A relation among DS^2, TS^2 and non-cylindrical ruled surfaces

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Abstract. *TS^2 is a differentiable manifold of dimension 4. For every $X \in TS^2$, if we set $X = (p, x)$ we have $< \vec{p}, \vec{x} >= 0$ since $\vec{p}$ is orthogonal to T_pS^2, therefore $\| \vec{p} \|= 1$. Those there could exist a one-to-one correspondence between TS^2 and DS^2. In this paper we gave and studied a one-to-one correspondence among TS^2, DS^2 and a non cylindrical ruled surface. We showed that for a restriction of an anti-symmetric linear vector field A along a spherical curve $\alpha(t)$ there exists a non-cylindrical ruled surface which corresponds to $\alpha(t)$ and has the following parametrization*

$$\alpha(t, \lambda) = \alpha(\vec{t}) + A(\alpha(t)) + \lambda\alpha(\vec{t})$$

So it is possible to study non-cylindrical ruled surfaces as the set of $(\alpha(t), A(\alpha(t)))$, where $\alpha(t) \in S^2$ and A is an anti-symmetric linear vector field in $\mathcal{R}^3$.

Key words: *dual unit sphere, non-cylindrical ruled surface, spherical curve, anti-symmetric linear vector field, tangent bundle*

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1. Anti-symmetric linear vector fields

Let $A = [a_{ij}]$ be a fixed real $n \times n$ matrix. For each such A we construct a vector field T_A on $\mathcal{R}^n$ by taking its value at each point $x \in \mathcal{R}^n$ to be the negative of the result of applying the matrix A to the vector X, i.e.

$$T_A(X) = -AX \tag{1}$$

Definition 1. *A vector field T_A is called linear vector field ([3]). If A is an anti-symmetric (symmetric, orthogonal, etc.) matrix then T_A is called an anti-symmetric (symmetric, orthogonal, etc.) linear vector field.*

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In this study we use an anti-symmetric linear vector field and S^2 as $\mathcal{R}^n$, because;

Theorem 1. *Let E^3 be a three-dimensional Euclidean vector space with the unit sphere S^2. Let an orthonormal base $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ be given in E^3. Then a linear vector field determines a vector field of tangent vectors on the sphere S^2 if and only if the matrix which is associated with the linear mapping A relative to the base $\{\vec{u_i}\}$ is given by a skew-symmetric matrix ([4]).*

2. Skew mappings

Definition 2. *Let V be a vector space of dimension n. An endomorphism φ of V is called skew if*

$$\varphi^* = -\varphi \quad ,$$

where φ^* denotes the adjoint of φ ([3]).

The above condition is equivalent to the relation

$$< \varphi(X), Y > + < X, \varphi(Y) > \; = \; 0, \quad X, Y \in V \tag{2}$$

It follows from (2) that the matrix of a skew mapping relative to an orthonormal base is skew-symmetric. Substitution of $Y = X$ in (1) yields the equation

$$< X, \varphi(Y) >= 0, \quad X \in V \tag{3}$$

showing that every vector is orthogonal to its image vector. Conversely, an endomorphism φ having this property is skew.

Consider the mapping $\psi = \varphi^2$. For this kind of φ there exists an orthonormal basis $\{\vec{u_i}\}$, $1 \leq i \leq n$, such that

$$\psi(u_i) = \lambda_i u_i, \; i = 1, \cdots, n$$

Furthermore, all eigenvalues λ_i, $1 \leq i \leq n$, are negative or zero. In fact, the equation $\psi(u) = \lambda u$ implies that

$$\lambda =< u, \psi(u) >=< u, \varphi^2(u) > = \; - < \varphi(u), \varphi(u) > \; \leq \; 0$$

Since the rank of φ is even and φ^2 has the same rank as φ, the rank of ψ must be even ([3]). Consequently, the number of negative eigenvalues is even and we can enumerate the vector u_i such that

$$\begin{aligned} \lambda_i &< 0 \;\text{ if } \; i = 1, \cdots, 2p \\ \lambda_i &= 0 \;\text{ if } \; i = 2p+1, \cdots, n \end{aligned}$$

Define the orthonormal basis $e_i, i = 1, \cdots, n$ by

$$e_{2i-1} = u_i,$$

$$e_{2i} = \frac{1}{c_i}\varphi(u_i), \;\; c_i = \sqrt{-\lambda_i}, \; i = 1, \cdots, p$$

and

$$e_i = u_i, \;\; i = 2p+1, \cdots, n.$$

Relative to this basis the matrix of φ has the form

$$\begin{bmatrix} 0 & x_1 & 0 & 0 & \cdots & \cdot & \cdot & \cdot & 0 \\ -x_1 & 0 & 0 & 0 & \cdots & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & x_2 & \cdots & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -x_2 & 0 & \cdots & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 & x_p & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & -x_p & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

3. Tangent bundle TM

Let M be a differentiable manifold of dimension n. The union of all tangent spaces of M is called the tangent bundle of M and is denoted by TM. TM admits a projection $\pi : TM \to M$, defined by

$$\pi(\vartheta) = m \Leftrightarrow \vartheta \in T_m M$$

If x is a chart of M with domain U, any vector $\vartheta \in \pi^{-1}(U)$ can be expressed uniquely as $\sum_i a_i \frac{\partial}{\partial x_i}|_m$ where $a = (a_1, \cdots, a_n) \in \mathcal{R}^n$. Therefore we have an injection

$$(\tilde{\psi\varphi}) : TM \to \mathcal{R}^{2n}$$

defined by $\vartheta \to (x(m), a)$, whose domain is $\pi^{-1}(U)$ and whose range is the open set $\psi(U) \times \mathcal{R}^n$.

For $M = S^2$ we have the tangent bundle TS^2. Furthermore, for every point $p \in S^2$, $\vec{p}$ is orthogonal to the vector space T_pS^2. So we can take $\vec{p}$ for the normal of T_pS^2. This relation gives us the permission to construct a one-to-one correspondence DS^2 and TS^2.

4. The dual unit sphere DS^2

Let $\mathcal{R}$ be the set of real numbers. We have on $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}$, for every

$$X = (x, x^*), Y = (y, y^*) \in \mathcal{R}^2 \quad \text{and} \quad \lambda \in \mathcal{R}$$

$$\begin{aligned} X \oplus Y &= (x + y, x^* + y^*) \\ \lambda . X &= (\lambda x, \lambda x^*) \\ X \odot Y &= (xy, xy^* + x^* y). \end{aligned}$$

The mathematical structure $(\mathcal{R}^2, \oplus, \odot)$ is a ring. The ring is denoted by D and called the ring of dual numbers. Every $X \in D$ is called a dual number. The element $(0, 1)$ has the property

$$(0, 1) \odot (0, 1) = (0, 0)$$

and is denoted by ε. Thus we have $\varepsilon^2 \cong 0$. Therefore by using the notation ε, we can write

$$X = x + \varepsilon x^*$$

for every $X = (x, x^*) \in D$, where $x \cong (x, 0)$, $x^* \cong (0, x^*)$.

Let D^3 be $D \times D \times D$. For every $X, Y \in D^3$ such that $X(a_1, a_2, a_3)$, $Y = (b_1, b_2, b_3)$, $a_i = x_i + \varepsilon x_i^*$, $b_i = y_i + \varepsilon y_i^*$, $i = 1, 2, 3$.

Define

$$X + Y = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \qquad \text{(sum)}$$

$$< X, Y >= \sum_{i=1}^{3} a_i \cdot b_i \qquad \text{(dot product)}.$$

Then we can write

$$< X, Y >=< x, y > +\varepsilon(< x, y^* > + < x^*, y >),$$

where $x = (x_1, x_2, x_3)$, $x^* = (x_1^*, x_2^*, x_3^*)$, $y = (y_1, y_2, y_3)$ and $y^* = (y_1^*, y_2^*, y_3^*)$. So we have the norm of a vector $X \in D^3$ as

$$\| X \|=\| x \| +\varepsilon \frac{< x, x^* >}{\| x \|}$$

For $X \in D^3$ if $\| X \|= (1, 0)$ then X is called a dual unit vector. The set

$$\{X \in D^3 :\| X \|= (1, 0) \in D\}$$

is called the dual unit sphere and is denoted by DS^2. ([5]).

Theorem 2. *There exists a one-to-one correspondence between the oriented lines in $\mathcal{R}^3$ and the points of the dual unit sphere ([5]).*

5. TS^2, DS^2 and non-cylindrical ruled surfaces

Let X be an element of TS^2 where $TS^2 = \cup_p T_p S^2$. Then $X = (x_1, x_2, x_3, x_1^*, x_2^*, x_3^*)$.

Thus if we set $x = (x_1, x_2, x_3)$, $x^* = (x_1^*, x_2^*, x_3^*)$ then it is clear that

$$\begin{aligned} \| x \| &= 1 \\ < x, x^* > &= 0 \end{aligned}$$

So we can write $X = (x, x^*) \in DS^2$, isomorphically. Conversely, for every $X = (x, x^*) \in DS^2$ we have

$$\| x \|= 1, < x, x^* >= 0.$$

So $X = (x_1, x_2, x_3, x_1^*, x_2^*, x_3^*) \in TS^2$. Thus we have the following :

Theorem 3. *There is a one-to-one correspondence TS^2 and DS^2.*

We know that every curve on DS^2 can be associated to a ruled surface in $\mathcal{R}^3$ ([1]). Now we will ask how a curve on TS^2 can be associated to a ruled surface in $\mathcal{R}^3$ and answer the question.

Let $P = (p, p^*) \in TS^2$, then p is orthogonal p^*. It is well known from vector algebra that the equation

$$p \times x = p^*, p, x, p^* \in \mathcal{R}^3$$

with $< p, p^* >= 0$ has the set of solutions

$$x(\lambda) = -\frac{1}{\| p \|^2} p \times p^* + \lambda p, \qquad \lambda \in \mathcal{R}.$$

The solution $x(\lambda)$ represents a straight line in the direction of the vector $\vec{p}$. Since $\| p \| = 1$, so

$$x(\lambda) = -p \times p^* + \lambda p \ .$$

Let α be a curve on S^2 such that $\alpha : I \subseteq \mathcal{R} \longrightarrow S^2$, $t \longrightarrow \alpha(t)$ and A be an antisymmetric vector field. The restriction of A on $\alpha(I)$ will be denoted by A_α, $A_\alpha = A_\alpha(\alpha(I))$. For every $t_0 \in I$. We have the straight line

$$x_{t_0}(\lambda) = \alpha(t_0) \times A_\alpha(\alpha(t_0)) + \lambda\alpha(t_0).$$

So the equation

$$x_t(\lambda) = \alpha(t) \times A_\alpha(\alpha(t)) + \lambda\alpha(t), \qquad t \in I, \lambda \in \mathcal{R}$$

describes a surface. We set

$$\varphi(t, \lambda) = \alpha(t) \times A_\alpha(\alpha(t)) + \lambda\alpha(t), \quad t \in I, \lambda \in \mathcal{R}. \tag{5}$$

Equation (5) defines a non-cylindrical ruled surface.

Conversely, let a non-cylindrical ruled surface in $\mathcal{R}^3$ be given by the equation

$$\sigma(u, \vartheta) = \beta(u) + \vartheta d(u).$$

The spherical representation of the unit direction vectors $d(u)$ describes a curve on S^2.

Suppose that this curve is denoted by α, $\alpha : I \longrightarrow S^2$, we can define a mapping A along the curve α by the following equation,

$$A(\alpha(u)) = -\alpha(u) \times \beta(u),$$

where the sign $\times$ denotes the wedge product in $\mathcal{R}^3$. It is clear that $A(\alpha(u))$ is an anti-symmetric vector field. Therefore we have

$$\begin{aligned} \| \vec{\alpha}(u) \| &= 1\,, \\ < \vec{\alpha}(u), A(\alpha(u)) > &= 0 \end{aligned}$$

and so $< \vec{\alpha}(u), A(\alpha(u)) > \ \in DS^2$. That is to say $< \vec{\alpha}(u), A(\alpha(u)) >$ is an element of TS^2. So we have

Theorem 4. *There exists a one-to-one correspondence between a restriction of an anti-symmetric vector field along a spherical curve and a non-cylindrical ruled surface in* $\mathcal{R}^3$.

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