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A regularized iterative algorithm for limited-angle inverse Radon transform

M. DEFRISE

Theoretische Natuurkunde, Vrije Universiteit Brussel,
B-1050 Brussels, Belgium

C. DE MOL A5/5.

Département de Mathématique, Université Libre de Bruxelles,
B-1050 Brussels, Belgium

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Abstract. The tomography problem is investigated when the available projections are restricted to a limited angular domain. It is shown that a previous algorithm proposed for extrapolating the data to the missing cone in Fourier space is unstable in the presence of noise because of the ill-posedness of the problem. A regularized algorithm is proposed, which converges to stable solutions. The efficiency of both algorithms is tested by means of numerical simulations.

1. Introduction

For some problems in tomographical imaging, data collection is restricted to a limited angle of view. For instance, in radiology and nuclear medicine imaging, this situation can arise in the presence of obstacles or because of the necessity of reducing radiation dosage or increasing imaging speed. Limited-angle tomography also has important non-medical applications in, for instance, the reconstruction of three-dimensional fields by the holographic interferometry technique [1].

Let us briefly recall that in two dimensions and for parallel beam projection the problem of inverting the Radon transform can be formulated as follows [2]: we find the object function $f(x, y)$ from the set of projections $g(s, \theta)$,

$$g(s,\theta)=\int_{-\infty}^{+\infty} du\, f(s\cos\theta-u\sin\theta,\, s\sin\theta+u\cos\theta) \tag{1}$$

where the angle θ giving the direction of projection lines varies in the interval $[-\theta_{\max}, +\theta_{\max}]$. For the usual full-angle tomography, $\theta_{\max}=\pi/2$. Let us denote by R the operator mapping the solution f on to the data $g=Rf$. When both the solution and the data are assumed to be square integrable on their respective domains of definition, the operator R can be viewed as a linear operator from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}\times[-\theta_{\max}, +\theta_{\max}])$. Let us now define the usual back-projection operator with angular integration limited to $[-\theta_{\max}, +\theta_{\max}]$,

$$(\bar{R}g)(x,y)=\int_{-\theta_{\max}}^{+\theta_{\max}} d\theta\, g(x\cos\theta+y\sin\theta,\theta), \tag{2}$$

and consider the problem of reconstructing f from the back-projected data $\bar{R}g$. The equation to be solved can be easily written in Fourier space. Indeed, when denoting by $(\mathscr{F}_2 f)(\underline{v})$ the two-dimensional Fourier transform of the function f of $L^2(\mathbb{R}^2)$, we have

$$\mathscr{F}_2(\bar{R}Rf)(\underline{v})=\chi_\Omega\frac{1}{v}(\mathscr{F}_2 f)(\underline{v}) \tag{3}$$

where $v=\sqrt{|\underline{v}|^2}$ and the characteristic function χ_Ω takes the value 1 in the angular sector Ω defined by $|\tan\theta|\leqslant\tan\theta_{max}$ and 0 elsewhere. Let us now remark that because of non-uniqueness and instability the solution of this equation is an ill-posed problem and that even if $\theta_{max}=\pi/2$, some form of regularization is needed for a numerical solution of the problem. From equation (3), we see that the part of $\mathscr{F}_2 f$ with support in the complementary sector $|\tan\theta|>\tan\theta_{max}$, called the 'missing cone', cannot be retrieved from the data. The problem has a unique solution only if $\theta_{max}=\pi/2$. However, this non-uniqueness problem in the limited-angle case can be overcome when we know *a priori* that $f(x, y)$ vanishes outside some bounded domain. Indeed by the Paley–Wiener theorem, $\mathscr{F}_2 f$ is then an entire function which is uniquely determined by its values in the sector Ω. One is then tempted to analytically continue these values into the missing cone and then to apply a full-angle algorithm for the reconstruction of f. However, one should not overlook the instability problem; indeed, even if the solution is unique it will not in general depend continuously on the data and in such a situation the noise on the data will generate uncontrolled instabilities in the solution. In other words, the distance (measured in the norm of L^2) between two solutions of equation (1) reproducing the data g with a root-mean-square error of at most ε, can be arbitrarily large. Therefore some regularization procedure is needed, which makes explicit or implicit use of further prior knowledge about the solution. For instance in the full-angle case, the convolution algorithm is stabilized in the presence of noise by introducing a filter for the highest frequencies in Fourier space [2]. In the limited-angle case, the instability due to ill-posedness is much more severe [3, 4] so that more care will be needed for the choice of an effective regularization procedure.

2. The Gerchberg–Papoulis algorithm

In order to fill the missing cone with extrapolated data, it has been recently proposed [5, 6, 7] to apply an iterative algorithm initially developed by Gerchberg [8], Papoulis [9], and De Santis and Gori [10] for the extrapolation of band-limited signals in one dimension. This algorithm is based on repeated back and forth Fast Fourier Transform (FFT) between object and frequency space. We briefly describe this iterative procedure. Remember that the unknown function $f(x, y)$ is assumed to have a bounded support X. If we denote by D_X the corresponding domain-limiting operator, i.e. the multiplication by the characteristic function χ_X, we have $D_X f=f$. We can also define a band-limiting operator B_Ω which suppresses those Fourier components of f which are outside the 'band' Ω,

$$\mathscr{F}_2(B_\Omega f)(\underline{v})=\chi_\Omega\hat{f}(\underline{v}),$$

where we denote by $\hat{f}(\underline{v})$ the two-dimensional Fourier transform $\mathscr{F}_2 f$ of f. Similarly we can define the two related operators D_Ω and B_X acting in Fourier space,

$$D_\Omega\hat{f}=\chi_\Omega\hat{f}; \qquad B_X\hat{f}=\mathscr{F}_2\chi_X\mathscr{F}_2^{-1}\hat{f}. \tag{4}$$

As we have already seen, the Fourier components of the object f inside the sector Ω can be immediately retrieved from the data. As shown by equation (3), they are given by $v\cdot\mathscr{F}_2(\bar{R}g)(\underline{v})$. For noiseless data, this function must vanish outside Ω. This is why we take as starting point for the extrapolation of $\hat{f}$ to the missing cone the function

$$\hat{f}_0=D_\Omega[v\cdot\mathscr{F}_2(\bar{R}g)(\underline{v})]. \tag{5}$$

For noisy data, the operation D_Ω amounts to a pre-filtering of the data, which eliminates the noise components outside the sector Ω. This function $\hat{f}_0$ is then Fourier transformed in order to incorporate the prior knowledge of the boundedness of the support X of f (which guarantees the uniqueness of the extrapolation). The values of $\mathscr{F}_2^{-1}(\hat{f}_0)$ outside X are then set to zero by action of the operator D_X. This truncated function is Fourier transformed back into a new function which is no longer zero outside Ω (remember that the Fourier transform of a function having support in X is an entire function and hence cannot vanish on a domain without vanishing everywhere). Note that this function is simply given by $B_X\hat{f}_0$. The new estimate $\hat{f}_1$ will be equal to $B_X\hat{f}_0$ inside the missing cone and to $\hat{f}_0$ in Ω.

The whole process is then repeated again so that the nth iterated estimate for the extrapolation will be given by

$$\hat{f}_n = \hat{f}_0 + \bar{D}_\Omega B_X \hat{f}_{n-1} \tag{6}$$

where $\bar{D}_\Omega = 1 - D_\Omega$ is the operator equivalent to multiplication by the characteristic function of the missing cone. The corresponding estimate f_n for the object function $f(x, y)$ is simply obtained by Fourier transforming $\hat{f}_n$ and applying D_X.

The convergence of this iterative process follows for instance, from Von Neumann's alternating-projection theorem [11]. However, let us emphasize the fact that for noisy data, this theorem does not guarantee the convergence of the process to a meaningful solution. Indeed, because of the ill-posedness of the problem already stressed in § 1, instead of decreasing when the number of iterations n increases, the error on the reconstructed solution may grow indefinitely. A numerical example demonstrating this phenomenon is given in § 4.

In the next section we show how the iterative process can be modified according to regularization theory in order to converge to a solution which is stable with respect to noise. Let us also remark that other procedures for extrapolating data into the missing cone, if not regularized properly beforehand, can be subject to the same instability problems (see [12, 13] for instance).

3. A regularized iterative algorithm

Noise propagation in the iterative process can be controlled when formula (6) is modified into the following form:

$$\hat{\tilde{f}}_n = \hat{\tilde{f}}_0 + (1-\alpha)\bar{D}_\Omega B_X \hat{\tilde{f}}_{n-1} \tag{7}$$

where $\hat{\tilde{f}}_0 = \hat{f}_0$ and α is a small positive parameter called the regularization parameter. This modification has been proposed in [14] for regularizing the one-dimensional Gerchberg–Papoulis algorithm. The derivations and proofs given in that paper can be transposed straightforwardly to the present case and hence will not be reproduced here. Let us just mention that the basic idea is simply to incorporate into the formalism the prior knowledge that a certain functional of the solution is bounded by a prescribed constant E (in our case, this amounts essentially to an upper bound for the L^2-norm $\|f\|$). According to standard techniques in regularization theory [15, 16], this leads to the solving of a variational problem whose solution is no longer a solution of the original ill-posed problem but of a modified equation involving a regularization parameter. Formula (7) is obtained when solving this modified equation in an iterative way.

It is easy to check that the multiplicative factor $(1-\alpha)$ ensures the convergence of the iterative process even in the presence of noise and moreover general theorems ensure that $\hat{\tilde{f}} = \lim \hat{\tilde{f}}_n$ is a regularized solution to the extrapolation problem. This means that if noisy data tend to noise-free data (the root-mean-square error tending to zero), then the root-mean-square error on the corresponding regularized solution will also tend to zero.

Up to now, however, we have only regularized the extrapolation problem without taking care of the ill-posedness of the Radon transform inversion itself, for which noise can also provoke instabilities, even in the full-angle case. This can be understood from equation (3) which shows that Radon transform inversion requires a multiplication by ν in Fourier space. This operation will considerably amplify the noise in the highest frequency components of the object. Therefore, this reconstruction step should also be regularized and this is again done, with the same idea as before, by replacing the multiplication by ν (or division by $1/\nu$) with multiplication by $((1/\nu)+\alpha')^{-1}$. The role of the positive regularization parameter α' is equivalent here to a filter which suppresses the effect of the high-frequency components.

As for the choice of a value for the regularization parameter α (or α') different sophisticated methods have been proposed in the literature; in general, however, they present the drawback of requiring an accurate prior knowledge of the value of the upper bound E or of the root-mean-square error ε of the data (and sometimes of both). Therefore, in numerical practice, it is often easier to choose it empirically. Indeed, as it appeared from simulations, if the parameter α is set roughly to the correct order of magnitude, the regularized solution is almost insensitive to its precise value. Nevertheless, further investigations are currently underway in order to specify an optimal choice of α as a function of the noise level ε of the data and of the angular aperture θ_{max}.

Finally, let us remark that in principle the regularized iterative algorithm presented here could also be used for the reconstruction of a three-dimensional object from its projections along all planes perpendicular to the unit vectors contained in a given sector Ω. It can also be applied to the case where the set of available directions Ω is split into several disconnected angular regions.

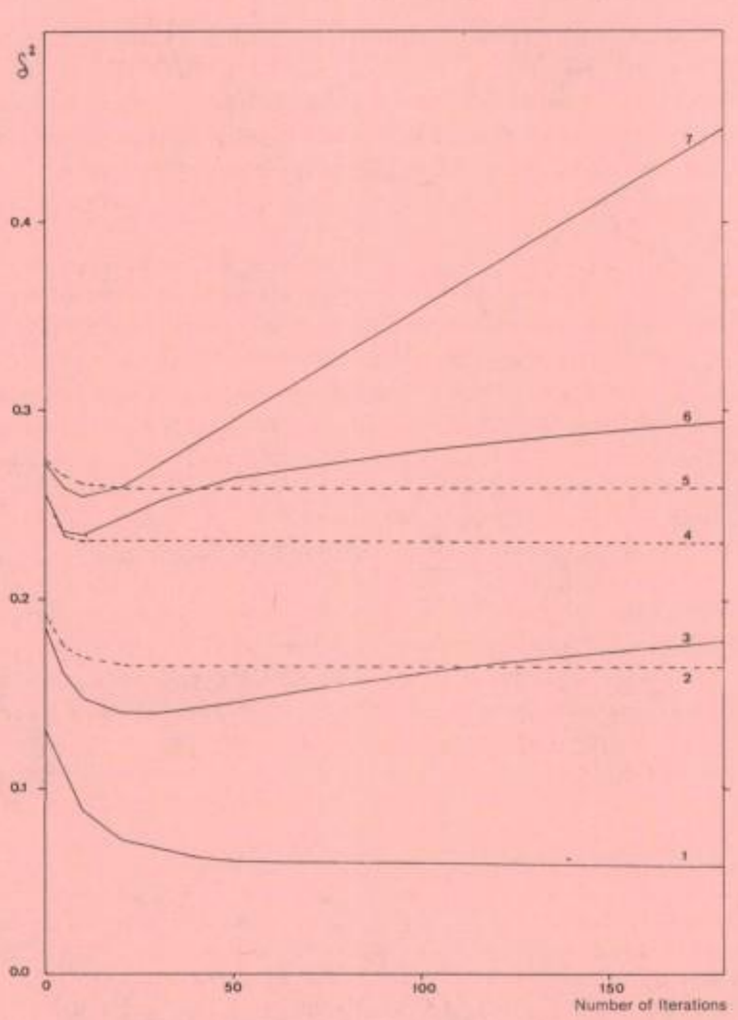
4. Numerical tests

The sensitivity to noise of the Gerchberg–Papoulis algorithm for limited-angle reconstruction and the efficiency of the proposed regularization method have been tested for several known objects. The numerical values presented here below were obtained for an object consisting of a circular 'skull' of radius 3·5 containing three discs. The reconstruction was done on a 64×64 square lattice of spacement 0·2, and the imposed support X coincided with the skull.

The input data consisted of 63 equally spaced projections for each of the 63 angular values which were uniformly distributed in the interval $(-1{\cdot}2\ \text{rad}–+1{\cdot}2\ \text{rad})$. Gaussian noise was added to the computed Radon transform, resulting in a root-mean-square error ε. The inversion algorithms described in §§ 2 and 3 were then applied using a standard two-dimensional Fast Fourier Transform routine. Moreover, the reconstructed object was normalized at each iteration by replacing the zero frequency component of its Fourier transform by a value which can be more accurately evaluated from the sampled Radon transform.

The figure shows the evolution during the successive iterations of the reconstruction error defined as the relative mean-square error,

$$\delta^2 = \frac{\sum_{i=1}^{64}\sum_{j=1}^{64}\{(\text{reconstructed pixel})_{i,j} - (\text{original pixel})_{i,j}\}^2}{\sum_{i=1}^{64}\sum_{j=1}^{64}\{(\text{original pixel})_{i,j}\}^2}. \tag{8}$$



Reconstruction error δ^2 versus number of iterations for $\theta_{max} = 1{\cdot}2$ rad. Dashed lines: regularized, $\alpha = 0{\cdot}1$; full lines: unregularized, $\alpha = 0$. Curve 1: noise-free data; Curves 2 and 3: $\eta = 0{\cdot}20$; $\alpha' = 0{\cdot}36$; with positivity constraint. Curves 4 and 6: $\eta = 0{\cdot}20$; $\alpha' = 0$; with positivity constraint. Curves 5 and 7: $\eta = 0{\cdot}20$; $\alpha' = 0$; without positivity constraint.

(We also tried other definitions of the reconstruction error, and obtained very similar results.)

The upper-most curve 7, obtained when using the unregularized algorithm ($\alpha = \alpha' = 0$), clearly demonstrates its sensitivity to noise: contrary to the noise-free case (see curve 1), the reconstruction error expands when the number of iterations is increased. Of course, the situation is not so bad for lower noise levels; let us remark, however, that large errors on the data are quite plausible for some non-medical applications. On the other hand, when the angular aperture θ_{max} decreases, we verified that the instability increases; as expected, the need for regularization is felt at much lower noise levels for smaller angular apertures.

The dashed curves, obtained with the regularized iterative algorithm ($\alpha \neq 0$), show rapid convergence to a solution for which the reconstruction error stabilizes to a given value when iterating further. In both the $\alpha = 0$ and $\alpha \neq 0$ cases, the additional constraint of positivity and the regularization of the ν-multiplication by taking $\alpha' \neq 0$ significantly improve the value of the reconstruction error δ^2. For $\alpha = 0$, however, they are not sufficient to fully stabilize the unregularized solution. The values of the regularization parameters chosen for this example are of the order of the 'noise-to-signal' ratio $\eta^2 = \varepsilon^2/\|g\|^2$ for the dimensionless parameter α, and of the ratio $\eta^2\|g\|^2/\|f\|^2$ for α' which has the dimension of a length. These orders of magnitude are in agreement with regularization theory [15, 16].

As an alternative to regularization theory, one could think of stopping the iteration (6) before noise gets amplified too much. However, when the object is unknown, it seems difficult, if not impossible, to give a general criterion for finding the optimal number of iterations which minimize the reconstruction error. As shown by the numerical results, noise propagation can be more easily and adequately controlled by regularization which is achieved by a small modification of the Gerchberg–Papoulis algorithm. This modification preserves the flexibility of the iterative procedures; the chief advantage of which is that it allows easy incorporation, at each iteration step, of further prior knowledge about the solution, such as positivity, leading to improved reconstructions of the object function.

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