A RELATION BETWEEN KILLING TENSOR FIELDS AND NEGATIVE PINCHED RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let M be a compact orientable Riemannian manifold. Let M_P be the tangent space of the manifold M at the point P. We denote by $\langle X, Y\rangle$ and $\|X\|$ the scalar product of two vectors $X, Y \in M_P$ and the norm of the vector X, respectively, where the scalar product on the tangent space M_P is induced by the Riemannian metric of the manifold M.

If X, Y are two vectors of the tangent space M_P, then the curvature tensor field R of the manifold M and the two vectors X, Y induce an endomorphism $R(X, Y)$of M_P into M_P. If X, Y, Z, T are four vectors of M_P, then the Riemannian curvature tensor R_1 at P can be considered as a quadrilinear mapping R_1: $M_P \times M_P \times M_P \times M_P \to \mathbf{R}$ which is defined by R_1: $(X, Y, Z, T) \to \langle R(X, Y)Z, T\rangle$. Let λ be a plane of the tangent space M_P which is spanned by two linearly independent vectors $X, Y \in M_P$. The sectional curvature of the plane λ is given by

$$\sigma(\lambda) = \sigma(X, Y) = -\frac{\langle R(X, Y)X, Y\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}.$$

We assume that the Riemannian manifold is compact orientable and negative δ-pinched, that means its sectional curvature $\sigma(\lambda)$ satisfies the inequalities

$$-\Lambda \leqq \sigma(\lambda) \leqq -\Lambda\delta,$$

for every $\lambda \in M_P$ and $\forall P \in M$.

We can normalize the metric on the manifold M such that the above inequalities become

$$-1 \leqq \sigma(\lambda) \leqq -\delta.$$

A Riemannian manifold M, whose sectional curvature satisfies the above inequalities, is called negative δ-pinched.

Now, our results can be stated as follows: Let M be a compact orientable negative δ-pinched manifold. If the dimension of M is even, $n=2m$ (resp. odd, $n=2m+1$) and $\delta > 1/4$ (resp. $\delta > 2(m-1)/(8m-5)$), then there exists neither Killing tensor field of order 2 nor conformal Killing tensor field of order 2.

2. Let M be a compact orientable negative δ-pinched manifold.

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We obtain a point P of the manifold M and consider a normal coordinate system on M with origin the point P.

We also consider an orthonormal basis $\{X_1, \cdots, X_n\}$ of the tangent space M_P. If $\{X_i, X_j, X_k, X_l\}$ is a set of four vectors of the orthonormal basis $\{X_1, \cdots, X_n\}$, then the following formulas hold:

$$(2.1) \qquad \langle R(X_i, X_j)X_k, X_l\rangle = R_{ijkl}, \qquad \sigma(X_i, X_j) = \sigma_{ij} = -R_{ijij},$$

where R_{ijkl} are the components of the Riemannian curvature.

If we apply the same technique as in [**1**, pp. 67–69], we obtain the following inequalities:

$$(2.2) \quad |R_{ijik}| \leqq (1-\delta)/2, \quad |R_{ijkl}| \leqq 2(1-\delta)/3, \quad i \neq j \neq k \neq l,$$

because all the computations made in [**1**] for the sectional curvature ranging $(\frac{1}{4}, 1]$ if $n=2m$, or $(2(m-1)/(9m-5), 1]$ if $n=2m+1$ are valid without any change of the curvature ranging in any interval $[a, b]$.

3. Let $\xi=\{\xi(X_i, X_j)=\xi_{ij}\}$ be an exterior 2-form. This exterior 2-form is called a Killing 2-form if it satisfies the relation:

$$\nabla_X\xi(Y, Z) = -\nabla_Y\xi(X, Z), \qquad \forall X, Y, Z \in T(M).$$

We consider the quadratic form $F(\xi)$ given by [**2**, p. 62]:

$$F(\xi) = \sum_{iji_2} R_{ij}\xi^{ii_2}\xi^{j}_{i_2} + \frac{1}{2}\sum_{ijkl} R_{ijkl}\xi^{ij}\xi^{kl}.$$

If the manifold is even dimension $n=2m$ (resp. odd dimension $n=2m+1$) and $\delta>1/4$ (resp. $\delta>2(m-1)/(8m-5)$), then by means of the second of (2.1) and (2.2) with the same method as in [**1**, p. 70], we have $F(\xi)<0$. It is well known [**2**, p. 67], if $F(\xi)<0$, it implies $\xi=0$.

Therefore, we can state the following theorem.

THEOREM I. *Let M be a compact orientable negative δ-pinched manifold. If the dimension of the manifold is even $n=2m$ (resp. odd $n=2m+1$) and $\delta>1/4$ (resp. $\delta>2(m-1)/(8m-5)$), then there exists no Killing tensor field of order 2 on the manifold M.*

4. Let $n=\{n(X_i, X_j)=n_{i_1 n_2}\}$ be an exterior 2-form on the manifold M. This exterior 2-form is called a conformal Killing 2-form, if it satisfies the following conditions [**2**, p. 73]:

$$\nabla_j n_{ii_2} + \nabla_i n_{ji_2} = 2\beta_{i_2}g_{ij}, \qquad \beta_{i_2} = g^{ij}\nabla_j n_{ii_2}/n.$$

We can prove with the same technique as in §3. If the manifold M

is compact orientable and negative δ-pinched, the dimension of the manifold is even $n=2m$, (resp. odd $n=2m+1$), and $\delta>1/4$ (resp. $\delta>2(m-1)/(8m-5)$), then the quadratic form $F(n)$ is negative definite.

From the above and the known theorem [2, p. 73], we obtain the following theorem:

THEOREM II. *Let M be a compact orientable negative δ-pinched Riemannian manifold. If the dimension of the manifold is even $n=2m$ (resp. odd $n=2m+1$) and $\delta>1/4$ (resp. $\delta>2(m-1)/(8m-5)$), then there exists no conformal Killing tensor field of order 2 on the manifold.*

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