemmas:

Lemma [II.3]. *Let* $p \geqq 2$, *and let* $f: \mathbb{R}^k \to \mathbb{R}$ *be a function of class* C^2 *satisfying*

$$|f(\xi)| \leqq L(1 + |\xi|^p), \qquad |Df(\xi)| \leqq L(1 + |\xi|^{p-1}).$$

Then for every $M > 0$ *there is a constant c, depending on M, such that if we set for any* $\lambda > 0$ *and* $A \in \mathbb{R}^k$ *with* $|A| \leqq M$

$$f_{A,\lambda}(\xi) = \lambda^{-2}[f(A + \lambda\xi) - f(A) - \lambda\, Df(A)\, \xi]$$

then

$$|f_{A,\lambda}(\xi)| \leqq c(|\xi|^2 + \lambda^{p-2} |\xi|^p),$$
$$|Df_{A,\lambda}(\xi)| \leqq c(|\xi| + \lambda^{p-2} |\xi|^{p-1}).$$

Proof. Set $K_M = \max\{|D^2 f(\xi)| : |\xi| \leqq M + 1\}$; then we have:

$$|\lambda\xi| \leqq 1 \Rightarrow |f_{A,\lambda}(\xi)| = \tfrac{1}{2} |D^2 f(A + \vartheta\lambda\xi)\, \xi\xi| \leqq \tfrac{1}{2} K_M |\xi|^2;$$

$$|\lambda\xi| > 1 \Rightarrow |f_{A,\lambda}(\xi)| \leqq \lambda^{-2} c(M)\, (1 + |\lambda\xi| + |\lambda\xi|^p) \leqq 3c(M)\, \lambda^{p-2} |\xi|^p,$$

and the first inequality is proven; the second is analogous. □

Lemma [II.4]. *Let* $p \geqq 2$, *and let* $g: \mathbb{R}^{nN} \to \mathbb{R}$ *be a function of class* C^1 *satisfying*

$$|g(\xi)| \leqq c_1(|\xi|^2 + \lambda^{p-2} |\xi|^p)$$

$$|Dg(\xi)| \leqq c_1(|\xi| + \lambda^{p-2} |\xi|^{p-1})$$

$$\int g(D\varphi)\, dx \geqq \gamma \int (|D\varphi|^2 + \lambda^{p-2} |D\varphi|^p)\, dx \quad \text{for all } \varphi \in C_0^1(\mathbb{R}^n; \mathbb{R}^N)$$

for suitable constants c_1, λ *and* γ.

Fix $\nu \geqq 0$ *and let* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ *satisfy*

$$\int_\Omega g(Du)\,dx \leqq \int_\Omega [g(Du + D\varphi) + \nu\,|D\varphi|]\,dx \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N).$$

Then there are $c_2, \delta > 0$, *depending only on* c_1, γ, *such that for every* $B_r \subset \Omega$

$$\fint_{B_{r/2}} (|Du|^2 + \lambda^{p-2}\,|Du|^p)^{1+\delta}\,dx \leqq c_2 \left[\fint_{B_r} (\nu^2 + |Du|^2 + \lambda^{p-2}\,|Du|^p)\,dx\right]^{1+\delta}.$$

Proof. Fix $B_r \subset \Omega$, let $\frac{1}{2}r < t < s < r$ and take a cut-off function $\zeta \in C_0^1(B_s)$ such that $0 \leqq \zeta \leqq 1$, $\zeta = 1$ on B_t and $|D\zeta| \leqq \dfrac{2}{s-t}$. If we set

$$\varphi_1 = [u - (u)_r]\,\zeta, \qquad \varphi_2 = [u - (u)_r]\,(1 - \zeta),$$

then $D\varphi_1 + D\varphi_2 = Du$, and

$$\gamma \int_{B_s} (|D\varphi_1|^2 + \lambda^{p-2}\,|D\varphi_1|^p)\,dx \leqq \int_{B_s} g(D\varphi_1)\,dx = \int_{B_s} g(Du - D\varphi_2)\,dx. \tag{2.10}$$

In addition, by the minimality of u,

$$\begin{aligned}\int_{B_s} g(Du)\,dx &\leqq \int_{B_s} g(Du - D\varphi_1)\,dx + \nu \int_{B_s} |D\varphi_1|\,dx \\ &\leqq \int_{B_s\setminus B_t} g(D\varphi_2)\,dx + \frac{\gamma}{2}\int_{B_s} |D\varphi_1|^2\,dx + \frac{\nu^2}{2\gamma}\,\text{meas}\,(B_r).\end{aligned}$$

Then

$$\begin{aligned}\int_{B_s} g(Du - D\varphi_2)\,dx &= \int_{B_s} g(Du)\,dx + \int_{B_s} [g(Du - D\varphi_2) - g(Du)]\,dx \\ &\leqq \int_{B_s\setminus B_t} g(D\varphi_2)\,dx + \frac{\gamma}{2}\int_{B_s} |D\varphi_1|^2\,dx + \frac{\nu^2}{2\gamma}\,\text{meas}\,(B_r) \\ &\quad + \int_{B_s\setminus B_t} |Dg(Du - \vartheta\,D\varphi_2)|\,|D\varphi_2|\,dx.\end{aligned} \tag{2.11}$$

By (2.10), (2.11) and the assumptions on g it then follows

$$\begin{aligned}\int_{B_t} (|Du|^2 + \lambda^{p-2}\,|Du|^p)\,dx &\leqq \int_{B_s} (|D\varphi_1|^2 + \lambda^{p-2}\,|D\varphi_1|^p)\,dx \\ &\leqq c(\gamma, c_1)\left[\nu^2 r^n + \int_{B_s\setminus B_t} [|Du|^2 + |D\varphi_2|^2 + \lambda^{p-2}(|Du|^p + |D\varphi_2|^p)\,dx\right] \\ &\leqq \tilde{c}\left[\nu^2 r^n + \int_{B_s\setminus B_t} (|Du|^2 + \lambda^{p-2}\,|Du|^p)\,dx\right. \\ &\quad \left. + \int_{B_s\setminus B_t}\left(\frac{|u - (u)_r|^2}{(s-t)^2} + \lambda^{p-2}\frac{|u - (u)_r|^p}{(s-t)^p}\right)dx\right].\end{aligned}$$

We fill the hole by adding to both sides the term

$$\tilde{c} \int_{B_t} (|Du|^2 + \lambda^{p-2} |Du|^p)\, dx;$$

then we divide by $\tilde{c} + 1$, thus obtaining

$$\int_{B_t} (|Du|^2 + \lambda^{p-2} |Du|^p)\, dx \leqq \vartheta \int_{B_s} (|Du|^2 + \lambda^{p-2} |Du|^p)\, dx$$

$$+ c \int_{B_r} \left[\nu^2 + \frac{|u - (u)_r|^2}{(s-t)^2} + \lambda^{p-2} \frac{|u - (u)_r|^p}{(s-t)^p} \right] dx,$$

with $\vartheta < 1$. Now a standard lemma (see *e.g.* [8] page 161 or [7] Lemma 3.2) yields

$$\fint_{B_{r/2}} (|Du|^2 + \lambda^{p-2} |Du|^p)\, dx \leqq c \fint_{B_r} \left(\nu^2 + \frac{|u - (u)_r|^2}{r^2} + \lambda^{p-2} \frac{|u - (u)_r|^p}{r^p} \right) dx$$

$$\leqq c \left[\fint_{B_r} (\nu^2 + |Du|^2 + \lambda^{p-2} |Du|^p)^{n/(n+2)}\, dx \right]^{(n+2)/n}; \tag{2.12}$$

we have used the Sobolev-Poincaré inequality.

The result follows from (2.12) by a modification of Gehring's theorem (see [8] page 122). □

The next lemma may be found in [3].

Lemma [II.5]. *Let G be a measurable subset of $\mathbb{R}^k$, with meas $(G) < +\infty$. Assume (M_h) is a sequence of measurable subsets of G such that, for some $\varepsilon > 0$, the following estimate holds*:

$$\text{meas}\,(M_h) \geqq \varepsilon \quad \text{for all } h \in \mathbb{N}.$$

Then a subsequence (M_{h_k}) can be selected such that $\bigcap_k M_{h_k} \neq \emptyset$.

By Lemmas [I.9], …, [I.12] of [1] one may deduce (see also [13] for a self-contained proof):

Lemma [II.6]. *Let Ω be a regular bounded open subset of $\mathbb{R}^n$, $q \geqq 1$ and $u \in W^{1,q}(\Omega; \mathbb{R}^N)$. For every $K > 0$ there is a $w \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ such that*

$$\|w\|_{1,\infty} \leqq K$$

$$\text{meas}\,\{x \in \Omega : u(x) \neq w(x)\} \leqq c \frac{\|u\|_{1,q}^q}{K^q},$$

and c is independent of K.

Proof of Theorem [II.1]

In this section we assume f satisfies (2.1), ..., (2.3) and we denote by $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ a minimizer of $I(u) = \int_\Omega f(Du)\,dx$. As in [5], we will prove a decay estimate (Proposition III.1]) from which the result will follow by a standard argument.

For every $B_r(x_0) \subset \Omega$ define

$$U(x_0, r) = \fint_{B_r(x_0)} (|Du - (Du)_r|^2 + |Du - (Du)_r|^p)\,dx.$$

Then we have

Proposition [III.1]. *Fix $M > 0$; there is a constant $C_M > 0$ such that for every $\tau < \frac{1}{2}$ there is an $\varepsilon = \varepsilon(\tau, M)$ such that if*

$$|(Du)_{x_0,r}| \leqq M \quad \text{and} \quad U(x_0, r) \leqq \varepsilon,$$

then

$$U(x_0, \tau r) \leqq C_M \tau^2 U(x_0, r).$$

Proof. Fix M and τ; we shall determine C_M later.

Reasoning by contradiction, we assume that there is a sequence $B_{r_h}(x_h)$ satisfying

$$B_{r_h}(x_h) \subset \Omega, \quad |(Du)_{x_h,r_h}| \leqq M, \quad \lim_h U(x_h, r_h) = 0$$

and

$$U(x_h, \tau r_h) > C_M \tau^2 U(x_h, r_h). \tag{3.1}$$

We introduce the following notations:

$$a_h = (u)_{x_h,r_h}, \quad A_h = (Du)_{x_h,r_h}, \quad \lambda_h^2 = U(x_h, r_h).$$

Since the proof is quite long, we divide it into several steps; moreover, we shall often pass to subsequences and still denote them by the same index h.

Step 1: Blow-up. We rescale the function u in each $B_{r_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$. Set

$$v_h(y) = \frac{1}{\lambda_h r_h}[u(x_h + r_h y) - a_h - r_h A_h y];$$

then

$$Dv_h(y) = \frac{1}{\lambda_h}[Du(x_h + r_h y) - A_h],$$

$$(v_h)_{0,1} = 0, \quad (Dv_h)_{0,1} = 0$$

and

$$\fint_{B_1(0)} (|Dv_h|^2 + \lambda_h^{p-2}|Dv_h|^p)\,dy = 1. \tag{3.2}$$

Without loss of generality we may then assume

$$v_h \to v \quad \text{weakly in } W^{1,2}(B_1;\mathbb{R}^N) \tag{3.3}$$

and, since $|A_h| \leqq M$,

$$A_h \to A. \tag{3.4}$$

Step 2: v satisfies a Linear System. We show that

$$\int_{B_1} \frac{\partial^2 f}{\partial\xi_\alpha^i\,\partial\xi_\beta^j}(A)\,D_\beta v^j\,D_\alpha\varphi^i\,dy = 0 \quad \text{for all } \varphi \in C_0^1(B_1;\mathbb{R}^N). \tag{3.5}$$

From the Euler system for u, rescaled in each $B_{r_h}(x_h)$, we deduce for every $\varphi \in C_0^1(B_1;\mathbb{R}^N)$

$$\int_{B_1} \frac{\partial f}{\partial\xi_\alpha^i}(A_h + \lambda_h Dv_h)\,D_\alpha\varphi^i\,dy = 0,$$

whence

$$\frac{1}{\lambda_h}\int_{B_1}\left[\frac{\partial f}{\partial\xi_\alpha^i}(A_h + \lambda_h Dv_h) - \frac{\partial f}{\partial\xi_\alpha^i}(A_h)\right] D_\alpha\varphi^i\,dy = 0. \tag{3.6}$$

Fixing φ, we split B_1 as follows:

$$E_h^+ \cup E_h^- = \{y \in B_1 : \lambda_h|Dv_h(y)| > 1\} \cup \{y \in B_1 : \lambda_h|Dv_h(y)| \leqq 1\}.$$

As for E_h^+, we get by (3.2)

$$\operatorname{meas}(E_h^+) \leqq \int_{B_1} \lambda_h^2|Dv_h|^2\,dy \leqq \lambda_h^2; \tag{3.7}$$

therefore, using (2.4),

$$\begin{aligned}\frac{1}{\lambda_h}\left|\int_{E_h^+}[Df(A_h + \lambda_h Dv_h) - Df(A_h)]\,D\varphi\,dy\right| &\leqq \frac{c}{\lambda_h}\int_{E_h^+}(1 + \lambda_h^{p-1}|Dv_h|^{p-1})\,dy\\ &\leqq c\left(\lambda_h + \int_{E_h^+}\lambda_h^{p-2}|Dv_h|^{p-1}\,dy\right)\\ &\leqq c\left(\lambda_h + \lambda_h^{(p-2)/p}[\operatorname{meas}(E_h^+)]^{1/p}\left(\int_{B_1}\lambda_h^{p-2}|Dv_h|^p\,dy\right)^{(p-1)/p}\right).\end{aligned}$$

Using (3.2), we obtain

$$\lim_h \frac{1}{\lambda_h}\int_{E_h^+}[Df(A_h + \lambda_h Dv_h) - Df(A_h)]\,D\varphi\,dy = 0. \tag{3.8}$$

On E_h^- we have

$$\frac{1}{\lambda_h}\int_{E_h^-}[Df(A_h+\lambda_h Dv_h)-Df(A_h)]\,D\varphi\,dy=\int_{E_h^-}\int_0^1 D^2f(A_h+s\lambda_h Dv_h)\,Dv_h\,D\varphi\,ds\,dy$$

$$=\int_{E_h^-}\int_0^1[D^2f(A_h+s\lambda_h Dv_h)-D^2f(A)]\,Dv_h\,D\varphi\,ds\,dy+\int_{E_h^-}D^2f(A_h)\,Dv_h\,D\varphi\,dy.$$

We observe that (3.7) implies that $\mathbb{1}_{E_h^-}\to\mathbb{1}_{B_1}$ in $L^q(B_1)$ for all $q<\infty$, and that by (3.3) we have

$$\lambda_h\,Dv_h(y)\to 0\quad\text{a.e. in } B_1.$$

Then by (3.3), (3.4), our choice of E_h^-, and the uniform continuity of D^2f on bounded sets, we get

$$\lim_h\frac{1}{\lambda_h}\int_{E_h^-}[Df(A_h+\lambda_h Dv_h)-Df(A_h)]\,D\varphi\,dy=\int_{B_1}D^2f(A)\,Dv\,D\varphi\,dy,$$

which together with (3.8) proves (3.5).

Assumption (2.3) ensures that

$$\gamma\,|\mu|^2\,|\eta|^2\leqq\frac{\partial^2 f}{\partial\xi_\alpha^i\,\partial\xi_\beta^j}(A)\,\mu_i\mu_j\eta_\alpha\eta_\beta\leqq c(M)\,|\mu|^2\,|\eta|^2,$$

therefore (see [8] Chapter 3) the solution v of (3.5) satisfies

$$\fint_{B_\tau}|Dv-(Dv)_\tau|^2\,dy\leqq c^*(M)\,\tau^2\quad\text{for every } \tau<1/2, \tag{3.9}$$

$$v\in C^\infty(B_1;\mathbb{R}^N), \tag{3.10}$$

$$\lambda_h^{(p-2)/p}(v_h-v)\to 0\quad\text{weakly in } W^{1,p}_{\mathrm{loc}}(B_1;\mathbb{R}^N); \tag{3.11}$$

we have used (3.2), (3.3).

Step 3: Higher Integrability of (v_h). If we set

$$f_h(\xi)=\lambda_h^{-2}[f(A_h+\lambda_h\xi)-f(A_h)-\lambda_h\,Df(A_h)\,\xi]$$

then by Lemma [II.3] we have

$$|f_h(\xi)|\leqq c(|\xi|^2+\lambda_h^{p-2}\,|\xi|^p) \tag{3.12}$$

$$|Df_h(\xi)|\leqq c(|\xi|+\lambda_h^{p-2}\,|\xi|^{p-1})$$

for a suitable constant $c=c(M)$, while (2.3) implies

$$\int_{B_1}f_h(D\varphi)\,dy\geqq\gamma\int_{B_1}(|D\varphi|^2+\lambda_h^{p-2}\,|D\varphi|^p)\,dy\quad\text{for all } \varphi\in C_0^1(B_1;\mathbb{R}^N). \tag{3.13}$$

Set for every $r<1$

$$I_r^h(w)=\int_{B_r}f_h(Dw)\,dy;$$

it is easily verified that v_h is a minimizer of each I_r^h. The assumptions of Lemma [II.4] are thus satisfied, with $\nu = 0$, and therefore

$$\int_{B_{1/2}} (|Dv_h|^2 + \lambda_h^{p-2} |Dv_h|^p)^{1+\delta} \, dy \leqq c \tag{3.14}$$

with c, δ depending on M.

From this estimate and (3.3) we obtain

$$v_h \rightharpoonup v \quad \text{weakly in } W^{1,2+2\delta}(B_{1/2}; \mathbb{R}^N).$$

Step 4: Upper bound. Fix $r < \frac{1}{2}$: it is not restrictive to assume that

$$\lim_h \, [I_r^h(v_h) - I_r^h(v)]$$

exists.

We prove that

$$\lim_h \, [I_r^h(v_h) - I_r^h(v)] \leqq 0. \tag{3.15}$$

Choose $s < r$ and take $\zeta \in C_0^\infty(B_r)$ such that $0 \leqq \zeta \leqq 1$, $\zeta = 1$ on B_s and $|D\zeta| \leqq 2/(r-s)$; if we set

$$\varphi_h = (v - v_h)\,\zeta,$$

by (3.10), (3.12) and the minimality of v_h follows

$$\begin{aligned} I_r^h(v_h) - I_r^h(v) &\leqq I_r^h(v_h + \varphi_h) - I_r^h(v) \\ &= \int_{B_r \setminus B_s} [f_h(Dv_h + D\varphi_h) - f_h(Dv)] \, dy \\ &\leqq c \int_{B_r \setminus B_s} \left(1 + |Dv_h|^2 + \lambda_h^{p-2} |Dv_h|^p + \frac{|v_h - v|^2}{(r-s)^2} + \lambda_h^{p-2} \frac{|v_h - v|^p}{(r-s)^p}\right) dy. \end{aligned}$$

But by (3.14), for every $E \subset B_{\frac{1}{2}}$

$$\int_E (|Dv_h|^2 + \lambda_h^{p-2} |Dv_h|^p) \, dy \leqq c \, [\text{meas}\,(E)]^{\delta/(1+\delta)}, \tag{3.16}$$

so that

$$I_r^h(v_h) - I_r^h(v) \leqq o(r-s) + \frac{c}{(r-s)^p} \int_{B_{1/2}} (|v_h - v|^2 + \lambda_h^{p-2} |v_h - v|^p) \, dy,$$

with $o(t)$ vanishing as $t \to 0$, and (3.15) follows by (3.3), (3.11) and since $s < r$ is arbitrary.

Step 5: Lower bound. We prove that

$$\lim_h \, [I_r^h(v_h) - I_r^h(v_h)] \geqq c(\gamma, p) \limsup_h \int_{B_r} (|Dv_h - Dv|^2 + \lambda_h^{p-2} |Dv_h - Dv|^p) \, dy. \tag{3.17}$$

Fix $K > 0$; by (3.14), using Lemma [II.6] with $q = 2 + 2\delta$, we may find a sequence $(w_h) \subset W^{1,\infty}(B_r; \mathbb{R}^N)$ such that

$$\|w_h\|_{1,\infty} \leqq K$$

$$\operatorname{meas}\{y \in B_r : v_h(y) \neq w_h(y)\} \leqq \frac{\hat{c}}{K^{2+2\delta}} \tag{3.18}$$

(we shall meet this $\hat{c}$ later); set $S_h = \{y \in B_r : v_h(y) \neq w_h(y)\}$. It is not restrictive to assume that

$$w_h \rightharpoonup w \quad \text{weakly* in } W^{1,\infty}(B_r; \mathbb{R}^N).$$

We have

$$\begin{aligned} I_r^h(v_h) - I_r^h(v) &= I_r^h(v_h) - I_r^h(w_h) \\ &\quad + I_r^h(w_h) - I_r^h(w) \\ &\quad + I_r^h(w) - I_r^h(v) \\ &= R_1^h + R_2^h + R_3^h. \end{aligned}$$

Now by (3.12), (3.16) and (3.18)

$$\begin{aligned} |R_1^h| &= \left| \int_{S_h} [f_h(Dv_h) - f_h(Dw_h)]\, dy \right| \\ &\leqq c \int_{S_h} (|Dv_h|^2 + \lambda_h^{p-2} |Dv_h|^p + K^2 + \lambda_h^{p-2} K^p)\, dy \\ &\leqq c \left(\frac{\hat{c}}{K^{2+2\delta}} \right)^{\delta/(1+\delta)} + \frac{c}{K^{2\delta}} + c\lambda_h^{p-2} K^{p-2-2\delta}; \end{aligned}$$

therefore

$$\limsup_h |R_1^h| \leqq \frac{c}{K^{2\delta}}. \tag{3.19}$$

Choose $s < r$ and take ζ as in Step 4. Define

$$\psi_h = (w_h - w)\, \zeta;$$

then

$$\begin{aligned} R_2^h &= I_r^h(w_h) - I_r^h(w + \psi_h) \\ &\quad + I_r^h(w + \psi_h) - I_r^h(w) - I_r^h(\psi_h) + I_r^h(\psi_h) \\ &= R_4^h + R_5^h + R_6^h. \end{aligned}$$

By (3.12) we obtain

$$\begin{aligned} |R_4^h| &= \left| \int_{B_r \setminus B_s} [f_h(Dw_h) - f_h(Dw + D\psi_h)]\, dy \right| \\ &\leqq c(K) \int_{B_r \setminus B_s} \left(1 + \frac{|w_h - w|^2}{(r-s)^2} + \lambda_h^{p-2} \frac{|w_h - w|^p}{(r-s)^p} \right) dy, \end{aligned}$$

so that

$$\limsup_h |R_4^h| \leqq (r-s)\, c(K). \tag{3.20}$$

To bound R_5^h, following [6] we remark that

$$f_h(A+B) - f_h(A) - f_h(B) = \int_0^1 \int_0^1 D^2 f_h(sA + tB)\, AB \, ds\, dt; \tag{3.21}$$

since

$$D^2 f_h(s\, Dw + t\, D\psi_h) = D^2 f(A_h + s\lambda_h\, Dw + t\lambda_h\, D\psi_h)$$

is bounded and converges to $D^2 f(A)$ uniformly, by (3.21) with $A = Dw$ and $B = D\psi_h$, and since $\psi_h \rightharpoonup 0$ weakly* in $W^{1,\infty}(B_r; \mathbb{R}^N)$,

$$\lim_h R_5^h = 0. \tag{3.22}$$

Now we use (3.13) to obtain

$$\begin{aligned} R_6^h &\geqq \gamma \int_{B_r} (|D\psi_h|^2 + \lambda_h^{p-2}\, |D\psi_h|^p)\, dy \\ &\geqq \gamma \int_{B_s} (|Dw_h - Dw|^2 + \lambda_h^{p-2}\, |Dw_h - Dw|^p)\, dy. \end{aligned}$$

Together with (3.20), (3.22) this implies

$$\liminf_h R_2^h \geqq \gamma \limsup_h \int_{B_s} (|Dw_h - Dw|^2 + \lambda_h^{p-2}\, |Dw_h - Dw|^p)\, dy - (r-s)\, c(K). \tag{3.23}$$

To deal with R_3^h we use a technique introduced in [1]: first we prove that (see (3.18) for $\hat{c}$)

$$\text{meas}\,\{y \in B_r : v(y) \neq w(y)\} \leqq \frac{2\hat{c}}{K^{2+2\delta}}. \tag{3.24}$$

Set $S = \{y \in B_r : v(y) \neq w(y)\}$ and

$$\tilde{S} = S \cap \{y \in B_r : v(y) = \lim_h v_h(y)\}:$$

then $\text{meas}\,(S) = \text{meas}\,(\tilde{S})$. We reason by contradiction: if

$$\text{meas}\,(S) > 2\hat{c}/K^{2+2\delta},$$

then by (3.18)

$$\text{meas}\,(\tilde{S} \setminus S_h) > \hat{c}/K^{2+2\delta}$$

for every h, and by Lemma [II.5] there is a $\bar{y} \in B_r$ such that

$$\bar{y} \in \tilde{S} \setminus S_h \quad \text{for infinitely many } h.$$

Passing to this subsequence, we have

$$v(\bar{y}) = \lim_h v_h(\bar{y}) = \lim_h w_h(\bar{y}) = w(\bar{y});$$

hence $\bar{y} \notin S$, which is a contradiction. This proves (3.24). Now, since $Dv = Dw$ a.e. in $B_r \setminus S$, by (3.10), (3.12), (3.24)

$$\begin{aligned}|R_3^h| &\leqq \int_S |f_h(Dw) - f_h(Dv)|\, dy \\ &\leqq c \int_S (K^2 + \lambda_h^{p-2} K^p + |Dv|^2 + \lambda_h^{p-2} |Dv|^p)\, dy \\ &\leqq \frac{c}{K^{2\delta}} + c\lambda_h^{p-2} K^{p-2-2\delta},\end{aligned}$$

so that

$$\limsup_h |R_3^h| \leqq \frac{c}{K^{2\delta}}. \tag{3.25}$$

Finally, we reduce the right hand side of (3.23) to the desired form:

$$\begin{aligned}&\int_{B_s} (|Dw_h - Dw|^2 + \lambda_h^{p-2} |Dw_h - Dw|^p)\, dy \\ &\quad \geqq 3^{1-p} \int_{B_s} (|Dv_h - Dv|^2 + \lambda_h^{p-2} |Dv_h - Dv|^p)\, dy + \\ &\quad - \int_{S_h} (|Dw_h - Dv_h|^2 + \lambda_h^{p-2} |Dw_h - Dv_h|^p)\, dy + \\ &\quad - \int_S (|Dw - Dv|^2 + \lambda_h^{p-2} |Dw - Dv|^p)\, dy.\end{aligned}$$

Therefore, arguing as we did for R_1^h and R_3^h, we obtain

$$\begin{aligned}&\limsup_h \int_{B_s} (|Dw_h - Dw|^2 + \lambda_h^{p-2} |Dw_h - Dw|^p)\, dy \\ &\quad \geqq 3^{1-p} \limsup_h \int_{B_s} (|Dv_h - Dv|^2 + \lambda_h^{p-2} |Dv_h - Dv|^p)\, dy - \frac{c}{K^{2\delta}}.\end{aligned}$$

Putting together (3.19), (3.23), (3.25) and this inequality, then letting $s \to r$ and $K \to \infty$, we get (3.17).

Step 6: Conclusion. Inequalities (3.15), (3.17) imply

$$\lim_h \int_{B_r} (|Dv_h - Dv|^2 + \lambda_h^{p-2} |Dv_h - Dv|^p)\, dy = 0;$$

going back to u and using (3.9) we have

$$\begin{aligned}\lim_h \frac{U(x_h, \tau r_h)}{\lambda_h^2} &= \lim_h \frac{1}{\lambda_h^2} \fint_{B_{\tau r_h}(x_h)} (|Du - (Du)_{\tau r_h}|^2 + |Du - (Du)_{\tau r_h}|^p)\,dx \\ &= \lim_h \fint_{B_\tau} (|Dv_h - (Dv_h)_\tau|^2 + \lambda_h^{p-2}|Dv_h - (Dv_h)_\tau|^p)\,dy \\ &= \fint_{B_\tau} |Dv - (Dv)_\tau|^2\,dy \\ &\leqq c^*(M)\,\tau^2,\end{aligned}$$

which contradicts (3.1) if we chose $C_M = 2c^*(M)$. □

The proof of Theorem [II.1] follows from Proposition [III.1] by a standard argument, see [8] Chapter 6 or [5] Section 7.

Proof of Theorem [II.2]

Throughout this section the function f satisfies (2.5), ..., (2.9). We need some additional lemmas.

Lemma [IV.1]. *Let (X, d) be a metric space, and $J: X \to [0, +\infty]$ a lower semicontinuous functional not identically $+\infty$. If*

$$J(u) < \alpha + \inf J,$$

there is a $v \in X$ such that

$$d(u, v) \leqq 1$$

and

$$J(v) \leqq J(w) + \alpha d(v, w) \quad \text{for every } w \in X.$$

The result above may be found in [4].

Lemma [IV.2]. *Let $p \geqq 1$, and let $f: \mathbb{R}^{nN} \to \mathbb{R}$ be a quasiconvex function of class C^1 satisfying*

$$|f(\xi)| \leqq L(1 + |\xi|^p).$$

Then for every $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ the functional $\int_\Omega f(Dw)\,dx$ is sequentially lower semicontinuous on the Dirichlet class $u + W_0^{1,p}(\Omega; \mathbb{R}^N)$ endowed with the weak topology of $W^{1,p}$.

Proof. It is enough to observe that f is separately convex, and thus (see 11) it satisfies also the condition

$$|f(\xi + \eta) - f(\xi)| \leqq c(1 + |\xi|^{p-1} + |\eta|^{p-1})\,|\eta|;$$

then the result follows from [12] Theorem 5. □

Lemma [IV.3.] *Let f satisfy* (2.6), (2.9) *and*

$$|f(x, u, \xi + \eta) - f(x, u, \xi)| \leqq c(1 + |\xi|^{p-1} + |\eta|^{p-1}) |\eta|,$$

and let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a minimizer of I. Then there are $q_0 > p$ and $C_0 > 0$, independent of u, such that $u \in W^{1,q_0}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and for every $B_r \subset \Omega$

$$\left(\fint_{B_{r/2}} |Du|^{q_0}\, dx \right)^{1/q_0} \leqq C_0 \left(\fint_{B_r} (1 + |Du|^p)\, dx \right)^{1/p}.$$

Proof. The argument is similar to Lemma [II.4]. Fix $B_r \subset \Omega$, let $\frac{1}{2} r < t < s < r$, take the cut-off function ζ of [II.4], and again set

$$\varphi_1 = [u - (u)_r]\, \zeta, \qquad \varphi_2 = [u - (u)_r]\,(1 - \zeta);$$

then $\varphi_1 + \varphi_2 = u - (u)_r$ and $D\varphi_1 + D\varphi_2 = Du$. Now, by (2.9)

$$\begin{aligned} \int_{B_s} [\gamma\, |D\varphi_1|^p + \psi(0)]\, dx &\leqq \int_{B_s} \psi(D\varphi_1)\, dx \leqq \int_{B_s} f(x, u, D\varphi_1)\, dx \\ &= \int_{B_s} f(x, u, Du - D\varphi_2)\, dx. \end{aligned} \tag{4.1}$$

By the minimality of u we have

$$\begin{aligned} \int_{B_s} f(x, u, Du)\, dx &\leqq \int_{B_s} f(x, u - \varphi_1, Du - D\varphi_1)\, dx \\ &= \int_{B_s} f(x, \varphi_2 + (u)_r, D\varphi_2)\, dx \\ &= \int_{B_s \setminus B_t} f(x, \varphi_2 + (u)_r, D\varphi_2)\, dx + \int_{B_t} f(x, (u)_r, 0)\, dx, \end{aligned}$$

so that by (2.6)

$$\int_{B_s} f(x, u, Du)\, dx \leqq L \int_{B_s \setminus B_t} |D\varphi_2|^p\, dx + cr^n,$$

and by (2.10)

$$\begin{aligned} \int_{B_s} f(x, u, Du - D\varphi_2)\, dx &= \int_{B_s} f(x, u, Du)\, dx + \int_{B_s} [f(x, u, Du - D\varphi_2) - f(x, u, Du)]\, dx \\ &\leqq \int_{B_s} f(x, u, Du)\, dx + c \int_{B_s \setminus B_t} (1 + |Du|^{p-1} + |D\varphi_2|^{p-1})\, |D\varphi_2|\, dx \\ &\leqq cr^n + c \int_{B_s \setminus B_t} (|D\varphi_2|^p + |Du|^p)\, dx \\ &\leqq cr^n + c \int_{B_s \setminus B_t} \left[|Du|^p + \frac{|u - (u)_r|^p}{(s - t)^p} \right] dx. \end{aligned}$$

Then by (4.1) we obtain

$$\int_{B_t} |Du|^p\,dx \leqq c \int_{B_s \setminus B_t} |Du|^p\,dx + c \int_{B_r} \left(1 + \frac{|u-(u)_r|^p}{(s-t)^p}\right) dx.$$

The conclusion follows as in Lemma [II.4]. □

Remark [*IV.4*]. Under the assumptions of Lemma [IV.3], if Ω is a ball B and u is more regular on ∂B, then the higher integrability goes up to the boundary. Precisely, assume there is a function $u_0 \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^N)$, with $q > p$, such that $u - u_0 \in W_0^{1,p}(B;\mathbb{R}^N)$: then there are q_0, C_0, with $p < q_0 < q$, such that $u \in W^{1,q_0}(B;\mathbb{R}^N)$ and

$$\left(\fint_B |Du|^{q_0}\,dx\right)^{1/q_0} \leqq C_0 \left(\left[\fint_B (1 + |Du|^p)\,dx\right]^{1/p} + \left[\fint_B |Du_0|^{q_0}\,dx\right]^{1/q_0}\right). \tag{4.2}$$

To prove this, adapt the proof of [IV.3] following [8], page 152.

Remark [*IV.5*]. The second inequality in (4.1), together with the analogous inequality in the proof of Remark [IV.4], is the only point in this paper where we need assumption (2.9). If f is independent of x or if the minimizer u happens to be continuous, instead of (2.9) we may just use (2.7) to show, if r is sufficiently small and x_0 is the center of B_r, that

$$\int_{B_s} f(x, u, D\varphi_1)\,dx \geqq \int_{B_s} f(x_0, u(x_0), D\varphi_1)\,dx - \varepsilon \int_{B_s} (1 + |D\varphi_1|^p)\,dx,$$

and the inequality follows using (2.8), if $\varepsilon < \gamma$.

Lemma [IV.6]. *Let f satisfy* (2.6), (2.8), *and fix $x_0 \in \Omega$ and $u_0 \in \mathbb{R}^N$. If B_r is any ball in $\mathbb{R}^n$, and $u \in W^{1,p}(B_r;\mathbb{R}^N)$, then the functional $\int_{B_r} f(x_0, u_0, Dw(x))\,dx$ is sequentially weakly lower semicontinuous on $u + W_0^{1,p}(B_r;\mathbb{R}^N)$, and satisfies*

$$\int_{B_r} f(x_0, u_0, Dw(x))\,dx \geqq \gamma \int_{B_r} |Dw|^p\,dx - c \int_{B_r} (1 + |Du|^p)\,dx. \tag{4.3}$$

Proof. The semicontinuity follows from Lemma [IV.2], since (2.8) implies quasi-convexity.

As for (4.3), let $\tilde{u} \in (u_r) + W_0^{1,p}(B_{2r};\mathbb{R}^N)$ be an extension of u such that $\int_{B_{2r}} |D\tilde{u}|^p\,dx \leqq c \int_{B_r} |Du|^p\,dx$; if we set for every $w \in u + W_0^{1,p}(B_r;\mathbb{R}^N)$

$$\tilde{w} = \begin{cases} w & \text{in } B_r \\ \tilde{u} & \text{in } B_{2r} \setminus B_r, \end{cases}$$

then by (2.8)

$$\int_{B_{2r}} [\gamma\,|Dw|^p + f(x_0, u_0, 0)]\,dx \leqq \int_{B_{2r}} f(x_0, u_0, D\tilde{w})\,dx$$

$$= \int_{B_r} f(x_0, u_0, Dw)\,dx + \int_{B_{2r}\setminus B_r} f(x_0, u_0, D\tilde{u})\,dx,$$

and (4.3) follows easily by (2.6). □

Lemma [IV.7]. *There are two constants, $0 < \beta_1 < \beta_2 < 1$, and for every $K > 0$ a constant $c_K > 0$, such that if u is a minimizer of I, $r < 1$, $B_{2r}(x_0) \subset \Omega$ and $(|Du|^p)_{x_0,2r} \leqq K$, then there is a $v \in u + W_0^{1,p}(B_r(x_0); \mathbb{R}^N)$ such that*

$$\left(\int_{B_{r/2}} |Dv - Du|^p \, dx\right)^{1/p} \leqq c_K r^{\beta_1}$$

and

$$\int_{B_r} f(x_0, (u)_{x_0,r}, Dv(x)) \, dx$$

$$\leqq \int_{B_r} f(x_0, (u)_{x_0,r}, Dv(x) + D\varphi(x)) \, dx + r^{\beta_2} \int_{B_r} |D\varphi(x)| \, dx$$

for every $\varphi \in C_0^1(B_r(x_0); \mathbb{R}^N)$.

Proof. By Lemma [IV.3] and the minimality of u follows the existence of $q_0 > p$ and $c_0 > 0$ such that $u \in W_{\mathrm{loc}}^{1,q_0}(\Omega)$ and

$$\left(\fint_{B_{s/2}} |Du|^{q_0} \, dx\right)^{1/q_0} \leqq c_0 \left(\fint_{B_s} (1 + |Du|^p) \, dx\right)^{1/p} \tag{4.4}$$

for every $B_s \subset \Omega$.

Now, by Lemma [IV.6] there is a minimum point $\bar{u}$ on $u + W_0^{1,p}(B_r)$ of the functional

$$I_r^0(w) = \fint_{B_r} f(x_0, (u)_r, Dw) \, dx;$$

by Remark [IV.4] there are numbers q_1 and c_1 with $p < q_1 < q_0$ and both independent of r, such that $\bar{u} \in W^{1,q_1}(B_r)$ and, by (4.2), (4.3),

$$\fint_{B_r} |D\bar{u}|^{q_1} \, dx \leqq c_1 \fint_{B_r} (1 + |Du|^{q_1}) \, dx.$$

Now, by use only of (2.7) and (4.4), the argument employed in [7] Lemma 4.1 yields

$$I_r^0(u) - I_r^0(\bar{u}) \leqq \tilde{c}(K) \, r^{\beta}, \tag{4.5}$$

where $\beta < 1$ depends only on σ, L, p. Consider the space $u + W_0^{1,1}(B_r)$ endowed with the metric

$$d(v, w) = (\tilde{c}(K) \, r^{\beta/2})^{-1} \fint_{B_r} |Dv - Dw| \, dx,$$

and set

$$J(w) = \begin{cases} I_r^0(w) & \text{if } w \in u + W_0^{1,p}(B_r) \\ +\infty & \text{otherwise.} \end{cases}$$

By Lemma [IV.6] the functional J is lower semicontinuous in the metric space above, and clearly

$$\inf J = I_r^0(\bar{u}),$$

therefore by (4.5) and Lemma [IV.1] there is a $v \in u + W_0^{1,1}(B_r)$ satisfying

$$\fint_{B_r} |Dv - Du| \, dx \leqq \tilde{c}(K) \, r^{\beta/2} \tag{4.6}$$

and

$$J(v) \leqq J(v + \varphi) + r^{\beta/2} \fint_{B_r} |D\varphi| \, dx$$

for every $\varphi \in W_0^{1,1}(B_r)$. In particular, $J(v)$ is finite, hence $v \in u + W_0^{1,p}(B_r)$; this proves the last assertion of the lemma, with $\beta_2 = \beta/2$. Moreover, by (4.3)

$$\begin{aligned}\gamma \fint_{B_r} |Dv|^p \, dx &\leqq I_r^0(Dv) + c \fint_{B_r} (1 + |Du|^p) \, dx \\ &\leqq I_r^0(Du) + r^{\beta/2} \fint_{B_r} |Dv - Du| \, dx + c \fint_{B_r} (1 + |Du|^p) \, dx \\ &\leqq c \fint_{B_r} (1 + |Du|^p) \, dx + r^{\beta/2} |Dv - Du| \, dx \\ &\leqq c(K). \end{aligned} \tag{4.7}$$

Consider the functional

$$w \mapsto I_r^0(w) + r^{\beta/2} \fint_{B_r} |Dv - Dw| \, dx .$$

Since its integrand $f(x_0, (u)_r, \xi) + r^{\beta/2} |Dv(x) - \xi|$ satisfies the assumptions of Lemma [IV.3], by the minimality of v there are numbers q and c, independent of K, r and satisfying $p < q < q_0$, such that $v \in W_{\text{loc}}^{1,q}(B_r)$ and

$$\left(\fint_{B_{r/2}} |Dv|^q \, dx \right)^{1/q} \leqq c \left(\fint_{B_r} (1 + |Dv|^p) \, dx \right)^{1/p}. \tag{4.8}$$

Now if $\vartheta = \dfrac{q - p}{(q - 1)\, p}$ we have $\dfrac{1}{p} = \vartheta + \dfrac{1 - \vartheta}{q}$, and so

$$\left(\fint_{B_{r/2}} |Dv - Du|^p \, dx \right)^{1/p} \leqq \left(\fint_{B_{r/2}} |Dv - Du| \, dx \right)^{\vartheta} \left(\fint_{B_{r/2}} |Dv - Du|^q \, dx \right)^{\frac{1-\vartheta}{q}}.$$

This inequality, together with (4,4), (4.6), (4.7), (4.8), implies

$$\left(\fint_{B_{r/2}} |Dv - Du|^p \, dx \right)^{1/p} \leqq c_K r^{\beta\vartheta/2},$$

and the result follows with $\beta_1 = \beta\vartheta/2 < \beta_2$. $\square$

The key to Theorem [II.2] is a statement similar to Proposition [II.1]: define for every $B_r(x_0) \subset \Omega$

$$U(x_0, r) = r^\delta + \fint_{B_r(x_0)} (|Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^p) \, dx,$$

for some positive $\delta < \beta_1$.

Proposition [IV.8]. *Fix* $M > 0$; *there is a constant* $C_M > 0$ *such that for every* $\tau < 1/8$ *there is an* $\varepsilon = \varepsilon(\tau, M)$ *such that if*

$$|(u)_{x_0,r}| \leqq M, \quad |(Du)_{x_0,r}| \leqq M, \quad U(x_0, r) \leqq \varepsilon$$

then

$$U(x_0, \tau r) \leqq C_M \tau^\delta U(x_0, r).$$

Proof. As in Proposition [III.1], fix M and τ (we shall determine C_M later), and assume

$$B_{4r_h}(x_h) \subset \Omega$$

$$|(u)_{x_h,4r_h}| \leqq M, \quad |(Du)_{x_h,4r_h}| \leqq M \tag{4.9}$$

$$U(x_h, 4r_h) = \lambda_h^2 \to 0 \tag{4.10}$$

and

$$U(x_h, 4\tau r_h) > C_M \tau^\delta \lambda_h^2. \tag{4.11}$$

By (4.9), (4.10) we have

$$\fint_{B_{4r_h}(x_h)} |Du|^p \, dx \leqq 2^{p-1}(M^p + \lambda_h^2) \leqq c, \tag{4.12}$$

so that by Lemma [IV.7] we may choose for every h a function $u_h \in u + W_0^{1,p}(B_{2r_h}(x_h); \mathbb{R}^N)$ satisfying

$$\left(\fint_{B_{r_h}(x_h)} |Du - Du_h|^p \, dx\right)^{1/p} \leqq c(M)\, r_h^{\beta_1} \tag{4.13}$$

$$\begin{aligned} &\fint_{B_{2r_h}(x_h)} f(x_h, (u)_{2r_h}, Du_h(x)) \, dx \\ &\quad \leqq \fint_{B_{2r_h}(x_h)} f(x_h, (u)_{2r_h}, Du_h + D\varphi(x)) \, dx + (2r_h)^{\beta_2} \fint_{B_{2r_h}(x_h)} |D\varphi| \, dx. \end{aligned} \tag{4.14}$$

By (4.12), (4.13) we have also

$$|(Du_h)_{x_h,r_h}| \leqq c(M),$$

and we may rescale in $B_{r_h}(x_h)$, setting

$$v_h(y) = \frac{1}{\lambda_h r_h} [u_h(x_h + r_h y) - (u_h)_{x_h,r_h} - r_h (Du_h)_{x_h,r_h} y].$$

After this, the proof goes on as in Proposition [III.1], with some changes. Those worth noting are the following.

Formula (3.6). Differentiating in (4.14), we show that the left-hand side of (3.6) is no longer equal to zero, but instead it vanishes as $h \to \infty$; indeed, it is dominated by $r_h^{\beta/2}/\lambda_h$, and by (4.10) and our choice of $\delta < \beta_1 < \beta_2$

$$r_h^{\beta_2} < c\,\lambda_h^2\,r_h^{\beta_2-\delta},$$

whence

$$r_h \to 0, \quad \frac{r_h^{\beta_2}}{\lambda} \to 0, \quad \frac{r_h^{\beta_2}}{\lambda_h^2} \to 0 \tag{4.15}$$

and similarly

$$\frac{r_h^{\beta_1}}{\lambda_h} \to 0. \tag{4.16}$$

Formula (3.14). Lemma [II.4] must now be used with $\nu = (2r_h)^{\beta_2}/\lambda_h^2$, after which the formula remains unchanged by (4.15).

Formula (3.15). The estimate begins with

$$I_r^h(v_h) - I_r^h(v) \leqq (I_r^h(v_h + \varphi_h) - I_r^h(v)) + \frac{(2r_h)^{\beta_2}}{\lambda_h^2} \int_{B_r} |D\varphi_h|\,dx.$$

The first term is dealt with as before, while the second term vanishes as $h \to \infty$ by (4.15) and since $(D\varphi_h)$ is bounded in L^2.

Step 6. In this case, since $4\tau < \frac{1}{2}$, we obtain

$$\lim_h \frac{1}{\lambda_h^2} \fint_{B_{4\tau r_h}(x_h)} (|Du_h - (Du_h)_{4\tau r_h}|^2 + |Du_h - (Du_h)_{4\tau r_h}|^p)\,dx \leqq c(M)\,\tau^2. \tag{4.17}$$

But by (4.13)

$$\frac{1}{\lambda_h^2} \fint_{B_{4\tau r_h}(x_h)} (|Du - Du_h|^2 + |Du - Du_h|^p)\,dx$$

$$\leqq \frac{c(\tau)}{\lambda_h^2}\left[\left(\fint_{B_{r_h}(x_h)} |Du - Du_h|^p\,dx\right)^{2/p} + \fint_{B_{r_h}(x_h)} |Du - Du_h|^p\,dx\right]$$

$$\leqq \frac{c(\tau)}{\lambda_h^2}(r_h^{2\beta_1} + r_h^{p\beta_1}),$$

which vanishes as $h \to \infty$ by (4.16).

This, together with (4.17), implies by (4.10)

$$\lim_h \frac{U(x_h, 4\tau r_h)}{\lambda_h^2} \leqq c\tau^\delta \limsup_h \frac{r_h^\delta}{\lambda_h^2} + c(M)\,\tau^2 \leqq c^*(M)\,\tau^\delta,$$

and the contradiction follows for $C_M = 2c^*(M)$. □

The conclusion of the proof of Theorem [II.2] may be attained by adapting [7] Section 6 to our simpler situation.

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