A Regularity Theory of Biharmonic Maps

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Abstract

In this article we prove the regularity of weakly biharmonic maps of domains in Euclidean four space into spheres, as well as the corresponding partial regularity result of stationary biharmonic maps of higher-dimensional domains into spheres. © 1999 John Wiley & Sons, Inc.

Introduction

In this article we consider the notion of biharmonic maps and begin an analytic study of the regularity properties of such maps in dimensions greater than or equal to four. To motivate our study, we observe that the conformal transformations of Euclidean spaces are not in general harmonic except in dimension two. The basic reason is that the energy integrand for harmonic maps is conformally invariant only in dimension two. Thus it is natural to study critical points of the conformally invariant energy functionals. There have been several studies of the energy integrand associated with the p-Laplacian (see, for example, [5, 11]). In dimension n the natural first-order functional is the conformally invariant n-energy: $\int |\nabla u|^n$. Unfortunately, the class of n-harmonic maps, although quite abundant, do not enjoy good regularity properties due to the possible degeneration of $|\nabla u|^n$ (see [9]). For this reason, it is of interest to study higher-order energy functionals that are conformally invariant and enjoy better regularity properties.

In this article we consider for simplicity the class of biharmonic maps from Euclidean domains to spheres. We realize the standard spheres $\mathbb{S}^k$ as unit vectors in $\mathbb{R}^{k+1}$, and consider maps $u:\Omega\to\mathbb{S}^k$ as vector-valued functions that are contained in $\mathbb{S}^k$. The energy functional for biharmonic maps is then $\int_\Omega |\Delta u|^2\,dx$. A locally defined biharmonic map is a map that is critical with respect to compactly supported variations. We note that in the case where the domain has dimension four, this energy functional is conformally invariant, and hence conformal maps of Euclidean four-space are biharmonic in this sense. We remark that this definition of biharmonic map depends on the embedding of the target space in Euclidean space. We

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do not use the more natural definition in which the energy integrand is replaced by the intrinsic $|(\Delta u)^T|^2$ where v^T denotes the tangential component of the vector v^T.

In analogy with the regularity theory of harmonic maps, we derive corresponding regularity results for biharmonic maps. Our main results are the following:

- Theorem 2.1: *Any biharmonic map in $W^{2,2}$ defined on a disk of dimension four to the standard sphere $\mathbb{S}^k$ is Hölder-continuous*.
- Theorem 4.1: *A stationary biharmonic map from an m-dimensional Euclidean disk* ($m \geq 5$) *to the sphere $\mathbb{S}^k$ is Hölder-continuous except on a set of* $(m-4)$*–dimensional Hausdorff measure zero*.
- Theorem 5.1: *If u is a weak solution of the biharmonic map equation and if u is continuous in B_1, then u is smooth*.

A companion article [2] to this one provides a simplified treatment of the analogues of the preceding results for harmonic maps and serves as an introduction to the techniques used here as well as references to previous work. Our method builds on the technique first introduced by Hélein [7] to write the nonlinearity in determinant form but proceeds more directly to exploit the special quadratic structure of the nonlinearity; thus we were able to avoid the deep structure theory of Hardy BMO duality. Our argument may allow flexibility to deal with other problems of this kind. We hope to return to the problem involving general targets in a future article. We mention here the related article [1] that proves regularity of minimizing solutions of semilinear scalar equations of fourth order with nonlinearity of similar structure to the biharmonic map equation. We also mention that Hardt and Mou also have some regularity results for locally minimizing biharmonic maps [6].

We remark here that Theorems 2.1 and 4.1 remain valid for maps from domains in a Riemannian manifold. In fact, the elliptic estimates we use remain valid provided we interpret all derivatives in the formula as covariant derivatives. Recently a result analogous to Theorem 2.1 with the extrinsic quantity Δu replaced by the intrinsic $(\Delta u)^T$ was also established by Y. Ku.

1 Derivation of the Euler Equation

Consider u a map $(M^m, g) \to (\mathbb{S}^k, h)$ with h the standard canonical metric on the unit sphere $\mathbb{S}^k$. Suppose $u = (u^1, \ldots, u^{k+1})$ is a critical point of the energy functional; define $E_2(u) \equiv \int_M \sum_{\alpha=1}^{k+1} (\Delta_g u^\alpha)^2 \, dV_g$. In this section we will derive the Euler-Lagrange equation for u.

PROPOSITION 1.1 *Suppose $u \in W^{2,2}$ is a critical point of the functional E_2; then u satisfies*

$$\Delta^2 u^\alpha = -u^\alpha \lambda, \quad \alpha = 1, 2, \ldots, k+1, \tag{1.1}$$

where $\lambda = \sum_{\beta=1}^{k+1} [(\Delta u^\beta)^2 + \Delta(|\nabla u^\beta|^2) + 2\nabla u^\beta \cdot \nabla \Delta u^\beta]$ *and* $\nabla \Delta u^\beta$ *exists in the* L^p *sense for all* $p < \frac{3}{4}$.

PROOF: Since $u : M^m \to \mathbb{S}^k$, the Euler equation of $E_2(u) = 0$ satisfies

$$(\Delta^2 u)^T = 0\,,$$

where $(\Delta^2 u)^T$ denotes the tangential component of $\Delta^2 u$. Therefore for some λ, $\Delta^2 u^\alpha = (\Delta^2 u^\alpha)^N = -u^\alpha \lambda$ where $(\Delta^2 u)^N$ denotes the normal component of $\Delta^2 u$.

It remains to compute λ. To do so, we observe that when the target manifold of the map is $\mathbb{S}^k$, we have $u^\beta \cdot u^\beta = 1$; hence $\nabla u^\beta \cdot u^\beta = 0$ and $\Delta u^\beta \cdot u^\beta = -|\nabla u^\beta|^2$ (where we treat u^β as a vector, and the equality holds by summing over β). Thus if we inner product both sides of (1.1) by u^α and sum over α, we get

$$\sum_{\alpha=1}^{k+1} \Delta^2 u^\alpha \cdot u^\alpha = -\lambda\,. \tag{1.2}$$

Multiplying both sides of (1.2) by a testing function $\varphi \in C_0^\infty(M)$ and integrating over M, we get

$$\begin{aligned}
-\int \lambda\varphi &= \sum_\alpha \int (\Delta^2 u^\alpha) u^\alpha \varphi \\
&= \sum_\alpha \int \Delta u^\alpha \Delta(u^\alpha \varphi) \\
&= \sum_\alpha \left[\int (\Delta u^\alpha)^2 \varphi + 2\int \Delta u^\alpha \nabla u^\alpha \nabla\varphi + \int \Delta u^\alpha u^\alpha \Delta\varphi \right] \\
&= -\sum_\alpha \int \left[(\Delta u^\alpha)^2 + 2\nabla\Delta u^\alpha \nabla u^\alpha \right] \varphi - \sum_\alpha \int |\nabla u^\alpha|^2 \Delta\varphi \\
&= -\sum_\alpha \int \left[(\Delta u^\alpha)^2 + 2\nabla\Delta u^\alpha \nabla u^\alpha \right] \varphi - \sum_\alpha \int \Delta|\nabla u^\alpha|^2 \varphi\,.
\end{aligned}$$

Thus

$$\lambda = \sum_\beta [(\Delta u^\beta)^2 + \Delta(|\nabla u^\beta|^2) + 2\nabla\Delta u^\beta \cdot \nabla u^\beta]$$

as claimed. □

In the following, we are going to rewrite the right-hand side of equation (1.2) in a “divergence" form. The purpose of doing so is to establish our regularity results later. (Some motivation for this approach is explained in [2]). We remark that for the purpose of establishing our regularity result (Theorem 2.1 below) for domain M^m with $m = \dim M = 4$, we only need a simpler form of the right-hand side of (1.2) than the form that appears in (1.3), which we will derive below. But for our approach to work for all $m \geq 4$, it is easier that we establish the right-hand side as it appears in (1.3).

We now fix a geodesic ball and assume it is a ball of radius 1, $B_1 = B_1(x_0)$. Fix an index α, and for each $\alpha = 1, \ldots, k+1$, denote $c^\alpha = ⨍_{B_1} u^\alpha(x)dx$, the average value of u^α over B_1. We are going to use the convention that the upper index α,

β, etc., denotes the component of u, the lower index i, j, etc., denotes the partial differentiation in the i, j, etc., direction. We also skip the summation over β and j with the understanding that β is summed over from 1 to $k+1$ and that j is summed over from 1 to m.

Definition. We denote a term *of type* I *by* T_1 if

$$T_1 \equiv \left(u_j^\alpha \Delta u^\beta (u^\beta - c^\beta)\right)_j \quad \text{or} \quad \left((u^\alpha - c^\alpha) u_i^\beta u_{ij}^\beta\right)_j \text{ terms}.$$

We denote a term of *type* II *by* T_2 if

$$T_2 \equiv \Delta\Big((u^\alpha - c^\alpha)|\nabla u^\beta|^2\Big), \Delta((u^\beta - c^\beta)\Delta u^\beta) \quad \text{or} \quad \Delta\Big(u^\alpha (u^\beta - c^\beta)\Delta u^\beta\Big) \text{ terms}.$$

We denote a term *of type* III *by* T_3 if

$$T_3 \equiv \left((u^\beta - c^\beta) u_j^\beta\right)_{jii}.$$

PROPOSITION 1.2 *Suppose* $u : M^m \to \mathbb{S}^k$ *satisfies equation* (1.1)*; then the right-hand side of* (1.1)

$$\begin{aligned} -u^\alpha \lambda &\equiv u^\alpha \left[(\Delta u^\beta)^2 + \Delta(|\nabla u^\beta|^2) + 2\nabla u^\beta \cdot \nabla \Delta u^\beta\right] \\ &= \textit{linear combination of terms of the form } T_1,\ T_2,\ \textit{and } T_3. \end{aligned} \tag{1.3}$$

We start with a technical lemma.

LEMMA 1.3 *For each fixed* α,

$$c^\alpha (\Delta |\nabla u^\beta|^2), \tag{1.4}$$

$$(u_j^\alpha (|\nabla u^\beta|^2))_j, \tag{1.5}$$

are a combination of T_ℓ *terms for* $\ell = 1, 2, 3$.

PROOF: To establish (1.4), we write

$$\begin{aligned} c^\alpha \Delta(|\nabla u^\beta|^2) &= c^\alpha \Delta(u_j^\beta u_j^\beta) \\ &= c^\alpha \Delta\left\{\left((u^\beta - c^\beta) u_j^\beta\right)_j - (u^\beta - c^\beta)\Delta u^\beta\right\} \\ &= c^\alpha \left((u^\beta - c^\beta) u_j^\beta\right)_{jii} - c^\alpha \left((u^\beta - c^\beta)\Delta u^\beta\right)_{ii} \\ &= T_2 + T_3 \text{ terms}. \end{aligned}$$

To establish (1.5), we have

$$\begin{aligned} (u_j^\alpha |\nabla u^\beta|^2)_j &= \left\{\left((u^\alpha - c^\alpha)|\nabla u^\beta|^2\right)_j - 2(u^\alpha - c^\alpha) u_i^\beta u_{ij}^\beta\right\}_j \\ &= \Delta\Big((u^\alpha - c^\alpha)|\nabla u^\beta|^2\Big) - 2\Big((u^\alpha - c^\alpha) u_i^\beta u_{ij}^\beta\Big)_j \\ &= T_2 + T_1 \text{ terms}. \end{aligned}$$

□

PROOF OF PROPOSITION 1.2: We name

$$S_1 = u^\alpha(\Delta u^\beta)^2\,, \quad S_2 = 2u^\alpha u^\beta_j(\Delta u^\beta)_j\,, \quad S_3 = u^\alpha \Delta|\nabla u^\beta|^2\,.$$

Then, using identities $u^\beta \cdot u^\beta_j = 0$ and $u^\alpha \Delta\Delta u^\beta = u^\beta \Delta\Delta u^\alpha$, we get

$$\begin{aligned}
\frac{S_2}{2} &= u^\alpha u^\beta_j (\Delta u^\beta)_j \\
&= \Big[u^\alpha(\Delta u^\beta)_j - u^\beta\,(\Delta u^\alpha)_j\Big]u^\beta_j \\
&= \Big[u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j - u^\alpha_j(\Delta u^\beta) + u^\beta_j(\Delta u^\alpha)\Big]u^\beta_j \\
&\quad + \Big[u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\Big]u^\beta_j \\
&= \Big\{\Big[u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j - u^\alpha_j(\Delta u^\beta) + u^\beta_j(\Delta u^\alpha)\Big](u^\beta - c^\beta)\Big\}_j \\
&\quad + \Big[u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\Big]u^\beta_j \\
&= \Big\{\Big[u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\Big](u^\beta - c^\beta)\Big\}_{jj} - \Big\{\Big[u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\Big]u^\beta_j\Big\}_j \\
&\quad - 2\Big\{\Big[u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\Big](u^\beta - c^\beta)\Big\}_j + \Big[u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\Big]u^\beta_j \\
&= -\Big\{\Big[u^\alpha(\Delta u^\beta) - u^\beta(\Delta u^\alpha)\Big]u^\beta_j\Big\}_j \\
&\quad + \Big[u^\alpha_j(\Delta u^\beta) - u^\beta_j(\Delta u^\alpha)\Big]u^\beta_j + T_1 + T_2 \text{ terms}\,.
\end{aligned} \tag{1.6}$$

For the S_3 term we have

$$\begin{aligned}
S_3 &= u^\alpha \Delta|\nabla u^\beta|^2 \\
&= (u^\alpha - c^\alpha)\Delta|\nabla u^\beta|^2 + c^\alpha\Delta(|\nabla u^\beta|^2) \quad \text{(by (1.4))} \\
&= \Delta((u^\alpha - c^\alpha)|\nabla u^\beta|^2) - 2u^\alpha_j(|\nabla u^\beta|^2)_j - (\Delta u^\alpha)|\nabla u^\beta|^2 + T_\ell \text{ terms} \\
&= -2(u^\alpha_j|\nabla u^\beta|^2)_j + (\Delta u^\alpha)|\nabla u^\beta|^2 + T_\ell \text{ terms (by (1.5))} \\
&= (\Delta u^\alpha)|\nabla u^\beta|^2 + T_\ell \text{ terms} \\
&= -(\Delta u^\alpha)u^\beta\Delta u^\beta + T_\ell \text{ terms}\,.
\end{aligned} \tag{1.7}$$

From (1.7) we have

$$\begin{aligned}
S_1 + S_3 &= \Big[u^\alpha\Delta u^\beta - u^\beta\Delta u^\alpha\Big]\Delta u^\beta + T_\ell \text{ terms} \\
&= \Big\{\Big[u^\alpha\Delta u^\beta - u^\beta\Delta u^\alpha\Big]u^\beta_j\Big\}_j - \Big[u^\alpha_j\Delta u^\beta - u^\beta_j\Delta u^\alpha\Big]u^\beta_j \\
&\quad - \Big[u^\alpha(\Delta u^\beta)_j - u^\beta(\Delta u^\alpha)_j\Big]u^\beta_j + T_\ell \text{ terms}
\end{aligned}$$

$$= -\frac{1}{2}S_2 - \frac{1}{2}S_2 + T_\ell \text{ terms} \quad \text{(by (1.6))}$$

$$= -S_2 + T_\ell \text{ terms}.$$

Thus $S_1 + S_2 + S_3 =$ combination of T_ℓ terms as claimed. □

2 Hölder Regularity on M^4

In this section, we will prove the following theorem:

THEOREM 2.1 *Any biharmonic map in $W^{2,2}$ defined on a disk of dimension four to the standard sphere $\mathbb{S}^k$ is Hölder-continuous.*

To prove the theorem, we start with some general inequalities for equations of type like that of (1.3).

LEMMA 2.2 *Fix a ball B on $\mathbb{R}^m$. Suppose $v \in W^{2,2}$ is a weak solution of*

$$\Delta^2 v = \operatorname{div}(F) = \sum_{j=1}^{m} \frac{\partial F^j}{\partial x_j} \quad \text{on } B \tag{2.1}$$

or

$$\Delta^2 v = \Delta G \quad \text{on } B \tag{2.2}$$

or

$$\Delta^2 v = \operatorname{div}(\Delta H^j) = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}(\Delta H^j) \quad \text{on } B, \tag{2.3}$$

with

$$\begin{cases} v = 0 & \text{on } \partial B \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial B. \end{cases}$$

Then for any $1 < q < \infty$, we have either

$$\|\nabla^3 v\|_{L^q(B)} \lesssim \|F\|_{L^q(B)} \tag{2.4}$$

or

$$\|\nabla^2 v\|_{L^q(B)} \lesssim \|G\|_{L^q(B)} \tag{2.5}$$

or

$$\|\nabla v\|_{L^q(B)} \lesssim \|H\|_{L^q(B)} \tag{2.6}$$

accordingly.

For any ball B of radius $r (B = B_r)$ in $\mathbb{R}^m$, any $p > 1$, and q with $\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$, denote

$$E(u)(B_r) \equiv \left(r^4 \fint_{B_r} |\nabla^2 u|^2 \right)^{\frac{1}{2}} + \left(r^q \fint_{B_r} |\nabla u|^q \right)^{\frac{1}{q}} \tag{2.7}$$
$$M_p(u)(B_r) \equiv \left(\fint_{B_r} |u - \overline{u}|^p \right)^{\frac{1}{p}} ,$$

where

$$\overline{u} = \fint_{B_r} u \quad \text{and} \quad D_p(u)(B_r) = \left(r^p \fint_{B_r} |\nabla u|^p \right)^{\frac{1}{p}} .$$

The following is the main technical lemma of this section:

LEMMA 2.3 *Let u be as in Theorem* 2.1 *and $m = 4$. Then, given any $0 < \beta < 1$, there exists some $\tau < \frac{1}{4}$ and $\varepsilon > 0$ so that if $E(u)(B_1) < \varepsilon$, we have*

$$(M_{p_0}(u) + D_{p_1}(u))(B_\tau) < \tau^\beta (M_{p_0}(u) + D_{p_1}(u))(B_1) , \tag{2.8}$$

where p_1 is any fixed number strictly between 2 *and* 4, *and $\frac{1}{p_0} = \frac{1}{p_1} - \frac{1}{4}$.*

PROOF OF LEMMA 2.3: We fixed a ball B, say $B = B_1$, on $\mathbb{R}^m$, and some $\frac{1}{2} \le r \le 1$ to be chosen later. Let k denote the biharmonic map $k = (k^1, \dots, k^k)$ defined on B_r with $\Delta^2 k^\alpha = 0$ on B_r and $k^\alpha = u^\alpha$ on ∂B_r, and $\frac{\partial k^\alpha}{\partial n} = \frac{\partial u^\alpha}{\partial n}$ on ∂B_r. Denote $v = u - k$; then v satisfies equation (1.3). We then define v_i, $i = 1, 2, 3$, to be the unique function satisfying $\Delta^2 v_i = T_i$ on B_r and $v_i = \frac{\partial v_i}{\partial n} = 0$ on ∂B_r; then $v = \sum_{i=1}^3 v_i$. We now apply Lemma 2.2 to each of the functions v_i and conclude that for any $1 < p_i < \infty$ and any constant $A_0 = (c^1, \dots, c^k)$ we have

$$\begin{aligned} &\|\nabla^3 v_1\|_{L^{p_3}(B_r)} + \|\nabla^2 v_2\|_{L^{p_2}(B_r)} + \|\nabla v_3\|_{L^{p_1}(B_r)} \\ &\quad \lesssim \|(u - A_0) \nabla u \nabla^2 u\|_{L^{p_3}(B_r)} + \|(u - A_0)|\nabla u|^2\|_{L^{p_2}(B_r)} \\ &\quad\quad + \|(u - A_0) \nabla u\|_{L^{p_1}(B_r)} . \end{aligned} \tag{2.9}$$

We choose p_2 and p_3 as $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{m}$, $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{m} = \frac{1}{p_1} + \frac{2}{m}$, and apply Sobolev embedding to the left-hand side of (2.9) to obtain

$$\begin{aligned} \|\nabla v\|_{L^{p_1}(B_r)} \lesssim &\|(u - A_0) \nabla u \nabla^2 u\|_{L^{p_3}(B_r)} + \|(u - A_0)|\nabla u|^2\|_{L^{p_2}(B_r)} \\ &+ \|(u - A_0) \nabla u\|_{L^{p_1}(B_r)} . \end{aligned} \tag{2.10}$$

We observe that by our assumption $u \in W^{2,2}$, we have $\nabla u \in W^{1,q}$ with $\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$. Thus, we may apply the Hölder inequality to the right-hand side of (2.10) and obtain

$$\begin{aligned} \|\nabla v\|_{L^{p_1}(B_r)} \lesssim &\left(\|\nabla^2 u\|^2_{L^2(B_r)} + \|\nabla u\|^2_{L^q(B_r)} + \|\nabla u\|_{L^q(B_r)} \right) \\ &\cdot \left(\|u - A_0\|_{L^s(B_r)} + \|u - A_0\|_{L^t(B_r)} \right) , \end{aligned} \tag{2.11}$$

where $\frac{1}{s} = \frac{1}{p_1} + \frac{3}{m} - 1,\ \frac{1}{t} = \frac{1}{p_1} - \frac{1}{q} = \frac{1}{p_1} + \frac{1}{m} - \frac{1}{2}$.

Notice that in the special case when $m = 4,\ q = 4$, we may apply Sobolev embedding on the left-hand side of (2.11) to obtain for $p_0 = s = t,\ \frac{1}{p_0} = \frac{1}{p_1} - \frac{1}{4}$,

$$\begin{aligned} &\|v\|_{L^{p_0}(B_r)} + \|\nabla v\|_{L^{p_1}(B_r)} \lesssim \\ &\quad \left(\|\nabla^2 u\|^2_{L^2(B_r)} + \|\nabla u\|^2_{L^4(B_r)} + \|\nabla u\|_{L^4(B_r)}\right)\|u - A_0\|_{L^{p_0}(B_r)}\,. \end{aligned} \tag{2.12}$$

We now choose p_1 to be any number strictly between 2 and 4 so that $p_0, p_2, p_3 > 1$, and choose r with $\frac{1}{2} \le r \le 1$ so that

$$\begin{aligned} &\left(\int_{\partial B_r} |u - A_0|^{p_0}\right)^{\frac{1}{p_0}} + \left(\int_{\partial B_r} |\nabla u|^{p_1}\right)^{\frac{1}{p_1}} \lesssim \\ &\qquad\qquad \left(\int_{B_1} |u - A_0|^{p_0}\right)^{\frac{1}{p_1}} + \left(\int_{B_1} |\nabla u|^{p_1}\right)^{\frac{1}{p_1}}\,. \end{aligned} \tag{2.13}$$

We then have for any $\tau,\ \tau < \frac{1}{4}$, and any $x \in B_\tau$ that the biharmonic function k satisfies

$$\begin{aligned} |\nabla k(x)| &\lesssim \int_{\partial B_r} |u - A_0| + \int_{\partial B_r} \left|\frac{\partial u}{\partial n}\right| \\ &\lesssim \left(\int_{B_1} |u - A_0|^{p_0}\right)^{\frac{1}{p_0}} + \left(\int_{B_1} |\nabla u|^{p_1}\right)^{\frac{1}{p_1}} \qquad \text{(by 2.13)} \\ &= M_{p_0}(u)(B_1) + D_{p_1}(u)(B_1)\,, \end{aligned} \tag{2.14}$$

where we take $A_0 = ⨍_{B_1} u$. Thus it follows for (2.12) and (2.14) that for any $\tau < \frac{1}{4}$,

$$\begin{aligned} &M_{p_0}(u)(B_\tau) + D_{p_1}(u)(B_\tau) \\ &\quad = \left(\tau^{-4}\int_{B_\tau} |u - \overline{u}|^{p_0}\right)^{\frac{1}{p_0}} + \left(\tau^{p_1 - 4}\int_{B_\tau} |\nabla u|^{p_1}\right)^{\frac{1}{p_1}} \\ &\quad = \tau^{-\frac{4}{p_0}}\|u - \overline{u}\|_{L^{p_0}(B_\tau)} + \tau^{1 - \frac{4}{p_1}}\|\nabla u\|_{L^{p_1}(B_\tau)} \\ &\quad \lesssim \tau^{-\frac{4}{p_0}}\|u - k(0)\|_{L^{p_0}(B_\tau)} + \tau^{1 - \frac{4}{p_1}}\|\nabla u\|_{L^{p_1}(B_\tau)} \\ &\quad \lesssim \tau^{-\frac{4}{p_0}}\left(\|v\|_{L^{p_0}(B_\tau)} + \|k - k(0)\|_{L^{p_0}(B_\tau)}\right) \\ &\qquad + \tau^{1 - \frac{4}{p_1}}\|\nabla v\|_{L^{p_1}(B_\tau)} + \tau^{1 - \frac{4}{p_1}}\|\nabla k\|_{L^{p_1}(B_\tau)} \\ &\quad \lesssim \tau^{1 - \frac{4}{p_1}}\ (E^2(u) + E(u))(B_1)\|u - A_0\|_{L^{p_0}(B_1)} + \tau\sup_{x \in B_\tau}|\nabla k(x)| \\ &\quad \lesssim \tau^{1 - \frac{4}{p_1}}\ \varepsilon\|u - A_0\|_{L^{p_0}(B_1)} + \tau\left(\|u - A_0\|_{L^{p_0}(B_1)} + \|\nabla u\|_{L^{p_1}(B_1)}\right)\,. \end{aligned}$$

Thus, if we choose τ sufficiently small and then ε small, we may conclude that when $E(u)(B_1) < \varepsilon$, then

$$(M_{p_0}(u)(B_\tau) + D_{p_1}(u)(B_\tau)) \leq \tau^\beta (M_{p_0}(u)(B_1) + D_{p_1}(u))(B_1),$$

which finishes the proof of Lemma 2.3. □

PROOF OF THEOREM 2.1: We claim that we may apply Lemma 2.3 iteratively to the function u. That is, if $E(u)(B_1) < \varepsilon$, then we have for each j

$$(M_{p_0}(u) + D_{p_1}(u))(B_{\tau^j}) \leq \tau^{j\beta} (M_{p_0}(u) + D_{p_1}(u))(B_1). \tag{2.15}$$

From (2.15) it follows from Morrey's estimate that u is Hölder-continuous.

To establish the iteration argument, it suffices to show that $E(u)(B_r) < \varepsilon$ whenever $E(u)(B_1) < \varepsilon$ where $r = \tau^j$ for all $j = 1, 2, \ldots$. Since in the case $m = 4$,

$$E(u)(B_r) = \left(\int_{B_r} |\nabla^2 u|^2\right)^{\frac{1}{2}} + \left(\int_{B_r} |\nabla u|^4\right)^{\frac{1}{4}},$$

it is clear $E(u)(B_r) < \varepsilon$ whenever $E(u)(B_1) < \varepsilon$. This establishes (2.15) and hence the theorem. □

3 Monotonicity Formula for Stationary Biharmonic Maps

In this section we will derive the monotonicity formula from the stationary assumption of a biharmonic map. We begin with a lemma.

LEMMA 3.1 *If u is a stationary biharmonic map on B_{2r}, then when we write $X = \Sigma x_i \frac{\partial}{\partial x_i}$, we have*

$$\int_{\partial B_r} |\Delta u|^2 X \cdot \frac{X}{r}\, d\sigma = \int_{B_r} (X(|\Delta u|^2) + m|\Delta u|^2) dx. \tag{3.1}$$

PROOF: Fix $\varepsilon > 0$ and let ψ_ε be a cutoff function defined on $[0, r]$ such that $\psi_\varepsilon(s) = 1$ for $0 \leq s \leq r - \varepsilon$, $\psi_\varepsilon(s) = 1 - \frac{s-(r-\varepsilon)}{\varepsilon}$ for $r - \varepsilon \leq s \leq r$. Consider the one-parameter (in t) family of diffeomorphisms $\varphi_\varepsilon(t) : B_{2r} \to B_{2r}$ with $\varphi_\varepsilon(0)(x) = x$ and

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_\varepsilon(t)(x) = \psi_\varepsilon(|x|)X(x) \quad \text{for all } x \in B_{2r}.$$

The stationary assumption implies

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \int_{B_{2r}} |\Delta u|^2 (\varphi_\varepsilon(t)(x) d(\varphi_\varepsilon^{-1}(t)(x)) \\ &= \int_{B_{2r}} \psi_\varepsilon X(|\Delta u|^2) + \operatorname{div}(\psi_\varepsilon X)|\Delta u|^2\, dx \\ &= \int_{B_{2r}} \left(\psi_\varepsilon X(|\Delta u|^2) + \psi_\varepsilon(\operatorname{div} X)|\Delta u|^2 + (\nabla\psi_\varepsilon \cdot X)|\Delta u|^2\right) dx. \end{aligned}$$

Let ε tend to zero and we get (3.1). □

Remark. In the proof of the above lemma, we need to justify that the term

$$\int_{B_r} (X(|\Delta u|^2)$$

makes sense for stationary harmonic functions for almost every r. This can be done using the method of the difference quotient. We also remark that here is the only place where we used the fact that u is stationary. In the proof of the next proposition, we will encounter terms such as $(\Delta u)_k$ that can also be justified by the same method.

PROPOSITION 3.2 (Monotonicity Formula) *For a stationary biharmonic map $u : B_{2r} \to N$, we have*

$$\frac{1}{r^{m-4}}\int_{B_r} |\Delta u|^2\,dx - \frac{1}{\rho^{m-4}}\int_{B_\rho} |\Delta u|^2\,dx = P + R \quad \text{for } 0 < \rho < r\,, \tag{3.2}$$

where

$$P = 4\int_{B_r\setminus B_\rho}\left(\frac{(u_\ell + x_i u_{i\ell})^2}{|x|^{m-2}} + \frac{(m-2)(x_i u_i)^2}{|x|^m}\right)$$

$$R = 2\int_{\partial B_r}\left(-\frac{x_i u_\ell u_{i\ell}}{|x|^{m-3}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) d\sigma$$
$$- 2\int_{\partial B_\rho}\left(-\frac{x_i u_\ell u_{i\ell}}{|x|^{m-3}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) d\sigma\,.$$

Thus P is a positive term and R is a boundary term.

PROOF: We first remark that in the computation below, every term that has subindices i, k, and ℓ is summed over these indices, but we will skip the summation sign for simplicity. We begin with

$$\begin{aligned}
r^{m-3}\frac{d}{dr}\frac{\int_{B_r}|\Delta u|^2\,dx}{r^{m-4}} &= r\int_{\partial B_r}|\Delta u|^2 d\sigma - (m-4)\int_{B_r}|\Delta u|^2\,dv \\
&= \int_{\partial B_r}|\Delta u|^2 X\cdot\frac{X}{r}d\sigma - (m-4)\int_{B_r}|\Delta u|^2\,dx \\
&= \int_{B_r}(X(|\Delta u|^2) + 4|\Delta u|^2)dx \quad \text{(by Lemma 3.1)} \\
&= \int_{B_r}(2x_i(\Delta u)_i(\Delta u) + 4|\Delta u|^2)dx \\
&= \int_{\partial B_r}\frac{2x_i x_k u_{ik}\Delta u}{r}\,d\sigma \\
&\quad + \int_{B_r}(4|\Delta u|^2 - 2|\Delta u|^2 - 2x_i u_{ik}(\Delta u)_k)dx \\
&= \int_{\partial B_r}\frac{2x_i x_k u_{ik}\Delta u - 2x_i x_k u_i(\Delta u)_k}{r}d\sigma
\end{aligned}$$

$$+\int_{B_r}(2|\Delta u|^2+2(\Delta u)_k u_k)dx \quad \text{(by equation (1.1))}$$
$$=\int_{\partial B_r}\frac{2x_ix_ku_{ik}\Delta u-2x_ix_ku_i(\Delta u)_k+2x_ku_k\Delta u}{r}\,d\sigma.$$

Hence

$$\frac{1}{r^{m-4}}\int_{B_r}|\Delta u|^2-\frac{1}{\rho^{m-4}}\int_{B_\rho}|\Delta u|^2=2\int_{B_r\setminus B_\rho}(\mathrm{I}+\mathrm{II}+\mathrm{III})dx$$

where

$$\mathrm{I}=\frac{x_ix_k(\Delta u)u_{ik}}{|x|^{m-2}},\quad \mathrm{II}=-\frac{x_ix_ku_i(\Delta u)_k}{|x|^{m-2}},\quad \text{and}\quad \mathrm{III}=\frac{x_ku_k\Delta u}{|x|^{m-2}}.$$

After several integrations by parts, we can rewrite

$$\begin{aligned}\int_{B_r\setminus B_\rho}\mathrm{I}\,dx&=\int_{\partial B_r}\left(\frac{x_ix_kx_\ell u_\ell u_{ik}}{|x|^{m-1}}-\frac{x_iu_\ell u_{i\ell}}{|x|^{m-3}}\right)d\sigma\\&\quad-\int_{\partial B_\rho}\left(\frac{x_ix_kx_\ell u_\ell u_{ik}}{|x|^{m-1}}-\frac{x_iu_\ell u_{i\ell}}{|x|^{m-3}}\right)d\sigma\\&\quad+\int_{B_r\setminus B_\rho}\left(\frac{x_iu_\ell u_{i\ell}}{|x|^{m-2}}+\frac{(m-2)x_ix_kx_\ell u_\ell u_{ik}}{|x|^m}+\frac{x_ix_ku_{\ell k}u_{i\ell}}{|x|^{m-2}}\right)dx\end{aligned}\tag{3.3}$$

$$\begin{aligned}\int_{B_r\setminus B_\rho}\mathrm{II}\,dx&=-\left(\int_{\partial B_r}\frac{x_ix_kx_\ell u_iu_{\ell k}}{|x|^{m-1}}d\sigma-\int_{\partial B_\rho}\frac{x_ix_kx_\ell u_iu_{\ell k}}{|x|^{m-1}}d\sigma\right)\\&\quad+\int_{B_r\setminus B_\rho}\left(\frac{x_ku_iu_{ik}}{|x|^{m-2}}+\frac{x_iu_i\Delta u}{|x|^{m-2}}\right.\\&\qquad\left.+\frac{(2-m)x_ix_kx_\ell u_iu_{\ell k}}{|x|^m}+\frac{x_ix_ku_{i\ell}u_{\ell k}}{|x|^{m-2}}\right)dx\end{aligned}\tag{3.4}$$

$$\begin{aligned}\int_{B_r\setminus B_\rho}\mathrm{III}\,dx&=\int_{\partial B_r}\frac{(x_ku_k)^2}{|x|^{m-1}}d\sigma-\int_{\partial B_\rho}\frac{(x_ku_k)^2}{|x|^{m-1}}d\sigma\\&\quad+\int_{B_r\setminus B_\rho}\left(-\frac{|\nabla u|^2}{|x|^{m-2}}+\frac{(m-2)x_kx_\ell u_ku_\ell}{|x|^m}-\frac{x_ku_\ell u_{\ell k}}{|x|^{m-2}}\right)dx.\end{aligned}\tag{3.5}$$

Combining the terms in (3.3), (3.4), and (3.5), we find

$$\begin{aligned}&\frac{1}{r^{m-4}}\int_{B_r}|\Delta u|^2-\frac{1}{\rho^{m-4}}\int_{B_\rho}|\Delta u|^2\\&=2\int_{\partial B_r}\left(-\frac{x_iu_\ell u_{i\ell}}{|x|^{m-3}}+\frac{(x_iu_i)^2}{|x|^{m-1}}\right)d\sigma-2\int_{\partial B_\rho}\left(-\frac{x_iu_\ell u_{i\ell}}{|x|^{m-3}}+\frac{(x_iu_i)^2}{|x|^{m-1}}\right)\\&\quad+2\int_{B_r\setminus B_\rho}\left[\frac{x_iu_\ell u_{\ell i}}{|x|^{m-2}}+\frac{2(x_iu_{i\ell})^2}{|x|^{m-2}}+\frac{(x_iu_i)\Delta u}{|x|^{m-2}}-\frac{|\nabla u|^2}{|x|^{m-2}}\right.\\&\qquad\left.+\frac{(m-2)(x_iu_i)^2}{|x|^m}\right]dx.\end{aligned}\tag{3.6}$$

After integrating by parts and using the identity

$$\int_{B_r} \frac{|\nabla u|^2}{|x|^{m-2}}\,dx + \int_{B_r} \frac{x_i u_\ell u_{i\ell}}{|x|^{m-2}}\,dx = \frac{1}{2}\int_{\partial B_r} \frac{|\nabla u|^2}{|x|^{m-3}}\,d\sigma\,, \tag{3.7}$$

we find

$$\begin{aligned}
&\frac{1}{r^{m-4}}\int_{B_r} |\Delta u|^2\,dx - \frac{1}{\rho^{m-4}}\int_{B_\rho} |\Delta u|^2\,dx \\
&\quad = 2\int_{\partial B_r}\left(-\frac{x_i u_\ell u_{i\ell}}{|x|^{m-3}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) d\sigma \\
&\qquad - 2\int_{\partial B_\rho}\left(-\frac{x_i u_\ell u_{i\ell}}{|x|^{m-3}} + 2\frac{(x_i u_i)^2}{|x|^{m-1}} - 2\frac{|\nabla u|^2}{|x|^{m-3}}\right) d\sigma \\
&\qquad + 4\int_{B_r\setminus B_\rho}\left(\frac{(u_\ell + x_i u_{i\ell})^2}{|x|^{m-2}} + \frac{(m-2)(x_i u_i)^2}{|x|^m}\right) dx\,.
\end{aligned} \tag{3.8}$$

This finishes the proof of Proposition 3.2. □

Remark. If we use the formula

$$\frac{d}{dr}\int_{\partial B_r} f\,d\sigma = \frac{1}{r}\int_{\partial B_r} x_i f_i\,d\sigma + \frac{m-1}{r}\int_{\partial B_r} f\,d\sigma\,,$$

we may rewrite our monotonicity formula as

$$\sigma(r) = \frac{1}{r^{m-4}}\int_{B_r} |\Delta u|^2\,dx + \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r^{m-5}}\int_{\partial B_r} |\nabla u|^2\right) - 4\int_{\partial B_r} \frac{(x_i u_i)^2}{r^{m-1}}\,d\sigma\,,$$

which is a monotonically increasing function in r. Actually,

$$\sigma(r) - \sigma(\rho) = P = 4\int_{B_r\setminus B_\rho}\left(\frac{(u_\ell + x_i u_{i\ell})^2}{|x|^{m-2}} + \frac{(m-2)(x_i u_i)^2}{|x|^m}\right).$$

One also observes that $\sigma(r) - \sigma(\rho) = 0$ when and only when $u(x) = u(r\frac{x}{|x|})$ for $x \in B_r \setminus B_\rho$.

4 Regularity Result for Stationary Biharmonic Maps

In this section, we will establish the following regularity result for stationary biharmonic maps:

THEOREM 4.1 *A stationary biharmonic map from an m-dimensional Euclidean disk* $(m \geq 5)$ *to the sphere* $\mathbb{S}^k$ *is Hölder-continuous except on a set of* $(m-4)$*–dimensional Hausdorff measure zero.*

As in the proof of the corresponding result for stationary harmonic maps in [2], our proof below is patterned after the proof in Section 2 of the case for the four-dimensional argument. In the case when the dimension of the domain manifold $m \geq 5$, the exponents resulting from the Sobolev inequalities (2.11) and (2.12) do not match, so we will show instead that the BMO norm of the map decays when the energy is small. In fact, we have to show the decay of the map in every scale. The monotonicity formula makes the control in every scale possible.

An added difficulty in the proof is how to handle the extra term R (which may not be positive) in the monotonicity formula of Proposition 3.2. We will show that the size of R is small compared to the size of the energy term E as defined in (2.7). We start with some technical lemmas.

Throughout this section we assume u is a stationary biharmonic map defined on the disk B_2 on $\mathbb{R}^m$.

LEMMA 4.2 *For each $r < 1$, denote $E_2(u)(B_r) = \left(r^4 \fint_{B_r} |\nabla^2 u|^2\, dx\right)^{1/2}$. We have for all $0 < \rho < r$,*

$$
\begin{aligned}
E_2^2(u)(B_\rho) \le E_2^2(u)(B_r)& \\
&+ c\left[\left(r^4 \fint_{\partial B_r} |\nabla^2 u|^2 d\sigma\right)^{\frac12}\left(r^2 \fint_{\partial B_r} |\nabla u|^2 d\sigma\right)^{\frac12}\right. \\
&\qquad \left. + r^2 \fint_{\partial B_r} |\nabla u|^2 d\sigma\right] \\
&+ c\left[\left(\rho^4 \fint_{\partial B_\rho} |\nabla^2 u|^2 d\sigma\right)^{\frac12}\left(\rho^2 \fint_{\partial B_\rho} |\nabla u|^2 d\sigma\right)^{\frac12}\right. \\
&\qquad \left. + \rho^2 \fint_{\partial B_\rho} |\nabla u|^2 d\sigma\right],
\end{aligned}
\tag{4.1}
$$

where c is a universal constant depending only on dimension m.

PROOF: We first observe that if we denote $\tilde{E}_2(u) = \left(r^4 \fint_{B_r} (\Delta u)^2 dx\right)^{1/2}$, then (4.1) with E_2 replaced by $\tilde{E}_2$ is a direct consequence of the monotonicity formula in Proposition 3.2. To compare E_2 with $\tilde{E}_2$, we apply the Bochner identity

$$\frac{1}{2}\Delta|\nabla u|^2 = (u_{ik})^2 + (\Delta u)_i u_i$$

and integrate over ball B_r on both sides to obtain

$$\int_{B_r} |\nabla^2 u|^2\, dx = \int_{B_r} (\Delta u)^2 + \frac{1}{r}\int_{\partial B_r} u_{ik} u_i x_k - \frac{1}{r}\int_{\partial B_r} (\Delta u) u_i x_i\,. \tag{4.2}$$

Thus

$$E_2^2(u)(B_r) \le \tilde{E}_2^2(u)(B_r) + c\left(r^4 \fint_{\partial B_r} |\nabla^2 u|^2\, d\sigma\right)^{\frac12}\left(r^2 \fint_{\partial B_r} |\nabla u|^2\, d\sigma\right)^{\frac12}. \tag{4.3}$$

Also, applying (4.3) to B_ρ, we obtain (4.1). □

Definition. Fixing $0 < r \le 1$, if $1/2^{k+1} \le r < 1/2^k$ for some k, we denote $r^* = 1/2^k$. We say ∂B_r is a *good slice* if it satisfies both

$$
\begin{cases}
r \int_{\partial B_r} |\nabla^2 u|^2 d\sigma \le 8 \int_{B_{r^*}} |\nabla^2 u|^2\, dx \\
r \int_{\partial B_r} |\nabla u| d\sigma \le 8 \int_{B_{r^*}} |\nabla u| dx\,.
\end{cases}
\tag{4.4}
$$

We remark that such a good slice always exists for all $k \geq 0$.

LEMMA 4.3 *There exists some constant c such that for all good slices ∂B_ρ, ∂B_r, $\rho < r < \frac{1}{2}$, for all $\eta > 0, \eta$ sufficiently small, we have*

$$E(u)(B_\rho) \leq cE(u)(B_r^*) + \eta E(u)(B_{\rho^*}) + C_\eta M(u)(B_{\rho^*})\,, \tag{4.5}$$

where $C_\eta = c\eta^{-(3+m)}$, $M(u) = M_1(u)$, and $E(u)$ and $M_1(u)$ are defined as in (2.7).

PROOF: We first observe that by an interpolating inequality of L. Nirenberg [10], we have

$$D_2(u)(B_r) \lesssim E_2(u)(B_r)M_1^{1-a}(u)(B_r) + M_1(u)(B_r)\,, \tag{4.6}$$

where $\frac{1}{2} - \frac{1}{m} = a(\frac{1}{2} - \frac{2}{m}) + (1-a)$ (thus $a = \frac{2+m}{4+m}, \frac{1}{2} < a < 1$). By combining (4.6) and (4.1), we obtain that for all good slices ρ, r where $0 < \rho < r$,

$$\begin{aligned} E_2^2(u)(B_\rho) \leq\ & E_2^2(u)(B_r) + cE^2(u)(B_{r^*}) \\ & + cE_2^{1+a}(u)(B_{\rho^*})M_1^{1-a}(u)(B_{\rho^*}) \\ & + cE_2^{2a}(u)(B_{\rho^*})M_1^{2(1-a)}(u)(B_{\rho^*}) + cM_1^2(u)(B_{\rho^*}) \\ & + cE_2(u)(B_{\rho^*})M_1(u)(B_{\rho^*})\,. \end{aligned} \tag{4.7}$$

We now apply the inequality $x^a y^{1-a} \leq a\eta x + (1-a)\bar{C_\eta} y$ for all $x, y > 0,\ 0 < a < 1$, where $\bar{C_\eta} = \eta^{-a/(1-a)} = \eta^{-(1+m/2)}$. We similarly apply $x^{1+a}y^{1-a} \leq \frac{1+a}{2}(\eta x)^2 + \frac{1-a}{2}(C_\eta y)^2$ with $c_\eta = \eta^{-(3+m)}$ to (4.7), with $x = E_2(u)(B_{\rho^*}), y = M_1(u)(B_{\rho^*})$. We obtain

$$E_2(u)(B_\rho) \leq cE(u)(r^*) + c\eta E(u)(B_{\rho^*}) + cC_\eta M(u)(B_{\rho^*})\,.$$

We now observe that we can estimate $D_q(u)(B_\rho)$ via Sobolev embedding and (4.6). Thus, we obtain (4.5) after adjusting the constant η. □

As an immediate corollary of Lemma 4.3, we have the following:

COROLLARY 4.4 *Suppose u is a stationary biharmonic map on B_2. Then there is a constant c such that for all $0 < 4\rho < r < 1$ and η sufficiently small, we have*

$$E(u)(B_{\rho^*}) \leq cE(u)(B_{2r^*}) + \eta E(u)(B_{2\rho^*}) + C_\eta M(u)(B_{2\rho^*})\,. \tag{4.8}$$

PROOF: Given any ρ and r with $4\rho < r < \frac{1}{2}$, say $\frac{1}{2^{k+1}} \leq \rho < \frac{1}{2^k}$, we may choose ρ_1 with $\frac{1}{2^k} \leq \rho_1 < \frac{1}{2^{k-1}}$ a good slice, and r_1 a good slice similarly chosen with $\frac{r}{2} \leq r_1^* < r$ so that $\rho_1 < r_1$. We then apply (4.7) to ρ_1, r_1 and observe that $\rho_1^* = 2\rho^*$, $r_1^* = 2r^*$. Equation (4.8) then follows. □

The following lemma is the version of Lemma 2.3 for $m \geq 5$:

LEMMA 4.5 *There exists some* $\tau < \frac{1}{4}$ *and* c *a dimensional constant so that for all* $r < 1$,

$$(4.9)\qquad \begin{aligned}(M_{p_0}(u) + D_p(u))(B_{\tau r}) \leq\, & c\tau^{1-\frac{m}{p}} E^2(u)(B_r) M_s(u)(B_r) \\ & + c\tau^{1-\frac{m}{p}} D_q(u)(B_r) M_t(u)(B_r) \\ & + \tau (M_s(u) + D_p(u))(B_r),\end{aligned}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$, $\frac{1}{s} = \frac{1}{p} + \frac{3}{m} - 1$, $\frac{1}{t} = \frac{1}{p} + \frac{1}{m} - \frac{1}{2}$, $\frac{1}{p_0} = \frac{1}{p} - \frac{1}{m}$, *and* p *is a suitably chosen constant bigger than* 1.

PROOF: We choose $\frac{1}{2} < r < 1$ with ∂B_r a good slice and run through exactly the same argument (and same notation) as in the proof of Lemma 2.3. We obtain for any $p = p_1 > 1$ suitably chosen,

$$(2.11)\qquad \begin{aligned}\|\nabla v\|_{L^p(B_r)} \lesssim\, & \left(\|\nabla^2 u\|_{L^2(B_r)} + \|\nabla u\|_{L^q(B_r)}\right) \|u - A_0\|_{L^s(B_r)} \\ & + \|\nabla u\|_{L^q(B_r)} \|u - A_0\|_{L^t(B_r)},\end{aligned}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{m}$, $\frac{1}{s} = \frac{1}{p} + \frac{3}{m} - 1$, $\frac{1}{t} = \frac{1}{p} + \frac{1}{m} - \frac{1}{2}$, and A_0 is any constant. Thus, if p is chosen with $\frac{1}{m} < 1 - \frac{3}{m} < \frac{1}{p} < 1 < \frac{3}{2} - \frac{1}{m} < 2 - \frac{3}{m}$, then $m \geq 5$ implies that such $1 < p < m$ exists with $s, t > 1$. We now apply Sobolev embedding to the left-hand side of (2.11) and obtain for $\frac{1}{p_0} = \frac{1}{p} - \frac{1}{m}$ and any constant B,

$$(4.10)\qquad \begin{aligned}\|v\|_{L^{p_0}(B_r)} + \|\nabla v\|_{L^p(B_r)} \lesssim\, & \left(\|\nabla^2 u\|^2_{L^2(B_r)} + \|\nabla u\|^2_{L^q(B_r)}\right) \|u - A_0\|_{L^s(B_r)} \\ & + \|\nabla u\|_{L^q(B_r)} \|u - A_0\|_{L^t(B_r)}.\end{aligned}$$

We now choose $\tau < \frac{1}{4}$; then for all $x \in B_\tau$ and ∂B_r a good slice that the biharmonic function k satisfies

(4.11)

$$|\nabla k(x)| \leq \int_{\partial B_r} |u - A_0| + \int_{\partial B_r} \left|\frac{\partial u}{\partial n}\right| dx \lesssim M_1(u)(B_1) + D_1(u)(B_1) \quad \text{for all } |x| \leq \tau.$$

Thus, we have from (4.10) and (4.11)

$$(4.12)\qquad \begin{aligned} & M_{p_0}(u)(B_\tau) + D_p(u)(B_\tau) \\ & \lesssim \tau^{-\frac{m}{p_0}} \left(\int_{B_\tau} |u - \bar{u}_\tau|^{p_0}\right)^{\frac{1}{p_0}} + \left(\tau^p \fint_{B_\tau} |\nabla u|^p\right)^{\frac{1}{p}} \\ & \lesssim \tau^{-\frac{m}{p_0}} \left(\|v\|_{L^{p_0}(B_\tau)} + \|k - k(0)\|_{L^{p_0}(B_\tau)}\right) \\ & \quad + \tau^{-\frac{p-m}{p}} \left(\|\nabla v\|_{L^p(B_\tau)} + \|\nabla k\|_{L^p(B_\tau)}\right)\end{aligned}$$

$$\lesssim \tau^{-\frac{m}{p_0}} E^2(u)(B_1)\|u-A_0\|_{L^s(B_r)} + \tau^{-\frac{m}{p_0}} (D_q u)(B_1)\|u-A_0\|_{L^t(B_r)}$$
$$+\tau^{-\frac{m}{p_0}}\tau^{1+\frac{m}{p_0}} \sup_{x\in B_\tau} |\nabla k(x)| + \tau^{\frac{p-m}{p}}\tau^{\frac{m}{p}} \sup_{x\in B_\tau}|\nabla k(x)|$$
$$\lesssim \tau^{1-\frac{m}{p}} E^2(u)(B_1)M_s(u)(B_1) + \tau^{1-\frac{m}{p}} D_q(u)(B_1)M_t(u)(B_1)$$
$$+\tau(M_1(u)(B_1)+D_1(u))(B_1)\,.$$

We observe that every term scales in an invariant way in (4.12); therefore we may rewrite (4.12) in the form of (4.9). This finishes the proof of Lemma 4.5. ☐

COROLLARY 4.6 *Let r and τ be as in Lemma* 4.5. *Then*

$$(M_{p_0}(u)+D_p(u))(B_{\tau r}) \le \Big(\tau^{1-\frac{m}{p}}(E^2(u)+E(u))(B_r)+\tau\Big)(M_s(u)+D_p(u))(B_r)\,. \tag{4.13}$$

Our next observation is that by our choices of p_0, p, s, t, and q, we have $1<p<q<m$, $1<p_0<t<s<m$, and $\frac{1}{p}>1-\frac{3}{m}$; thus $M_t(u)(B_r)\lesssim D_q(B_r)\le E(u)(B_r)$. Taking this together with the trivial estimate that $\|u\|_\infty\le 1$, which implies $M_s(u)(B_r)\le 2$ for all $r<1$, we obtain directly from (4.12) the following estimate:

COROLLARY 4.7 *Let r,τ be as in Lemma* 4.5; *then*

$$\begin{aligned} M_1(u)(B_{\tau r}) &\le c\tau^{1-\frac{m}{p}}E^2(u)(B_r)+c\tau\left(M_1(u)+D_1(u)\right)(B_r)\\ &\le c\tau^{1-\frac{m}{p}}E^2(u)(B_r)+c\tau E(u)(B_r)\,. \end{aligned} \tag{4.14}$$

We now combine estimate (4.14) with the monotonicity formula (4.5) to derive the following estimate:

LEMMA 4.8 *For ε and ρ_0 sufficiently small, there exists some constant C so that if $E(u)(B_1)<\varepsilon$, then*

$$E(u)(B)\le CE(u)(B_1) \quad \textit{for all balls } B\subseteq B_{\rho_0}\subset B_1\,. \tag{4.15}$$

PROOF: For simplicity we now write $E(u)(B_\rho)=E(\rho)$ and $M_1(u)(B_\rho)=M(\rho)$. We notice that from (4.8) and (4.14) we have

$$\forall 2\rho<r<1\,,\qquad E(\rho^*)\le cE(2r^*)+\eta E(2\rho^*)+C_\eta M(2\rho^*)\,, \tag{4.16}$$

$$\forall \tau\le\frac{1}{4},\ r<1\,,\quad M(\tau r)\le c\tau^{1-\frac{m}{p}}E^2(r)+c\tau E(r)\,. \tag{4.17}$$

We will now apply (4.16) and (4.17) to establish (4.15). To do this, we fix $\tau_0=2^{-\ell}$ (ℓ large to be chosen later) and consider $\rho_k=\tau_0^{2k}$ for each $\rho=\rho_k$. We estimate

$E(\rho)$ by applying (4.16) as

$$(4.18)\quad \begin{aligned} E(\rho_k^*) &\le cE(1)+C_\eta M(2\rho_k^*)+\eta E(2\rho_k^*) \\ &\le cE(1)+C_\eta M(2\rho_k^*)+\eta(cE(1)+\eta E(4\rho_k^*))+C_\eta M(4\rho_k^*) \\ &= c(1+\eta)E(1)+C_\eta M(2\rho_k^*)+C_\eta \eta M(4\rho_k^*)+\eta^2 E(4\rho_k^*) \\ &\le \frac{c}{1-\eta}E(1)+M^k+\eta^{2l}E(\rho_{k-1}^*) \quad \text{(inductively)}, \end{aligned}$$

where

$$M^k = C_\eta \sum_{j=0}^{2\ell-1} \eta^j M(2^{j+1}\rho_k^*).$$

We claim that there exists some $\delta<1$ so that following estimates hold for all $k\ge 1$:

$$\text{(a)}_k \qquad M^k \le \frac{C_\eta}{1-\eta}\left(b\frac{c}{1-\delta}+c\tau_0+2\eta^{2\ell}\right)E(\rho_{k-1}^*)$$

$$\text{(b)}_k \qquad E(\rho_k^*) \le \frac{2c}{1-\eta}E(1)(1+\delta+\cdots+\delta^k),$$

where $b=c(\tau_0^2)^{1-\frac{m}{p}}E(1)$ and we set ρ_0^* as $\rho_0^*=1$.

We establish (a)$_k$ and (b)$_k$ inductively. *When* $k=1$.

(a) Estimate of M^1: For each $0\le j\le 2\ell-1$, we apply (4.17) to $M(2^{j+1}\rho_1^*)$ to obtain

$$M(2^{j+1}\rho_1^*) \le c(2^{j+2}\rho_1)^{1-\frac{m}{p}}E^2(1)+c2^{j+2}\rho_1 E(1).$$

Notice

$$\tau_0^2 \le 4\rho_1 \le 2^{j+2}\rho_1 \le 2^{\ell+1}\rho_1 \le 2\tau_0 \quad \text{for } 0\le j\le \ell-1$$

and

$$\tau_0^2 \le 4\rho_1 \le 2^{j+2}\rho_1 \le 2^{2\ell+1}\rho_1 \le 2 \quad \text{for } \ell\le j\le 2\ell-1.$$

Thus, denoting $b=c(\tau_0^2)^{(1-\frac{m}{p})}E(1)$, we have

$$(4.19)\quad \begin{aligned} M^1 &\le C_\eta \sum_{j=0}^{\ell-1}\eta^j(b+2c\tau_0)E(1)+C_\eta\sum_{j=\ell}^{2\ell-1}\eta^j(b+2c)E(1) \\ &\le C_\eta\frac{1}{1-\eta}(b+2c\tau_0+2c\eta^\ell)E(1). \end{aligned}$$

(b) Thus, if we choose δ_1 so that

$$(4.20)\qquad C_\eta(b+2c\tau_0+2c\eta^\ell)\le\delta_1,$$

then $M^1\le\frac{c}{1-\eta}\delta_1 E(1)$. It then follows from (4.18) that if $\delta_1\le\delta$, $E(\rho_1^*)\le \frac{2c}{1-\eta}(1+\delta)E(1)$.

For general k. We assume $(a)_j$ and $(b)_j$ for $j \le k-1$.

(a) Estimate of M^k:

$$M^k = C_\eta \sum_{j=0}^{2\ell-1} \eta^j M(2^{j+1}\rho_k^*).$$

We now estimate $M(2^{j+1}\rho_k^*)$ by (4.17) with $2^{j+1}\rho_k^* = \tilde{\tau}_j \rho_{k-1}^*$. Thus $\tilde{\tau}_j = 2^{j+1}\tau_0^2$ and $\tau_0^2 \le 2\tau_0^2 \le \tilde{\tau}_j \le 2^\ell \tau_0^2 = \tau_0$ for $0 \le j \le \ell-1$, and $\tau_0^2 \le 2\tau_0^2 \le \tilde{\tau}_j \le 2^2\ell\tau_0^2 = 1$ for $\ell \le j \le 2\ell-1$. Hence, for $0 \le j \le \ell-1$ we have:

$$\begin{aligned}
M(2^{j+1}\rho_k^*) &\le c\tilde{\tau}_j^{1-\frac{m}{p}} E^2(\rho_{k-1}^*) + c\tilde{\tau}_j E(\rho_{k-1}^*) \\
&\le c(\tau_0^2)^{1-\frac{m}{p}} E^2(\rho_{k-1}^*) + c\tau_0 E(\rho_{k-1}^*) \quad \text{(for } p<m) \\
&\le \left[c(\tau_0^2)^{1-\frac{m}{p}} E(1)(1+\delta+\cdots+\delta^{k-1}) \frac{2c}{1-\eta} + c\tau_0 \right] E(\rho_{k-1}^*) \\
&\hspace{20em} \text{(by } (b)_{k-1}) \\
&\le \left(b\frac{2c}{1-\eta}\frac{1}{1-\delta} + c\tau_0 \right) E(\rho_{k-1}^*).
\end{aligned}$$

Similarly, for $\ell \le j \le 2\ell-1$ we have

$$\begin{aligned}
M(2^{j+1}\rho_k^*) &\le c(\tau_0^2)^{1-\frac{m}{p}} E^2(\rho_{k-1}^*) + cE(\rho_{k-1}^*) \\
&\le \left(b\frac{2c}{1-\eta}\frac{1}{1-\delta} + c \right) E(\rho_{k-1}^*).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
M^k &= C_\eta \sum_{j=0}^{2\ell-1} \eta^j M(2^{j+1}\rho_k^*) \\
&\le \frac{C_\eta}{1-\eta}\left(b\frac{2c}{1-\eta}\frac{1}{1-\delta} + c\tau_0 \right) E(\rho_{k-1}^*) \\
&\quad + \frac{C_\eta}{1-\eta}\eta^\ell \left(b\frac{2c}{1-\eta}\frac{1}{1-\delta} + c \right) E(\rho_{k-1}^*).
\end{aligned} \tag{4.21}$$

(b) For fixed $\eta < 1$, choose ℓ sufficiently large so that both $\tau_0 = 2^{-\ell}$ and η^ℓ are sufficiently small; then choose ε sufficiently small so that $b = c(\tau_0^2)^{1-m/p}E(1)$ with $E(1) < \varepsilon$ small. Thus,

$$\delta = \max\left(\frac{C_\eta}{1-\eta}\left(4bc\frac{1}{1-\eta} + c\tau_0 + c\eta^\ell \right) \delta_1 \right) < 1.$$

Then, inductively we obtain from (4.20) that

$$M^k \le \delta E(\rho_{k-1}^*)$$

$$\leq \delta \frac{2c}{1-\eta} E(1)(1+\delta+\cdots+\delta^{k-1}) \quad (\text{by } (b)_{k-1})$$
$$\leq \frac{2c}{1-\eta} E(1)(\delta+\delta^2+\cdots+\delta^{k-1}).$$

By (4.18) we have thus established (b)$_k$.

We now remark that once (b)$_k$ holds for all k, we have $E(\rho) \lesssim \frac{c}{1-\eta}\frac{1}{1-\delta}E(1)$ for all $\rho \leq \tau_0^2$. From this it follows also that $E(u)(B) \leq CE(1)$ for all $B \subseteq B_{\rho_0}$ and for constant C where $\rho_0 = \tau_0^2$ by some simple covering argument. We have thus finished the proof of the lemma.

□

PROOF OF THEOREM 4.1: We follow the same line of proof as theorem 2.5 in [2]. For $\varepsilon > 0$ sufficiently small, with $E(u)(B_1) < \varepsilon$, we have from Lemma 4.3

$$E(u)(B) \leq CE(u)(B_1) \leq C\varepsilon \quad \text{for all } B \subseteq B_{\rho_0}.$$

Thus, it follows from (4.13) that there exists some $\rho < 1$ with

$$\sup_{B \subseteq B_\rho} \left(M_{p_0}(u) + D_p(u)\right)(B) \leq \left(C\rho^{1-\frac{m}{p}}\varepsilon + C\rho\right)\left(M_s(u) + D_p(u)\right)(B_1).$$

If we apply the John-Nirenberg [8] inequality, we then conclude that there exists some universal constant M such that

$$\begin{aligned} &\|u\|_{\mathrm{BMO}_s(B_\rho)} + D_{p_1}(u)(B_\rho) \leq \\ &\quad M(C\rho^{1-\frac{m}{p}}\varepsilon + C\rho)\left(\|u\|_{\mathrm{BMO}_s(B_1)} + D_{p_1}(u)(B_1)\right), \end{aligned} \tag{4.22}$$

where

$$\|u\|_{\mathrm{BMO}_s(B)} = \sup_{B^1 \subseteq B} \inf_{\mathrm{const}\, c} \left(⨍_{B^1} |u-c|^s \right)^{\frac{1}{s}}.$$

Now, for any $\beta < 1$ there is a $\rho = \rho_0$ small such that $MC\rho_0 < \frac{\rho^\beta}{2}$, and then ε small such that $MC\rho_0^{1-m/p}\varepsilon \leq \rho_0^\beta/2$. Accordingly, we have

$$\|u\|_{\mathrm{BMO}_s(B_{\rho_0})} + D_{p_1}(u)(B_{\rho_0}) \leq \rho_0^\beta\left(\|u\|_{\mathrm{BMO}_s(B_1)} + D_{p_1}(u)(B_1)\right). \tag{4.23}$$

An iteration of (4.23) leads to

$$\|u\|_{\mathrm{BMO}_s(B_{\rho_0^k})} + D_{p_1}(u)(B_{\rho_0^k}) \leq \rho_0^{k\beta}\left(\|u\|_{\mathrm{BMO}_s(B_1)} + D_{p_1}(u)(B_1)\right),$$

for each $k = 1, 2, \ldots$. The above inequality proves that

$$\|u\|_{\mathrm{BMO}_s(B_r)} + D_{p_1}(u)(B_r) \leq Cr^\beta\left(\|u\|_{\mathrm{BMO}_s(B_1)} + D_{p_1}(u)(B_1)\right) \tag{4.24}$$

for all $0 \leq r \leq 1$.

It follows from (4.24) and the standard covering argument (e.g., as in Evans [3]) that the singularity set of the stationary map is a set of $(m-4)$–Hausdorff dimension zero. We have thus finished the proof of Theorem 4.1. □

Remark. We remark that we actually have proved that for all $\beta < 1$,

$$\|u\|_{\mathrm{BMO}_s(B_{\rho_0})} + D_{p_1}(u)(B_{\rho_0}) \le C\rho^\beta$$

whenever the energy is small near the center of the ball. Hence we proved a Hölder regularity for any exponent $\beta < 1$. Actually, one has for any $p_1 < 4$ and any exponent $\beta < 1$, there exists some constant C such that

$$D_{p_1}(u)(B_{\rho_0}(x)) \le C\rho_0^\beta$$

holds for any x in the regular set of u for some ρ_0 sufficiently small. Thus it follows from the Sobolev embedding and Hölder inequality that

$$|u(x) - u(y)| \le C|x-y|^\beta$$

when y is sufficiently close to x.

5 Further Smoothness

In this section we show that the solution is actually smooth once it is continuous. We remark that, according to the classical regularity theory, it suffices to prove that the solution is $C^{2,\alpha}$ for some $\alpha > 0$.

In the previous section we showed that u is Hölder-continuous with any exponent $\beta < 1$, that is,

$$|u(x) - u(y)| \le C|x-y|^\beta \tag{5.1}$$

and

$$D_{p_1}(u)(B_{\rho_0}) \le C\rho_0^\beta, \tag{5.2}$$

when the center of the ball B_{ρ_0} is in the regular set of u for some $p_1 < 4$. Actually, (5.2) implies (5.1).

THEOREM 5.1 *If u is a Hölder-continuous biharmonic map satisfying* (5.1) *and* (5.2) *in B_1, then u is locally smooth.*

We remark that one can modify the proof of Theorem 5.1 given below to prove that any continuous biharmonic map is in fact smooth. But for simplicity, we will prove the result only in the setting of Theorem 5.1.

THEOREM 5.2 *If the biharmonic map is Hölder-continuous as*

$$r^2 \fint_{B_r(x)} |\nabla u|^2 + \|u - u(x)\|^2_{L^\infty(B_r(x))} \le Cr^{2\beta} \tag{5.3}$$

for all $0 < r \le \rho$ *and for some $x \in R^m$, then*

$$\rho^4 \fint_{B_\rho(x)} |\nabla^2 u|^2 \le C\rho^{2\beta}. \tag{5.4}$$

PROOF: Let $\eta(x)$ be a cutoff function in $B_\rho = B_\rho(x)$ such that $0 \le \eta \le 1$, $|\nabla\eta| \le \frac{C}{\rho}$, $|\nabla^2\eta| \le \frac{C}{\rho^2}$ with $\eta = 1$ in $B_{\rho/2}$ and $\eta = 0$ near ∂B_ρ. Multiplying the biharmonic map equation (1.1) by $\eta^4(u-u(x))$ and integrating by parts, we have that for any $\varepsilon < 1$,

$$
\begin{aligned}
&-\int_{B_\rho}\eta^4(u-u(x))\lambda u\\
&\quad=\int_{B_\rho}\Delta(\eta^4(u-u(x))\Delta u\\
&\quad\le\int_{B_\rho}\eta^4|u-u(x)||\nabla^2u|^2+\eta^4(u-u(x))\nabla(\nabla u*\nabla^2u)\\
&\quad\lesssim\int_{B_\rho}\rho^\beta\eta^4|\nabla^2u|^2+\eta^2|\eta^2\nabla u+2(\nabla\eta^2)(u-u(x))||\nabla u*\nabla^2u|\\
&\quad\le\left(2\varepsilon\int_{B_\rho}\eta^4|\nabla^2u|^2\right)+C_\varepsilon\rho^{m-4+4\beta}+C_\varepsilon\|\nabla u\|^4_{L^4(B_\rho)}\\
&\quad\le\left(2\varepsilon\int_{B_\rho}\eta^4|\nabla^2u|^2\right)+C_\varepsilon\rho^{m-4+4\beta}+C_\varepsilon\|\nabla u\|^4_{L^4(B_\rho)}\,,
\end{aligned}
\tag{5.5}
$$

where we have taken ρ to be sufficiently small. Now we apply the Gagliardo and Nirenberg inequality [10] and we have

$$
\begin{aligned}
\|\nabla u\|^4_{L^4(B_\rho)}&\le C\int_{B_\rho}|\nabla^2u|^2\|u-u(x)\|^2_{L^\infty(B_\rho)}+C\rho^{m-4}\|u-u(x)\|^4_{L^\infty(B_\rho)}\\
&\le C\rho^{2\beta}\int_{B_\rho}|\nabla^2u|^2+C\rho^{m-4+4\beta}.
\end{aligned}
\tag{5.6}
$$

On the other hand, we have the following standard treatment for the left-hand side of the above inequality:

$$
\begin{aligned}
&\int_{B_\rho}\Delta(\eta^4(u-u(x))\Delta u\\
&\quad=\int_{B_\rho}\eta^4(\Delta u)^2+4\eta^2(\nabla\eta^2)\nabla u\Delta u+(\Delta\eta^4)(u-u(x))\Delta u\\
&\quad=\int_{B_\rho}\eta^4(\Delta u)^2+4\eta^2\Delta u(\nabla\eta^2)\nabla u\\
&\qquad+2\eta^2\Delta u(u-u(x))(\Delta\eta^2+4|\nabla\eta|^2)\\
&\quad\ge\frac12\int_{B_\rho}(|(\eta^2\Delta u|^2-C\int_{B_\rho}\left(|(\nabla\eta^2)\nabla u|^2+C\rho^{-4}|u-u(x)|^2\right)\\
&\quad\ge\frac12\int_{B_\rho}|\eta^2\Delta u|^2-C(\rho^{m-2}+\rho^{m-4+2\beta})\\
&\quad\ge\frac12\int_{B_\rho}|\eta^2\Delta u|^2-C\rho^{m-4+2\beta},
\end{aligned}
\tag{5.7}
$$

whereas,

(5.8)
$$\begin{aligned}\int_{B_\rho}|\eta^2\Delta u|^2 &= \int_{B_\rho}|\eta^2\Delta(u-u(x))|^2\\ &\geq \frac{1}{2}\int_{B_\rho}|\Delta(\eta^2(u-u(x))|^2-2C|(\nabla\eta^2)\nabla u|^2-C|(\Delta\eta^2)(u-u(x))|^2\\ &\geq \frac{1}{4}\int_{B_\rho}\eta^4|\nabla^2 u|^2-C|(\nabla\eta^2)\nabla u|^2-C|\nabla^2\eta^2|^2|u-u(x)|^2\\ &\geq \frac{1}{4}\int_{B_\rho}\eta^4|\nabla^2 u|^2-C\rho^{m-2}-C\rho^{m-4+2\beta}.\end{aligned}$$

Combining the inequalities (5.5) with (5.8), we have

$$\begin{aligned}\int_{B_{\rho/2}}|\nabla^2 u|^2 &\leq \int_{B_\rho}\eta^4|\nabla^2 u|^2\\ &\leq \left(8\int_{B_\rho}\Delta(\eta^4(u-u(x))\Delta u\right)+C\rho^{m-4+2\beta}\\ &\leq \left((16\varepsilon+C\rho^{2\beta})\int_{B_\rho}|\nabla^2 u|^2\right)+C\rho^{m-4+2\beta}.\end{aligned} \tag{5.9}$$

Then an iteration process in [4, p. 86] shows that if

$$\sigma(\tfrac{\rho}{2})\leq\varepsilon\sigma(\rho)+C\rho^\beta \quad \text{for } 0<\rho\leq A \text{ with } \varepsilon<2^{-\beta},$$

then

$$\sigma(\rho)\lesssim C\rho^\beta \qquad \text{for all } 0<\rho\leq A.$$

Taking ε and $\rho\leq\rho_0$ small so that $16\varepsilon+C\rho_0^{2\beta}$ is small, it follows from the iteration process above that

$$\int_{B_{\rho 2}}|\nabla^2 u|^2\leq C\rho^{m-4+2\beta}. \tag{5.10}$$

The theorem follows. ☐

COROLLARY 5.3 *Under the condition of the above theorem, we have*

$$\int_{B_\rho}|\nabla u|^4\leq C\rho^{m-4+4\beta}. \tag{5.11}$$

PROOF: The proof follows from the Gagliardo-Nirenberg inequality (5.6) as in the proof of the above theorem. ☐

The $C^{1,\alpha}$ regularity for u is a corollary of the following Campanato space estimates:

THEOREM 5.4 *Assume $\gamma > 0$ is a noninteger and $p > 1$. Suppose u is a weak solution of*

$$\Delta^2 u = f + \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}$$

with conditions that

$$\fint_{B_r} |f| \le C r^{\gamma-4} \quad \text{and} \quad \left(\fint_{B_r} |g_i|^p \right)^{\frac{1}{p}} \le C r^{\gamma-3}.$$

Then, u is $C^{[\gamma],\{\gamma\}}$ at x, the center of the balls B_r in the $W^{3,1}$ norm, where $[\gamma]$ denotes the integer part and $\{\gamma\}$ denotes the fractional part of γ; i.e., there is a polynomial of order $[\gamma]$ such that

$$\fint_{B_r} |u-P| + r|\nabla(u-P)| + r^2|\nabla^2(u-P)| + r^3|\nabla^3(u-P)| \le C r^{\gamma},$$

where C depends on the estimates on f, g_i, and $|u|_{W^{3,1}}$.

The above theorem can be proved using an argument similar to that of theorem 2.2 in [4, p. 84]; we will skip the proof here.

THEOREM 5.5 *Let u be a biharmonic map that is Hölder-continuous with exponent $\frac{1}{2} < \beta < 1$ in the following fashion*:

$$(5.12)\quad r^4 \fint_{B_r(x)} |\nabla^2 u|^2 + \left(r^4 \fint_{B_r(x)} |\nabla u|^4 \right)^{\frac{1}{2}} \le C r^{2\beta} \quad \textit{for any } x \in B_{\frac{1}{2}},\ 0 < r < 1.$$

Then, u is $C^{1,2\beta-1}$ in $B_{1/2}$ in the sense that for each $x \in B_{1/2}$ there is a linear function L such that

(5.13)

$$\fint_{B_r(x)} |u-L| + r|\nabla(u-L)| + r^2|\nabla^2 u| + r^3|\nabla^3 u| \le C r^{2\beta} \quad \textit{for all } 0 < r < 1.$$

PROOF: Since u is a biharmonic map, applying equation (1.1), we write

$$\Delta^2 u = \nabla^2 u * \nabla^2 u + \nabla(\nabla u * \nabla^2 u) = f + \nabla g.$$

From (5.12) we see immediately that

$$\fint_{B_r} |f^{\alpha}| \le C r^{2\beta-4} \quad \text{and} \quad \left(\fint_{B_r} |g^{\alpha}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \le C r^{2\beta-3}.$$

Thus, we may apply Theorem 5.4 with $\gamma = 2\beta$ and conclude that u is $C^{1,2\beta-1}$. □

Our next step is to show that any $C^{1,\alpha}$ solution is $C^{2,\alpha}$.

LEMMA 5.6 *If a function is $C^{1,\alpha}$ in $W^{3,1}$ in the sense that for any x there is a linear function L, then for any $B_r = B_r(x)$, $0 < r \le 1$,*

$$(5.14)\quad \fint_{B_r} |u-L| + r|\nabla(u-L)| + r^2|\nabla^2 u| + r^3|\nabla^3 u| \le C r^{1+\alpha}.$$

Then, u is $C^{1,\alpha}$ *in* $W^{2,2}$*, i.e.,*

$$\fint_{B_r} r^4 |\nabla^2 u|^2 \le C r^{2+2\alpha}\,. \tag{5.15}$$

PROOF: First, we have that u is in the classical $C^{1,\alpha}$-space by the Morrey embedding theorem. Then we can apply the Gagliardo-Nirenberg inequality of the form that

$$r^4 \fint_{B_r} |\nabla^2 u|^2 \le C r^4 \|\nabla(u-L)\|_{L^\infty} \fint_{B_r} |\nabla^3 u| + C r^2 \|\nabla(u-L)\|^2_{L^\infty}\,,$$

and thus (5.15) follows directly from (5.14). □

PROOF OF THEOREM 5.1: We will establish Theorem 5.1 in several steps. Assume that u is a biharmonic map satisfying both conditions (5.1) and (5.2) with some $\frac{1}{2} < \beta$. Then, from Theorem 5.5 we conclude that u is in $C^{1,\alpha}$ for $\alpha = 2\beta - 1$. Lemma 5.6 then asserts that u is also $C^{1,2\beta-1}$ in the $W^{2,2}$ sense, as in (5.15). We may then apply Theorems 5.4 and 5.5 again to obtain that u is in fact in $C^{[4\beta],\{4\beta\}}$, and hence in $C^{2,4\beta-2}$. Thus, u is smooth by the classical regularity theory. □

Remark. We remark that our scheme above actually indicates that once a biharmonic map satisfies conditions (5.1) and (5.2) for some $\beta > 0$, then we may iterate to conclude that it satisfies (5.1) and (5.2) for 2β. Thus, we may iterate the above scheme finite many times to prove that any biharmonic map that is Hölder-continuous is in fact smooth.

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