PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 3, July 1983

A RELATION BETWEEN POINTWISE CONVERGENCE OF FUNCTIONS AND CONVERGENCE OF FUNCTIONALS

HAÏM BREZIS AND ELLIOTT LIEB[1]

ABSTRACT. We show that if $\{f_n\}$ is a sequence of uniformly L^p-bounded functions on a measure space, and if $f_n \to f$ pointwise a.e., then $\lim_{n\to\infty}\{\|f_n\|_p^p - \|f_n - f\|_p^p\} = \|f\|_p^p$ for all $0 < p < \infty$. This result is also generalized in Theorem 2 to some functionals other than the L^p norm, namely $\int |j(f_n) - j(f_n - f) - j(f)| \to 0$ for suitable $j\colon \mathbf{C} \to \mathbf{C}$ and a suitable sequence $\{f_n\}$. A brief discussion is given of the usefulness of this result in variational problems.

1. Introduction. Let (Ω, Σ, μ) be a measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex valued measurable functions which are uniformly bounded in $L^p = L^p(\Omega, \Sigma, \mu)$ for some $0 < p < \infty$. Suppose that $f_n \to f$ pointwise almost everywhere (a.e.). What can be said about $\|f\|_p$?

The simplest tool for estimating $\|f\|_p$ is Fatou's lemma, which yields

$$\|f\|_p \leqslant \liminf_{n\to\infty} \|f_n\|_p.$$

The purpose of this note is to point out that much more can be said, namely

$$\lim_{n\to\infty}\left\{\|f_n\|_p^p - \|f_n - f\|_p^p\right\} = \|f\|_p^p. \tag{1}$$

More generally, if $j\colon \mathbf{C} \to \mathbf{C}$ is a continuous function such that $j(0) = 0$, then, when $f_n \to f$ a.e. and $\int |j(f_n(x))|\, d\mu(x) \leqslant C < \infty$, it follows that

$$\lim_{n\to\infty} \int [j(f_n) - j(f_n - f)] = \int j(f) \tag{2}$$

under suitable conditions on j and/or $\{f_n\}$.

Heuristically, (2) says the following. If we write $f_n = f + g_n$ with $g_n \to 0$ a.e., then, for large n, $\int j(f + g_n)$ decouples into two parts, namely $\int j(f)$ and $\int j(g_n)$.

Equation (1) is not merely an idle exercise, but it is actually useful in the calculus of variations to prove the existence of maximizing (resp. minimizing) functions in some cases in which compactness is not available. In fact (1) was first used by one of us (E. Lieb), but with a different notion of convergence than pointwise convergence of $f_n \to f$, to solve a variational problem **[1]**. Later, it was also used in another variational problem **[2]**. At the end of this note we shall give a brief account of how (1) can be used.

Received by the editors August 9, 1982 and, in revised form, November 17, 1982.
1980 *Mathematics Subject Classification*. Primary 28A20, 35J60, 46E30.
Key words and phrases. Convergence of functionals, pointwise convergence, L^p spaces.
[1]Work partially supported by U.S. National Science Foundation Grant PHY-8116101.

0002-9939/82/0000-1190/02.25

Two theorems will be stated: (i) the L^p case $(0 < p < \infty)$, (ii) the general case (2). Although (i) is a corollary of (ii) we state it separately because it is an important special case and because the assumptions are especially transparent.

E. Lieb is most grateful to the Institute for Advanced Study for its support and hospitality. Both authors thank the Summer Research Institute for bringing them together in Melbourne, Australia, where this note had its origin.

2. The L^p case $(0 < p < \infty)$.

THEOREM 1. *Suppose $f_n \to f$ a.e. and $\| f_n \|_p \leq C < \infty$ for all n and for some $0 < p < \infty$. Then the limit in* (1) *exists and the equality in* (1) *holds.*

REMARKS. (i) By Fatou's lemma, $f \in L^p$.

(ii) In case $0 < p \leq 1$, and if we assume that $f \in L^p$, then we do not need the hypothesis that $\| f_n \|_p$ is uniformly bounded. [This follows from the inequality $\big| |f_n|^p - |f_n - f|^p \big| \leq |f|^p$ and the dominated convergence theorem.] However, when $1 < p < \infty$, the hypothesis that $\| f_n \|_p$ is uniformly bounded is really necessary (even if we assume that $f \in L^p$) as a simple counterexample shows.

(iii) When $1 < p < \infty$, the hypotheses of Theorem 1 imply that $f_n \rightharpoonup f$ weakly in L^p. [By the Banach-Alaoglu theorem, for some subsequence, f_{n_k} converges weakly to some g; but $g = f$ since $f_n \to f$ a.e.] However, weak convergence in L^p is insufficient to conclude that (1) holds, except in the case $p = 2$. When $p \neq 2$ it is easy to construct counterexamples to (1) under the assumption only of weak convergence. When $p = 2$ the proof of (1) is trivial under the assumption of weak convergence.

3. The general case. In order to prove (2), some conditions are needed on the function j and the sequence $\{f_n\}$. To make this point clear we shall later give an example for which (2) fails. On the other hand, we shall not attempt to find the most general conditions for which (2) holds but shall, instead, content ourselves here with conditions which are reasonably simple, yet general enough to cover many examples.

Let j: $\mathbf{C} \to \mathbf{C}$ be a continuous function with $j(0) = 0$. In addition let j satisfy the following hypothesis:

For every sufficiently small $\varepsilon > 0$ there exist two continuous, nonnegative functions φ_ε and ψ_ε such that

$$|j(a + b) - j(a)| \leq \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b) \tag{3}$$

for all $a, b \in \mathbf{C}$.

THEOREM 2. *Let j satisfy the above hypothesis and let $f_n = f + g_n$ be a sequence of measurable functions from Ω to $\mathbf{C}$ such that*:

(i) $g_n \to 0$ *a.e.*

(ii) $j(f) \in L^1$.

(iii) $\int \varphi_\varepsilon(g_n(x))\, d\mu(x) \leq C < \infty$, *for some constant C, independent of ε and n.*

(iv) $\int \psi_\varepsilon(f(x))\, d\mu(x) < \infty$ *for all $\varepsilon > 0$.*

Then, as $n \to \infty$,

$$\int |j(f + g_n) - j(g_n) - j(f)| d\mu \to 0. \tag{4}$$

REMARKS. (i) It is not assumed that $j(f_n)$ or $j(g_n)$ are separately in L^1.

(ii) Note that the convergence in (4) is in the strong L^1 topology. This is a stronger statement than (2).

PROOF OF THEOREM 2. Fix $\varepsilon > 0$ and let

$$W_{\varepsilon,n}(x) = \left[|j(f_n(x)) - j(g_n(x)) - j(f(x))| - \varepsilon\varphi_\varepsilon(g_n(x))\right]_+,$$

where $[a]_+ = \max(a, 0)$. As $n \to \infty$, $W_{\varepsilon,n}(x) \to 0$ a.e. On the other hand,

$$\begin{aligned}|j(f_n) - f(g_n) - j(f)| &\leq |j(f_n) - j(g_n)| + |j(f)| \\ &\leq \varepsilon\varphi_\varepsilon(g_n) + \psi_\varepsilon(f) + |j(f)|.\end{aligned}$$

Therefore, $W_{\varepsilon,n} \leq \psi_\varepsilon(f) + |j(f)| \in L^1$. By dominated convergence, $\int W_{\varepsilon,n}\, d\mu \to 0$ as $n \to \infty$. However,

$$|j(f_n) - j(g_n) - j(f)| \leq W_{\varepsilon,n} + \varepsilon\varphi_\varepsilon(g_n)$$

and, thus,

$$I_n \equiv \int |j(f_n) - j(g_n) - j(f)|\, d\mu \leq \int [W_{\varepsilon,n} + \varepsilon\varphi_\varepsilon(g_n)]\, d\mu.$$

Consequently, $\limsup_{n\to\infty} I_n \leq \varepsilon C$. Now let $\varepsilon \to 0$. □

EXAMPLES. (a) $j(t) = |t|^p$, $0 < p < \infty$. Here (3) is satisfied with $\varphi_\varepsilon(t) = |t|^p$ and $\psi_\varepsilon(t) = C_\varepsilon |t|^p$ for some C_ε sufficiently large. Therefore hypotheses (ii)—(iv) are simply that $f \in L^p$ and the $\{g_n\}$ are uniformly bounded in L^p. This proves Theorem 1.

(b) Suppose that j is a continuous, convex function from $\mathbf{C}$ to $\mathbf{R}$ with $j(0) = 0$. Choose some number $k > 1$. Then (3) holds for $\varepsilon k < 1$ with

$$\varphi_\varepsilon(t) = j(kt) - kj(t) \quad \text{and} \quad \psi_\varepsilon(t) = |j(C_\varepsilon t)| + |j(-C_\varepsilon t)|,$$

with $1/C_\varepsilon = \varepsilon(k-1)$. This is proved in Lemma 3 below. Therefore, the hypotheses of Theorem 2 are satisfied if there is some fixed $k > 1$ such that $[j(kg_n) - kj(g_n)]$ is uniformly bounded in L^1, and if $j(Mf)$ is in L^1 for every real M.

(c) The condition in example (b) that $j(kg_n) - kj(g_n)$ is uniformly bounded in L^1 for some constant $k > 1$ can be essential, not only for the hypotheses of Theorem 2 but for the conclusion as well. Let $\Omega = [0, 1]$, $j(t) = e^{|t|} - 1$, $d\mu = dx$, $f(x) = 1$, $g_n(x) = \ln(1 + n)$ if $0 < x < 1/n$, and $g_n(x) = 0$ otherwise. Then $\int j(f_n) = 2e - 1$, $\int j(g_n) = 1$ and $\int j(f) = e - 1$. In this example we see that (2) does *not* hold even though $j(g_n)$ is uniformly bounded in L^1 and $j(Mf) \in L^1$ for all real M. Note that for this sequence $\{g_n\}$, $j(kg_n)$ is *not* uniformly bounded when $k > 1$. However since $j(t)$ is convex, (b) above tells us that the conclusion of Theorem 2 would be valid for any other sequence $\tilde{g}_n$ such that $j(k\tilde{g}_n)$ is uniformly bounded in L^1 for some $k > 1$.

LEMMA 3. *Let* $j\colon \mathbf{C} \to \mathbf{R}$ *be convex and let* $k > 1$. *Then*

$$|j(a + b) - j(a)| \leq \varepsilon[j(ka) - kj(a)] + |j(C_\varepsilon b)| + |j(-C_\varepsilon b)|$$

for all $a, b \in \mathbf{C}$, $0 < \varepsilon < 1/k$ *and* $1/C_\varepsilon = \varepsilon(k-1)$.

PROOF. Let $\alpha = 1 - k\varepsilon$, $\beta = \varepsilon$, $\gamma = (k-1)\varepsilon$. Then $\alpha + \beta + \gamma = 1$ and $(a+b) = \alpha a + \beta(ka) + \gamma(C_\varepsilon b)$. By convexity,

$$j(a+b) \leqslant \alpha j(a) + \beta j(ka) + \gamma j(C_\varepsilon b).$$

This implies that

$$j(a+b) - j(a) \leqslant \varepsilon[j(ka) - kj(a)] + |j(C_\varepsilon b)|.$$

For the reverse inequality let

$$\alpha = 1/(1+k\varepsilon), \quad \beta = \varepsilon/(1+k\varepsilon), \quad \gamma = \varepsilon(k-1)/(1+k\varepsilon),$$

whence $a = \alpha(a+b) + \beta(ka) + \gamma(-C_\varepsilon b)$. Then

$$j(a) - j(a+b) \leqslant \varepsilon[j(ka) - kj(a)] + \varepsilon(k-1)j(-C_\varepsilon b). \quad \square$$

4. Applications. In the calculus of variations an oft-met problem is to show that an infimum or supremum is achieved. We shall outline by two examples how Theorem 1 can be used for this purpose.

(A) If K is the sharp constant in the inequality $\|Af\|_q \leqslant K\|f\|_p$, where A is a bounded linear operator from L^p to L^q, can one find f such that equality holds? We shall assume that $\infty > q \geqslant p \geqslant 1$. In fact, the problem in **[1]** that motivated Theorem 1 was the Hardy-Littlewood-Sobolev inequality on $L^p(\mathbf{R}^n, dx)$. Namely, A is the integral kernel $A(x,y) = |x-y|^{-\lambda}$, $0 < \lambda < n$ and $p^{-1} + \lambda/n = 1 + q^{-1}$. Let $K = \sup\{R(f) \mid f \in L^p,\ f \neq 0\}$, where $R(f) = \|Af\|_q / \|f\|_p$. The problem we address here is to prove the existence of a maximizing f, i.e. $R(f) = K$. Suppose that an L^p-bounded sequence $\{f_n\}$ can be found such that (i) $R(f_n) \to K$, (ii) $f_n \to f$ a.e., (iii) $f \neq 0$. (For the H.L.S. inequality, this can be done by using a rearrangement inequality.) The difficulty that one faces is to show $R(f) = K$. This difficulty can be overcome by Theorem 1 if we make the additional assumption that $Af_n \to Af$ a.e. (This can also be verified for the H.L.S. problem.) With these assumptions we have that

$$K^p = \lim_{n\to\infty} \frac{\|Af_n\|_q^p}{\|f_n\|_p^p} = \lim_{n\to\infty} \frac{\{\|Af\|_q^q + \|Ag_n\|_q^q\}^{p/q}}{\{\|f\|_p^p + \|g_n\|_p^p\}}$$

with $f_n = f + g_n$ as before. Since $p/q \leqslant 1$ and $(a+b)^t \leqslant a^t + b^t$ for $a, b \geqslant 0$ and $t \leqslant 1$, and since $\|Ag_n\|_q \leqslant K\|g_n\|_p$ (by definition), it follows that $K^p \leqslant \|Af\|_p^p / \|f\|_p^p$. Thus f is maximizing, as desired. For further details see **[1]**.

(B) This is taken from **[2]**. Let $\Omega \subset \mathbf{R}^n$, $n \geqslant 3$, be a bounded domain. Let $\lambda \geqslant 0$ and let

$$R_\lambda(f) = \frac{\int|\nabla f|^2 - \lambda\int|f|^2}{\|f\|_p^2} \quad \text{with } p = \frac{2n}{n-2}.$$

The problem is to show that $K_\lambda = \inf\{R_\lambda(f) \mid f \in H_0^1(\Omega), f \neq 0\}$ is achieved.

Suppose that we know that $K_\lambda < K_0$ (this is indeed the case for every $\lambda > 0$ when $n \geqslant 4$, and for λ sufficiently large when $n = 3$; see **[2]**); then K_λ is achieved.

To prove this, let $\{f_n\}$ be a minimizing sequence with $\|f_n\|_p = 1$. Since f_n is bounded in $H^1(\Omega)$ we may assume that $f_n \rightharpoonup f$ weakly in H^1, $f_n \to f$ strongly in L^2 and

$f_n \to f$ a.e. We have

$$\int |\nabla f_n|^2 - \lambda \int |f_n|^2 = K_\lambda + o(1),$$

and since $\int |\nabla f_n|^2 \geqslant K_0 \| f_n \|_p^2 = K_0$ (by definition of K_0), it follows that $\lambda \int |f|^2 \geqslant K_0 - K_\lambda > 0$. Therefore $f \neq 0$. On the other hand, let $g_n = f_n - f$. We have

$$\int |\nabla f_n|^2 - \lambda \int |f_n|^2 = K_\lambda \|f_n\|_p^2 + o(1),$$

and since $g_n \rightharpoonup 0$ weakly in H^1, we obtain

$$\int |\nabla f|^2 + \int |\nabla g_n|^2 - \lambda \int |f|^2 = K_\lambda \|f_n\|_p^2 + o(1).$$

Consequently,

$$\int |\nabla f|^2 + K_0 \|g_n\|_p^2 - \lambda \int |f|^2 \leqslant K_\lambda \|f_n\|_p^2 + o(1).$$

On the other hand, it follows from Theorem 1 that

$$\|f_n\|_p^p = \|f\|_p^p + \|g_n\|_p^p + o(1).$$

Since $p \geqslant 2$ we deduce that

$$\|f_n\|_p^2 \leqslant \|f\|_p^2 + \|g_n\|_p^2 + o(1).$$

If $K_\lambda \geqslant 0$, we conclude that

$$K_\lambda \|f_n\|_p^2 \leqslant K_\lambda \|f\|_p^2 + K_0 \|g_n\|_p^2 + o(1)$$

and, therefore,

$$\int |\nabla f|^2 - \lambda \int |f|^2 \leqslant K_\lambda \|f\|_p^2 + o(1),$$

i.e. f is minimizing, as desired.

If $K_\lambda < 0$, we have

$$\int |\nabla f|^2 - \lambda \int |f|^2 \leqslant K_\lambda + o(1) \leqslant K_\lambda \|f\|_p^2 + o(1)$$

since $\| f \|_p \leqslant 1$. Here again, f is minimizing, as desired. For further details see [**2**].

References

1. E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (to appear).

2. H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. (to appear).

Département de Mathématiques, Université Paris VI, 4, pl. Jussieu 75230 Paris Cedex 05, France

The Institute for Advanced Study, Princeton, New Jersey 08540

Permanent address: Departments of Mathematics and Physics, Princeton University, Jadwin Hall, P.O.B. 708, Princeton, New Jersey 08544