

## A RELATION BETWEEN THE TOTAL CURVATURE AND THE MEASURE OF RAYS, II

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**1. Introduction.** The total curvature of complete, noncompact, connected and oriented Riemannian 2-manifold  $M$  is defined to be an improper integral  $\int_M G dv$  of the Gaussian curvature  $G$  of  $M$  with respect to the Riemannian volume  $dv$  over  $M$ . It is well known that the total curvature of such an  $M$  is not a topological invariant but it depends on the choice of the Riemannian structure. The pioneering work of Cohn-Vossen on the total curvature states that if  $M$  is finitely connected and if  $M$  admits the total curvature, then  $\int_M G dv \leq 2\pi\chi(M)$ , where  $\chi(M)$  is the Euler characteristic of  $M$  (see [2, Satz 6]). It is interesting to investigate a geometric influence of the total curvature on the Riemannian structure of  $M$  which defines it. The first attempt in this direction of the work was made by Maeda in [5], [6] and [7]. He investigated some relations between the measure of rays emanating from a fixed point and the total curvature of a complete Riemannian manifold homeomorphic to  $\mathbf{R}^2$  whose Gaussian curvature is nonnegative everywhere.

From completeness and compactness of  $M$  it follows that through every point  $p$  on  $M$  there passes at least a ray  $\gamma: [0, \infty) \rightarrow M$ , where a ray is by definition a unit speed geodesic such that any subarc of it is a unique minimizing geodesic between the endpoints. Here all geodesics are parametrized by arc length unless otherwise mentioned. For a point  $p$  on  $M$  let  $T_pM$  and  $S_pM$  be the tangent space to  $M$  at  $p$  and the unit circle of  $T_pM$  centered at the origin.  $S_pM$  is endowed with a natural Lebesgue measure induced by the Riemannian structure of  $M$ . Let  $A(p)$  be the set of all unit vectors tangent to rays emanating from  $p$ .  $A(p)$  is closed because a limit geodesic of a sequence of rays is again a ray. Thus we are interested in the measure of the set  $A(p)$ . In a recent paper, Maeda has proved the following:

**THEOREM (Maeda [7]).** *If  $M$  is a complete Riemannian manifold homeomorphic to  $\mathbf{R}^2$  and if the Gaussian curvature  $G$  of  $M$  is nonnegative everywhere, then*

$$\int_M G dv = 2\pi - \inf_{p \in M} \text{meas}(A(p)) .$$

The purpose of the present note is to consider whether the above result is true for complete metrics on  $R^2$  on which  $G$  changes sign. We shall show that the equality in the above theorem does not hold in general where  $G$  changes sign. We shall furnish an example of a complete surface  $\Sigma$  homeomorphic to  $R^2$  embedded in the Euclidean 3-space  $E^3$  on which the equality in the above theorem does not hold. In fact we have

$$2\pi - \int_{\Sigma} G^+ dv < \inf_{p \in \Sigma} \text{meas}(A(p)) < 2\pi - \int_{\Sigma} G dv ,$$

where  $G^+(x) := \text{Max}\{G(x), 0\}$ ,  $x \in \Sigma$ . In general we have the following for a complete Riemannian manifold  $M$  homeomorphic to  $R^2$ :

**THEOREM 1.** *Let  $M$  be a complete Riemannian manifold homeomorphic to  $R^2$ . If  $M$  admits the total curvature, then*

$$2\pi - \int_M G^+ dv \leq \inf_{p \in M} \text{meas}(A(p)) \leq 2\pi - \int_M G dv .$$

The infimum of  $\text{meas}(A(p))$  is attained when  $G$  has compact support. However it is not certain whether the infimum is attained if the support of  $G$  is noncompact. But in the special case where  $G > 0$  everywhere, we have the following as a direct consequence of the above Theorem 1:

**THEOREM 2.** *Let  $M$  be a complete Riemannian manifold whose Gaussian curvature is positive everywhere. If the infimum of  $\text{meas}(A(p))$  is attained at some point on  $M$ , then the total curvature of  $M$  is equal to  $2\pi$ .*

However the author does not know if the converse of Theorem 2 is true or not. Other geometric significance of the total curvature has been investigated by Innami [4] and Shiohama [9], [10]. Basic tools used in the proofs of our results will be given in §2 and the proofs are stated in §3. The example stated above will be furnished in §3.

**2. Preliminaries.** Let  $M$  be a connected, complete and noncompact Riemannian 2-manifold without boundary. The total curvature of  $M$  is defined as follows.

**DEFINITION 1.**  $M$  admits the total curvature if and only if for every monotone increasing sequence of compact domains  $\{V_j\}$  of  $M$  such that  $\bigcup_{j \geq 1} V_j = M$ , the sequence  $\left\{ \int_{V_j} G dv \right\}$  has a limit in  $[-\infty, \infty]$ .

It turns out that the limit of  $\left\{ \int_{V_j} G dv \right\}$  does not depend on the choice of  $\{V_j\}$  if the total curvature of  $M$  exists.

REMARK.  $M$  admits the total curvature if and only if either  $\int_M G^+ dv < \infty$  or  $\int_M G^- dv > -\infty$  is fulfilled, where  $G^- = G - G^+$ . If  $M$  does not admit the total curvature, then for any given real number  $A$  there exists a monotone increasing sequence  $\{V_j\}$  of compact domains in  $M$  such that  $\bigcup_{j \geq 1} V_j = M$  and that  $\lim_{j \rightarrow \infty} \int_{V_j} G dv = A$  (see [2, the footnote on p. 79]).

A geodesic  $\gamma: R \rightarrow M$  is by definition a straight line if any subarc of  $\gamma$  is minimizing. The following theorem was proved by Cohn-Vossen which plays an essential role for the proofs of our results.

THEOREM A ([3, Satz 6]). *Let  $M$  be a complete Riemannian manifold homeomorphic to  $R^2$ . If  $M$  admits the total curvature and if there exists a straight line  $\gamma: R \rightarrow M$ , then*

$$\int_M G dv \leq 0.$$

The proof of Theorem A is based upon the following Lemma B which Cohn-Vossen discussed in [3, §5], and is valid in all dimensions.

LEMMA B. *Let  $N$  be an  $n$ -dimensional ( $n \geq 2$ ) complete noncompact Riemannian manifold and let  $\gamma: [0, \infty) \rightarrow N$  be a ray and let  $x$  be a fixed point on  $N$ . For any positive  $\varepsilon$ , there exist a divergent sequence  $\{t_j\}$  and minimizing geodesics  $\sigma_j: [0, l_j] \rightarrow N$  such that  $\sigma_j(0) = x$ ,  $\sigma_j(l_j) = \gamma(t_j)$ ,  $j = 1, \dots$ , and they satisfy  $\sphericalangle(\dot{\sigma}_j(l_j), \dot{\gamma}(t_j)) < \varepsilon$  for all  $j = 1, \dots$ .*

PROOF OF LEMMA B. Set  $f(t) := d(x, \gamma(t))$ , where  $d$  is the distance function on  $N$  induced by the Riemannian metric.  $f$  is Lipschitz continuous with Lipschitz constant 1, and hence it is differentiable almost everywhere.  $f$  is differentiable at  $t_0 > 0$  if and only if every minimizing geodesic joining  $x$  to  $\gamma(t_0)$  makes a constant angle with  $\gamma$  at  $\gamma(t_0)$ . It then turns out that the constant is equal to  $\cos^{-1}(f'(t_0))$ .  $f$  is non-differentiable at  $t_0 > 0$  if and only if there are two distinct minimizing geodesics joining  $x$  to  $\gamma(t_0)$  such that their angles with  $\gamma$  at  $\gamma(t_0)$  are not equal. Thus setting  $\theta(t) := \cos^{-1}(f'(t))$  where it is defined, we have  $t - f(t) = \int_0^t [1 - \cos \theta(u)] du - f(0)$ . It follows from the triangle inequality that  $t - f(t) \leq d(x, \gamma(0))$  for all  $t \geq 0$ . Therefore the integrand of the above equality is bounded above for all  $t > 0$ . Thus  $\lim_{u \rightarrow \infty} \inf [1 - \cos \theta(u)] = 0$ , and the proof is complete.

In the proof of Theorem A and later in §3, Lemma B is used on a closed unbounded domain  $D$  of  $M$  which is bounded by a geodesic polygon. The distance function  $\hat{d}(x, y)$  on  $D$  is defined to be the infimum of lengths of all curves in  $D$  joining  $x$  and  $y$ . Every two points on  $D$  can be joined by a  $\hat{d}$ -minimizing segment in  $D$  whose length realizes the distance between the two points. The existence of such segments was already established Cohn-Vossen [2, §10, §11].

PROOF OF THEOREM A. Let  $\{W_j\}$  be a monotone increasing sequence of compact domains satisfying  $\bigcup_{j \geq 1} W_j = M$  such that for each  $j$ ,  $W_j - \text{int}(W_j)$  is a simply closed geodesic polygon. Let  $\varepsilon$  be an arbitrary small positive number and fix  $j$ . By Lemma B, we can choose large  $t_j$  and  $s_j$  in such a way that  $\gamma(t_j)$  and  $\gamma(-s_j)$  are joined by two segments  $T_j$  and  $S_j$  in  $M - \text{int}(W_j)$  satisfying the following properties:  $T_j$  and  $S_j$  are not homotopic to each other in  $M - \text{int}(W_j)$ , they have the same minimal lengths among all curves in  $M - \text{int}(W_j)$  having the same endpoints and belonging to the same homotopy classes, and their angles at the endpoints are less than  $\varepsilon/2$ . The minimizing property of  $S_j$  and  $T_j$  in  $M - \text{int}(W_j)$  implies that if  $x$  is a non-differentiable point of  $S_j$ , then  $x$  belongs to  $\partial W_j$  and the angle of  $S_j$  at  $x$  is not smaller than  $\pi$  if it is measured with respect to  $M - \text{int}(W_j)$ . Thus  $S_j \cup T_j$  bounded a convex domain, say,  $D_j$ . The Gauss-Bonnet theorem implies that

$$\int_{D_j} Gdv < \varepsilon.$$

Now by choosing a subsequence  $\{D_k\}$  of  $\{D_j\}$  if necessary, we may assume that  $\{D_k\}$  is a monotone increasing sequence with  $\bigcup_k D_k = M$ . Then  $\lim_{k \rightarrow \infty} \int_{D_k} Gdv = \int_M Gdv \leq \varepsilon$ , and the proof is complete since  $\varepsilon$  is arbitrary.

From now on, let  $M$  be a complete Riemannian manifold homeomorphic to  $R^2$ . Let  $q$  be an arbitrary fixed point on  $M$ . Since  $A(q)$  is closed in  $S_q(M)$ ,  $S_q(M) - A(q)$  consists of a disjoint union of open subarcs of  $S_q(M)$ . Set  $\bigcup_{\lambda \in A} F_\lambda = S_q(M) - A(q)$ , where  $A$  is an index set, and each  $F_\lambda$  is an open subarc of  $S_q(M)$ . For each  $\lambda \in A$  let  $D_\lambda(q)$  be a unique component of the set  $D(q) := M - \{\exp_q tu; u \in A(q), t \geq 0\}$  which contains  $\{\exp_q tv; v \in F_\lambda, 0 < t < \text{the convexity radius at } q\}$ . Clearly  $\bar{D}_\lambda(q) - D_\lambda(q)$  consists of rays, say,  $\sigma, \tau: [0, \infty) \rightarrow M$  with  $\sigma(0) = \tau(0) = q$ . The following fact was first proved by Maeda [5] under the assumption that  $G \geq 0$ , and later the assumption  $G \geq 0$  was removed by the author. Here a proof simpler than that in [8] will be stated.

LEMMA C (compare [8, Lemma 4]). *With the same notations as above, we have:*

(1) For any  $\varepsilon \in (0, \angle(\dot{\sigma}(0), \dot{\tau}(0))/2)$  there exists an  $R = R(\varepsilon)$  such that if  $U$  is the unique unbounded component of the set  $D_\lambda(q) \cap \{x \in M; d(q, x) > R\}$ , then for any  $x \in U$  and for any minimizing geodesic  $\alpha: [0, l] \rightarrow M$  with  $\alpha(0) = q$ ,  $\alpha(l) = x$ , either  $\angle(\dot{\sigma}(0), \dot{\alpha}(0)) < \varepsilon$  or else  $\angle(\dot{\tau}(0), \dot{\alpha}(0)) < \varepsilon$  holds.

(2) For any fixed  $t > R$  if  $c_t: [0, 1] \rightarrow U$  is a curve whose image bounds a unique unbounded component of  $\{x \in U; d(q, x) > t\}$ , then there are a point  $x$  on  $c_t([0, 1])$  and two distinct minimizing geodesics  $\beta, \gamma: [0, t] \rightarrow M$  such that their images are in  $D_\lambda$  and that they satisfy

$$\angle(\dot{\beta}(0), \dot{\sigma}(0)) < \varepsilon, \quad \angle(\dot{\gamma}(0), \dot{\tau}(0)) < \varepsilon.$$

PROOF OF LEMMA C. First of all it follows from the construction of the boundary of  $\bar{D}_\lambda$  that every minimizing geodesic joining  $q$  to any point in  $D_\lambda$  does not intersect its boundary and has its image in  $\bar{D}_\lambda$ . Therefore (1) is a direct consequence of the fact that there is no ray emanating from  $q$  whose initial tangent vector belongs to  $F_\lambda$ .

To prove (2) note that there is an open set around  $\sigma(t)$  in which every point can be joined to  $q$  by a unique minimizing geodesic which makes an angle with  $\sigma$  at  $q$  less than  $\varepsilon$ . Note also that for every  $u \in [0, 1]$  every minimizing geodesic joining  $q$  to  $c_t(u)$  does not meet  $c_t([0, 1])$  except at  $c_t(u)$ . Let  $I_\sigma = \{u \in [0, 1]; \text{every minimizing geodesic joining } q \text{ to } c_t(u) \text{ makes an angle with } \sigma \text{ at } q \text{ less than } \varepsilon\}$ . Then it follows from what is noted above that  $I_\sigma \neq \emptyset$  and that if  $u \in I_\sigma$ , then  $u' \in I_\sigma$  holds for all  $u' \in [0, u]$ . Similarly we define  $I_\tau = \{u \in [0, 1]; \text{every minimizing geodesic joining } q \text{ to } c_t(u) \text{ makes an angle with } \tau \text{ at } q \text{ less than } \varepsilon\}$ . Then we conclude that  $I_\tau$  is a nonempty subinterval of  $[0, 1]$  containing 1. If there is a  $u_0 \in [0, 1] - I_\sigma \cup I_\tau$ , then the point  $x = c_t(u_0)$  has the desired property. Suppose that  $[0, 1] = I_\sigma \cup I_\tau$ . Then it follows from  $I_\sigma \cap I_\tau = \emptyset$  that one of the two intervals is open and the other is closed. Without loss of generality we may assume that  $I_\sigma$  is open. Set  $u_0 = \sup I_\sigma$ . Then a minimizing geodesic  $\beta: [0, t] \rightarrow M$  is obtained as a limit of minimizing geodesics  $\beta_j: [0, t] \rightarrow M$  with  $\beta_j(0) = q$ ,  $\beta_j(t) = c_t(u_j)$  such that  $u_j \in I_\sigma$  and  $\lim u_j = u_0$ . On the other hand, it follows from  $u_0 \in I_\tau$  that there is a minimizing geodesic  $\gamma$  joining  $q$  to  $c_t(u_0)$  with the desired property. Thus the proof is complete.

The following result was established in the previous work of the author and the proof is omitted here.

**THEOREM D** ([8, Theorem 2]). *Let  $M$  be a complete Riemannian manifold homeomorphic to  $\mathbf{R}^2$ . If  $M$  admits the total curvature, then for every point  $p$  on  $M$ ,*

$$\text{meas}(A(p)) \geq 2\pi - \int_M G^+ dv .$$

**3. The proof of Theorem 1.** By means of Theorem D we only need to prove that

$$(*) \quad \inf_{p \in M} \text{meas}(A(p)) \leq 2\pi - \int_M G dv .$$

If the total curvature of  $M$  is nonpositive, then  $(*)$  is obvious. Taking Theorem A into account, we may therefore assume that  $M$  admits no straight line. If the left hand side of  $(*)$  is zero, then  $(*)$  is nothing but the Cohn-Vossen theorem. Therefore we may assume that *through every point  $p$  on  $M$  there pass at least two (indeed more than two) distinct rays.*

Let  $\{V_j\}$  be a monotone increasing sequence of compact sets such that  $\bigcup_{j \geq 1} V_j = M$ . Let  $\varepsilon$  be an arbitrary fixed positive number, and fix  $j$ . The proof is divided into three steps as follows.

**STEP 1.** Since  $M$  has no straight line by assumption, there is an  $R_j > 0$  such that for each point  $x$  on  $M - B_{R_j}(V_j)$  all rays emanating from  $x$  do not intersect  $V_j$ . Indeed, we otherwise would have a divergent sequence of points  $\{x_k\}$  and rays  $\{\sigma_k\}$ , each  $\sigma_k$  emanating from  $x_k = \sigma_k(0)$  and passing through a point on  $V_j$ . Then the compactness of  $V_j$  would make it possible to choose a subsequence of  $\{\sigma_k\}$  which converges to a straight line, a contradiction. Let  $q$  be a point on  $M - B_{R_j}(V_j)$  and set  $S_q(M) - A(q) = \bigcup_{\lambda \in A} F_\lambda$  as before. Then there exists a  $\lambda \in A$  such that  $F_\lambda$  is a proper subarc of  $S_q(M)$  and that  $D_\lambda(q)$  contains  $V_j$ . Let  $\sigma, \tau: [0, \infty) \rightarrow M$  be distinct rays with  $\sigma(0) = \tau(0) = q$  which bound  $D_\lambda(q)$ . The existence of distinct rays emanating from  $q$  is guaranteed by the assumption that the left hand side of  $(*)$  is positive.

**STEP 2.** Since  $V_j$  is contained in  $D_\lambda(q)$ , there is a positive number  $\eta$  such that  $\eta = \inf \{ \angle(\dot{\sigma}(0), \dot{\alpha}(0)), \angle(\dot{\tau}(0), \dot{\alpha}(0)); \alpha \text{ is a minimizing geodesic joining } q \text{ to every point on } V_j \}$ . It follows from Lemma C, (2) that there exist a point  $p$  and two minimizing geodesics  $a, b: [0, l] \rightarrow D_\lambda(q)$  such that  $a(0) = b(0) = q$ ,  $a(l) = b(l) = p$  and  $\angle(\dot{a}(0), \dot{\sigma}(0)) < \eta/2$ ,  $\angle(\dot{b}(0), \dot{\tau}(0)) < \eta/2$ . Thus the subdomain of  $D_\lambda(q)$  which is bounded by  $a([0, l])$  and  $b([0, l])$  contains  $V_j$  in its interior.

Consider  $\bar{D}_\lambda(q)$  to be a complete Riemannian manifold with nonempty boundary. The distance function  $\hat{d}$  is naturally defined on  $\bar{D}_\lambda(q)$  by the metric on  $M$  restricted to  $\bar{D}_\lambda(q)$ . Then every two points can be joined by a  $\hat{d}$ -minimizing segment which may have a nondifferentiable point in its interior. It follows from Lemma B that there are large numbers  $s_j$  and  $t_j$  and  $\hat{d}$ -minimizing segments  $a_j$  and  $b_j$  joining  $p$  to  $\sigma(s_j)$  and  $\tau(t_j)$

respectively such that the angles between  $a_j$  and  $\sigma$  at  $\sigma(s_j)$  and between  $b_j$  and  $\tau$  at  $\tau(t_j)$  are less than  $\varepsilon/2$ . It should be checked that these angles are all positive. This is observed as follows. It suffices for the proof of  $\angle(\dot{a}_j, \dot{\sigma})|_{\sigma(s_j)} > 0$  to show that  $a_j$  is a geodesic in  $M$ . Suppose  $a_j$  has a nondifferentiable point in its interior. Then the break point coincides with  $q$ . (This might happen when  $\angle(\dot{\sigma}(0), \dot{\tau}(0))$  is close to  $2\pi$ .) Since  $a_j$  have  $\hat{d}$ -minimizing property and since  $a$  and  $b$  are minimizing, the subarc of  $a_j$  between  $p$  and  $q$  has the same length as  $a$  and  $b$ . Thus  $L(a_j) = \hat{d}(p, q) = L(a) + L(\sigma|[0, s_j])$ . But since  $\angle(\dot{a}(0), \dot{\sigma}(0)) < \eta/2$  we have  $L(a) + L(\sigma|[0, s_j]) > (s_j - r) + \hat{d}(\sigma(r), a(r)) + (L(a) - r)$  for a small  $r > 0$ , and this value is realized by a broken geodesic in  $D_\lambda(q)$  joining  $\sigma(s_j)$ ,  $\sigma(r)$ ,  $a(r)$  and  $p = a(l)$  in an obvious way. This contradicts the  $\hat{d}$ -minimizing property of  $a_j$ , proving the positivity of the angle.

STEP 3. Let  $D_j$  be the compact subdomain of  $\bar{D}_\lambda(q)$  which is bounded by  $a_j$ ,  $b_j$ ,  $\sigma([0, s_j])$  and  $\tau([0, t_j])$ . The above argument ensures that  $D_j$  contains the compact domain bounded by a geodesic biangle  $a$  and  $b$ , and hence  $D_j \supset V_j$ . By choosing a subsequence  $\{D_k\}$  of  $\{D_j\}$  if necessary, we may assume that  $\{D_k\}$  is monotone increasing and  $\bigcup_k D_k = M$ . For each  $k$  the Gauss-Bonnet theorem applies to yield

$$\begin{aligned} \int_{D_k} Gdv &< \text{meas}(\bar{F}_\lambda) + \varepsilon \leq 2\pi - \text{meas}(A(q)) + \varepsilon \\ &\leq 2\pi - \inf_{q \in M} \text{meas}(A(p)) + \varepsilon . \end{aligned}$$

This completes the proof of Theorem 1 since  $\varepsilon$  is an arbitrary positive number.

PROOF OF THEOREM 2. Let  $q$  be a point of  $M$  at which the infimum of the function  $x \mapsto \text{meas}(A(x))$  is attained. Let  $\bigcup_{\lambda \in A} F_\lambda = S_q(M) - A(q)$  and for each  $\lambda \in A$  let  $D_\lambda(q)$  be defined as in §2. It follows from Lemma C, (2) that  $D_\lambda(q)$  for each  $\lambda \in A$  is covered by a monotone increasing sequence of compact subdomains in  $D_\lambda(q)$  each of which is bounded by a geodesic biangle. Because of  $G > 0$  the total curvature of  $D_\lambda(q)$  exists and

$$\int_{D_\lambda(q)} Gdv \geq \text{meas}(\bar{F}_\lambda) .$$

Moreover we have

$$\int_M Gdv \geq \sum_{\lambda \in A} \int_{D_\lambda(q)} Gdv \geq \sum_{\lambda \in A} \text{meas}(\bar{F}_\lambda) = 2\pi - \text{meas}(A(q)) ,$$

where the first inequality is ensured by the assumption  $G > 0$ . Therefore all equalities hold by means of the Maeda theorem. The first equality

implies that  $M - \bigcup_{\lambda \in A} D_\lambda(q)$  has measure zero, in other words,  $A(q)$  has no interior in  $S_q(M)$ . Hence  $\text{meas}(A(q)) = 0$  implies the conclusion.

Finally we furnish an example of a surface  $\Sigma$  in  $E^3$ , on which both inequalities in Theorem 1 hold. The Gaussian curvature  $G$  of  $\Sigma$  has compact support and  $\Sigma$  has two "hills" on a plane. The construction is carried out as follows. For positive numbers  $a < b$  set  $f(t) = h(b-t)/(h(b-t) + h(t-a))$ , where  $h: R \rightarrow R$  is defined by

$$h(t) = \begin{cases} \exp(-1/t^2) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Computations show that  $f(t) = 0$  for  $t \geq b$ ,  $0 < f(t) < 1$  for  $a < t < b$  and  $f(t) = 1$  for  $a \geq t$ , and  $0 \geq f'(t) \geq f'((a+b)/2) = -8(b-a)^{-3}$ . For an arbitrary fixed point  $p = (p_1, p_2)$  with  $\|p\| > 2b$ , let  $\Sigma$  be defined as the graph of  $x_3 = x_3(x_1, x_2)$ ;

$$x_3 = \begin{cases} f(\|x\|) & \text{for } \|x - p\| > b \\ f(\|x - p\|) & \text{for } \|x - p\| \leq b, \end{cases}$$

where  $x = (x_1, x_2) \in R^2$ . It is elementary to verify that

$$\int_{\Sigma} G dv = 0 \quad \text{and} \quad \int_{\Sigma} G^+ dv = 4\pi(1 - \sin\{\tan^{-1}(b-a)^3/8\}).$$

Therefore we can choose  $a$  and  $b$  in such a way that

$$\pi < \int_{\Sigma} G dv < 2\pi$$

is satisfied. We can also choose  $p$  sufficiently far from the origin so that  $\text{meas}(A(x)) \geq \pi$  holds for any point  $x$  on  $\Sigma$ . This is possible because every compact set on  $E^2$  is contained in a cone of arbitrary small angle at the vertex is taken to be sufficiently far from the compact set. Therefore we have

$$2\pi - \int_{\Sigma} G^+ dv < \pi < \inf_{x \in \Sigma} \text{meas}(A(x)) < \pi = 2\pi - \int_{\Sigma} G dv.$$

It should be noted that the Cohn-Vossen theorem was extended to a finitely connected noncompact  $G$ -surface on which angular measure is defined. The total excess with respect to the angular measure plays the same role as the total curvature. For details see [1, §43 and §44].

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