A Relative Superior Julia Set and Relative Superior Tricorn and Multicorns of Fractals

Priti Dimri
Head
Dept. of Computer Science and Engineering
G.B.Pant Engineering College
Ghurdauri, Pauri

Shashank Lingwal
Dept. of Computer Science and Engineering
G.B.Pant Engineering College
Ghurdauri, Pauri

Ashish Negi
Associate Professor
Dept. of Computer Science and Engineering
G.B.Pant Engineering College
Ghurdauri, Pauri

ABSTRACT
In this paper we investigate the new Julia set and a new Tricorn and Multicorns of fractals. The beautiful and useful fractal images are generated using Ishikawa iteration to study many of their properties. The paper mainly emphasizes on reviewing the detailed study and generation of Relative Superior Tricorn and Multicorns along with Relative Superior Julia Set.

Keywords
Complex dynamics, relative superior Julia set, Ishikawa iteration, Relative superior Tricorn, Relative superior Multicorns.

1. INTRODUCTION
The term ‘Fractal’ was coined by Benoit B Mandelbrot, in 1975 to denote his generalisation of complex shapes. Fractal is derived from Latin word ‘fractus’ which describes the appearance of broken stone : irregular and fragmented. The object Mandelbrot set, given by Mandelbrot 1979 and its relative object Julia set due to their beauty and complexity of their nature have become elite area of research nowadays. The fractal graphics are generally obtained by “colouring” the escape speed of the seed points within the certain regions of complex plane that give rise to the unbounded orbits.

Many authors have presented the papers on several “orbit traps” rendering methods to create the artistic fractal images. An orbit trap is a bounded area in complex plane into which an orbiting point may fall. Motivated by this idea of “orbit traps”, Chauhan [4] introduces the different type of orbit traps for Ishikawa iteration procedure. They have considered Julia sets of $z_{n+1} = az_n^2 + c$, as orbit traps to explore their relevant fractal images, since, it is connected and bounded for a and c.

Recently Shizuo [15] has quoted the Multicorns as the generalized Tricorns or the Tricorns of higher order and presented various properties of them. Tricorn are being used for commercial purpose, e.g. Tricorn Mugs and Tricorn T shirts. Multicorns are symmetrical objects. Their symmetry has been studied by Lau and Schleicher [11]. Negi [16] have introduced a new class of Relative Superior Multicorns using Ishikawa iterates and also studied their corresponding Relative Superior Julia sets.

2. ELABORATION OF CONCEPT INVOLVED

2.1 Mandelbrot Set
Definition 1. [17] The Mandelbrot set M for the quadratic $Q_C(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of point 0 is bounded, that is,
$$M = \{c \in C : \{Q_c^n(0)\}; n = 0,1,2,3... \text{ is bounded}\}$$

An equivalent formulation is
$$M = \{c \in C : \{Q_c^n(0) \text{ does not tends to } \infty \text{ as } n \to \infty\}\}$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

2.2 Julia Set
Definition 2. [17] The set of points K whose orbits are bounded under the iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

2.3 Tricorn and Multicorns
Please The study of connectedness locus for antiholomorphic polynomials $\bar{z}^2 + c$ defined as Tricorn, coined by Milnor [14], plays intermediate role between quadratic and cubic polynomials.

Definition 3. [5] The Multicorns A_c , for the quadratic $A_c(z) = z'^n + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is, $A_c = \{c \in C : A_c(0)_{n=0,1,2,3...} \text{ is bounded}\}$. An equivalent formulation is $A_c = \{c \in C : A_c(0) \text{ not tends to } \infty \text{ as } n \to \infty\}$.

The Tricorn are special Multicorns when n=2. As quoted by Devany [7][8], iteration of function $A_c = z'^2 + c$, using the Escape Time Algorithm, results in many strange and surprising structures. Devany [7][8] has named it Tricorns and observed that f(z’), the conjugate function of f(z), is antipolynomial. Further, its second iterates is a polynomial of degree 4. The function $z'^2 + c$ is conjugate of $z'^2 + d$, where $d = e^{2\pi i/3}$, which shows that the Tricorn is symmetric under rotations through angle $2\pi / 3$. The critical point for A_c is 0, since $c = A_c(0)$ has only one preimage whereas any other $w \in C$, has two preimages.

2.4 Picard's Orbit

Definition 4. Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, the Picard's orbit is defined as the set of iterates of a point x_0, that is; $O(f,x) = \{x_n ; x_n = f(x_{n-1}), n = 1,2,3...\}$.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded, i.e. there exists M, such that $|Q^n(z)| \le M$, for all n, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence the boundary of a prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

2.5 Ishikawa Iteration

Definition 5. Ishikawa Iterates [17]**:** Let X be a subset of real or complex number and $f : X \to X$ for all $x_0 \in X$, we have the sequence $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = S'_n f(x_n) + (1 - S'_n)x_n$$
$$x_{n+1} = S_n f(y_n) + (1 - S_n)x_n$$

where $0 \le S'_n \le 1$, $0 \le S_n \le 1$ and S'_n & S_n are both convergent to non-zero number.

2.6 Relative Superior Mandelbrot Set

Now we define Mandelbrot set for the function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 6. [17] Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1,2,3,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e. $RSM = \{c \in C : Q_c^k(0) : k = 0,1,2,3...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exist a positive real number.

2.7 Relative Superior Julia Set

Definition 7. [17] The set of points RSK whose orbits are bounded under relative superior iteration of function Q(z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

3. GENERATING PROCESS

The basic principle of generating fractals employs the iterative formula: $z_{n+1} \leftarrow f(z_n)$ where z_0 = the initial valueof z, and z_i = the value of complex quantity z at the i^{th} iteration [17]. For example, the Mandelbrot's self-squared function for generating fractal is: f(z) = z^2+c , where z and c are both complex quantities. We propose the use of transformation function $z \to z^n + c, n \ge 2$ and $z \to (z^n + c)^{-1}$ for generating fractal images with respect to Ishikawa iterates, where z and c are the complex quantities and n is a real number. Each of these fractal images is constructed as two-dimensional array of pixel. Each pixel is represented by a pair of (x,y) coordinates. The complex quantities z and c can be represented as:

$$z = z_x + iz_y$$
$$c = c_x + ic_y$$

where $i = \sqrt{(-1)}$ and z_x , c_x are the real parts and z_y , c_y are the imaginary parts of z and c respectively. The pixel coordinates (x,y) may be associated with (c_x,c_y) or (z_x,z_y) .

Based on this concept, the fractal images can be classified as follows:

(a) **z-Plane** fractals, wherein (x,y) is a function of (z_x,z_y).
(b) **c-Plane** fractals, wherein (x,y) is a function of (c_x,c_y).

In the literature, the fractals for n=2 in z plane are termed as the Mandelbrot set while the fractals for n=2 in c plane are known as Julia sets [4, 5]

4. ESCAPE CRITERIA FOR RELATIVE SUPERIOR MANDELBROT AND JULIA SET

4.1 Escape Criterion for Quadratics

[13] Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

4.2 Escape Criterion for Cubics

[13] Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives the escape criterion for cubic polynomials.

4.3 General Escape Criterion

[13] Consider $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion.

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to (z^n + c)^{-1}$. However, instead of starting with z_0 = infinity, it is simpler to start with z_1 = c, which yields the same result. A critical point of $z \to F(z) + c$ is a point where $F'(z) = 0$.

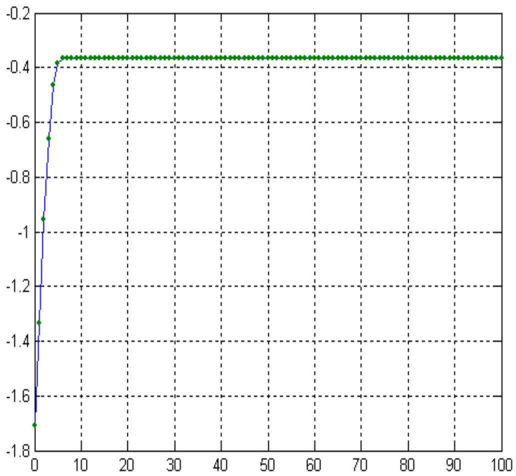
5. SIMULATION AND RESULTS

Fixed point of quadratic polynomial- [4]

Table 1 : Orbit of F(Z) for (z_0 = -1.71 -0.24i) at s=0.3 and s' = 0.4

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
1	1.7268	13	0.97304
2	1.3866	14	0.97304
3	1.2095	15	0.97304
4	1.132	16	0.97304
5	1.0602	17	0.97304
6	1.0051	18	0.97304
7	0.98257	19	0.97304
8	0.97564	20	0.97304
9	0.97308	21	0.97304
10	0.9732	22	0.97304
11	0.97308	23	0.97304
12	0.97305	24	0.97304

Fig. 1 : Observation : Here we observe that the value converges to a fixed point after 13 iterations

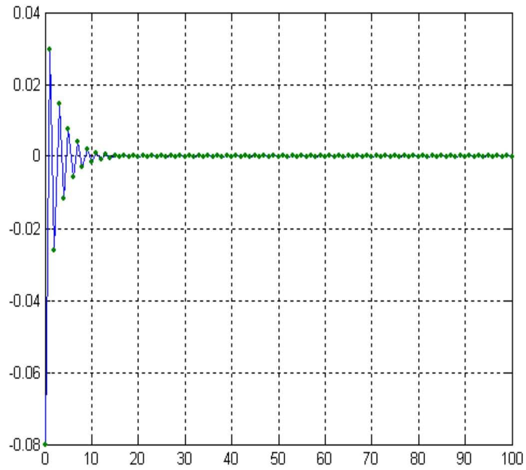


Fixed Point for Cubic Polynomial

Table 2 : Orbit of F(z) for (z_0 = -0.08+0.057i)at s=0.8 and s'=0.2

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
10	0.26107	25	0.26713
11	0.27154	26	0.26716
12	0.26406	27	0.26721
13	0.26942	28	0.26717
14	0.26558	29	0.2672
15	0.26833	30	0.26718
16	0.26637	31	0.26719
17	0.26777	32	0.26718
18	0.26677	33	0.26719
19	0.26749	34	0.26719
20	0.26697	35	0.26719
21	0.26734	36	0.26719
22	0.26708	37	0.26719
23	0.26717	38	0.26719
24	0.26727	39	0.26719

Fig. 2 : Observation : We skipped 09 iteration and value converges to a fixed point after 33 iterations



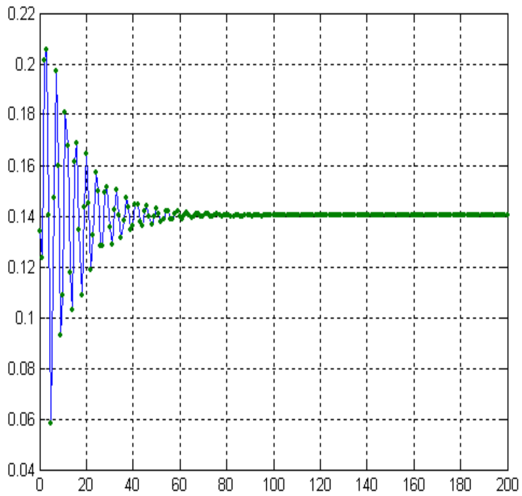
Fixed point for Bi-quadratic Polynomial

Table 3 : Orbit of F(z) for (z_0=0.134+0.128i) at s=0.3 and s'=0.4

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
81	0.9200	96	0.9199
82	0.9196	97	0.9200
83	0.9197	98	0.9200
84	0.9201	99	0.9198
85	0.9201	100	0.9198
86	0.9198	101	0.9200
87	0.9197	102	0.9200
88	0.9200	103	0.9199
89	0.9201	104	0.9198
90	0.9199	105	0.9199
91	0.9197	106	0.9200

92	0.9199	107	0.9199
93	0.9200	108	0.9199
94	0.9199	109	0.9199
95	0.9198	110	0.9199

Fig. 3: Observation : We skipped 81 iteration and after 107 iteration value converges to a fixed point



Generation of Relative Superior Julia Set

Relative Superior Julia set for Quadratic

Fig. 4 : Relative superior Julia set for s=1, s’=0.3, c= 0.430+0.18i

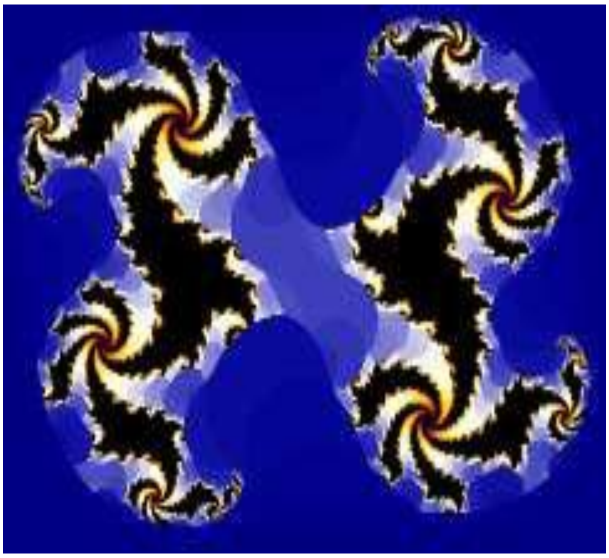
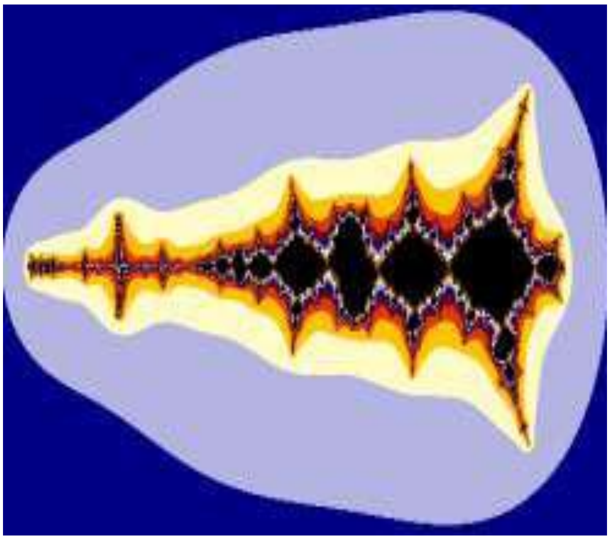


Fig. 5 : Relative Superior Julia Set for s=0.1, s’=0.4, c=-20.26+0.097i



Relative Superior Julia set for Cubic function :

Fig. 6 : Relative Superior Julia set for s=1, s’=0.5, c= -0146+1.54i

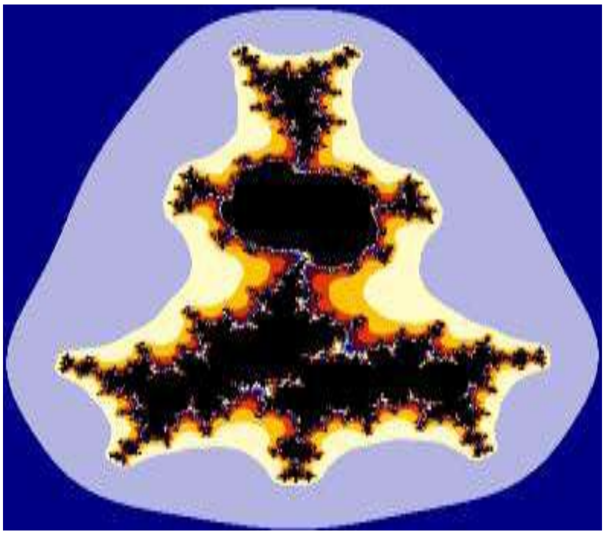
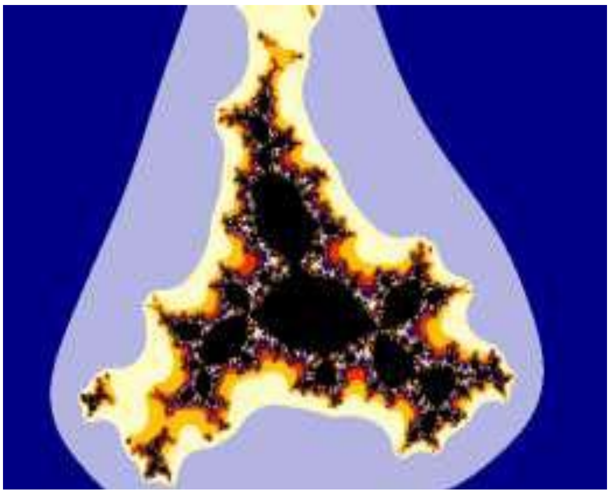


Fig. 7 : Relative Superior Julia Set for s=0.1, s’= 0.4, c=-1.6+6.7i



Relative Superior Julia Sets for Bi-Quadratic Function

Fig. 8 : Relative Superior Julia set for s= 1, s'= 0.5, c = -1.57

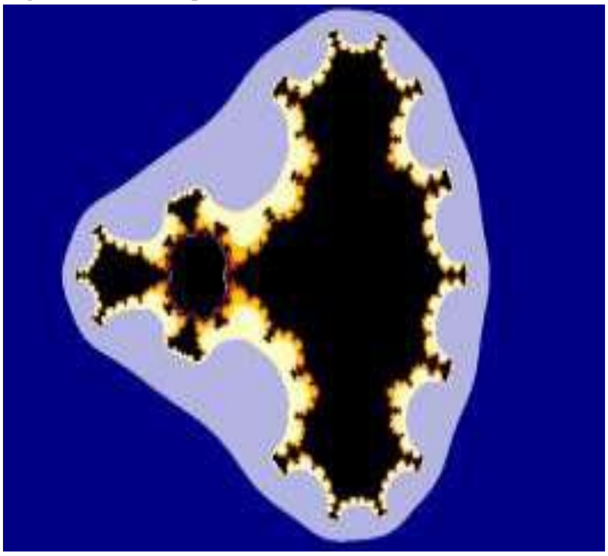
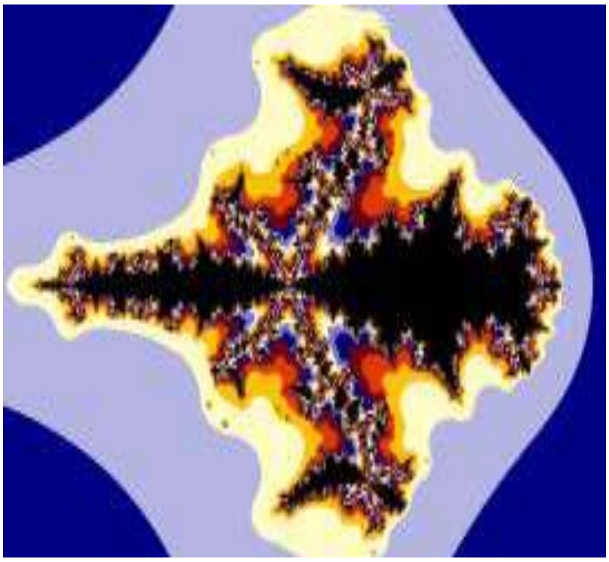


Fig. 9 : Relative Superior Julia Set for s=0.3, s'=0.4, c=-3.6

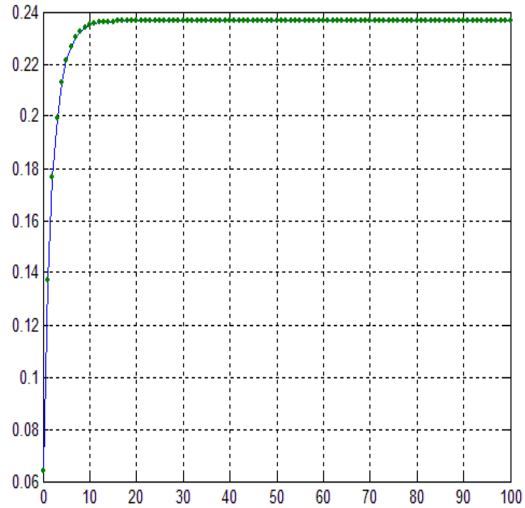


Fixed Points For Quadratic Polynomial – [5]

Table 4 : Orbit of F(z) for s=0.6, s'=0.4 at z_0=0.06415024553+0.03414122547i

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
1	0.07267	14	0.23604
2	0.13743	15	0.23621
3	0.17641	16	0.23638
4	0.19914	17	0.23643
5	0.21289	18	0.23646
6	0.22143	19	0.23648
7	0.22681	20	0.23649
8	0.23025	21	0.2365
9	0.23246	22	0.2365
10	0.23389	23	0.23651
11	0.23481	24	0.23651
12	0.2354	25	0.23651
13	0.23579	26	0.23651

Fig. 10 : The Value Converges to a fixed point after 23 iterations

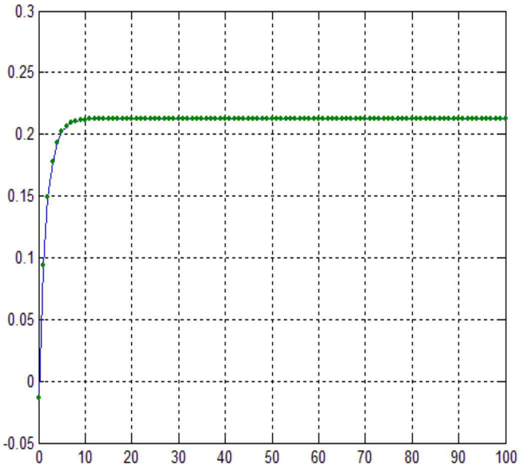


Fixed Point for Cubic Polynomial

Table 5 : Orbit of F(z) for s=0.5, s' =0.4 at z_0=-0.01341912254 + 0.001017666092i

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
1	0.013458	12	0.21207
2	0.09368	13	0.2122
3	0.14876	14	0.21228
4	0.17784	15	0.21232
5	0.19348	16	0.21235
6	0.20199	17	0.21236
7	0.20665	18	0.21236
8	0.20922	19	0.21236
9	0.21063	20	0.21237
10	0.21141	21	0.21237
11	0.21184	22	0.21237

Fig. 11 : Here value converges to a fixed point after 20 iterations

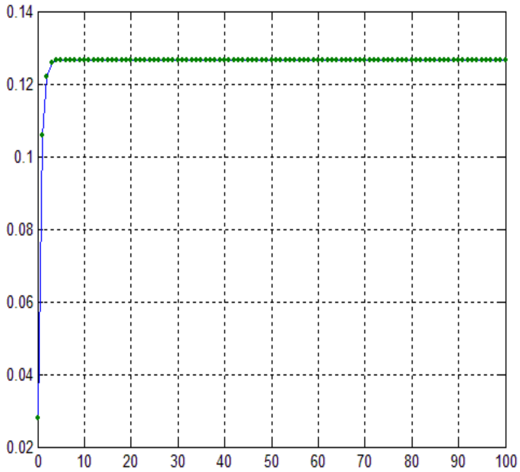


Fixed Points of Bi-Quadratic polynomial

Table 6 : Orbit of F(z) for s=0.8, s'=0.2 at z_0= 0.02786208647 – 0.03509673188i

Number of Iteration i	|F(z)|	Number of Iteration i	|F(z)|
1	0.044812	7	0.12663
2	0.10592	8	0.12664
3	0.12207	9	0.12664
4	0.12563	10	0.12664
5	0.12642	11	0.12664
6	0.12659	12	0.12664

Fig. 12 : Here the value converges to a fixed point after 8 iterations



Generation of Relative Superior Tricorns and Multicorns

Relative Superior Tricorn for Quadratic function

Fig. 13 : Relative superior Tricorn for s=s'=1

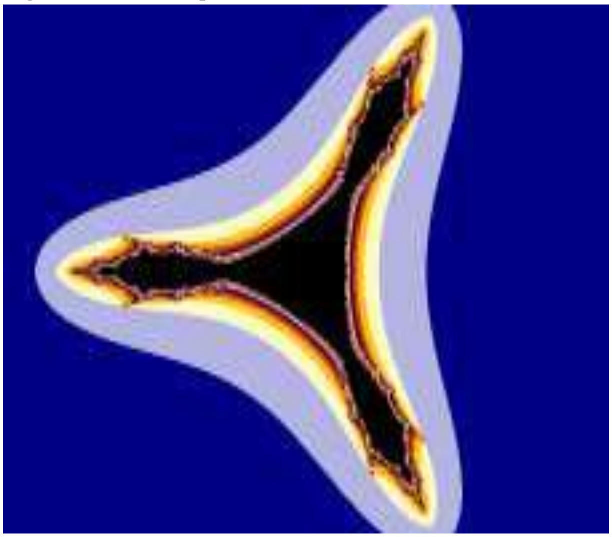
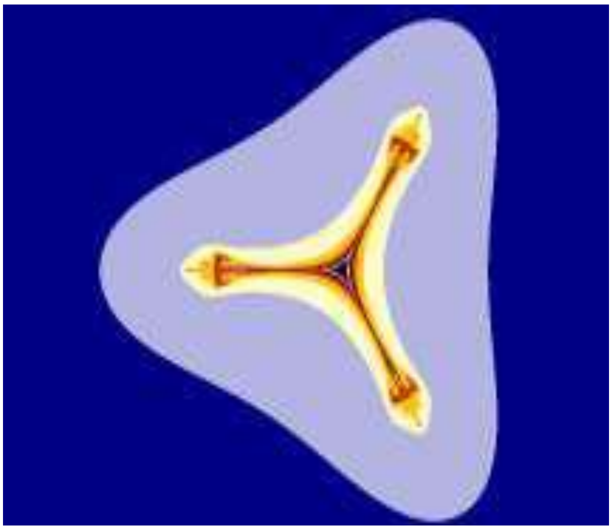


Fig. 14 : Relative Superior Tricorn for s=0.6, s'=0.4



Relative Superior Multicorns for Cubic Function

Fig. 15 : Relative Superior Multicorns for s=1, s' = 1

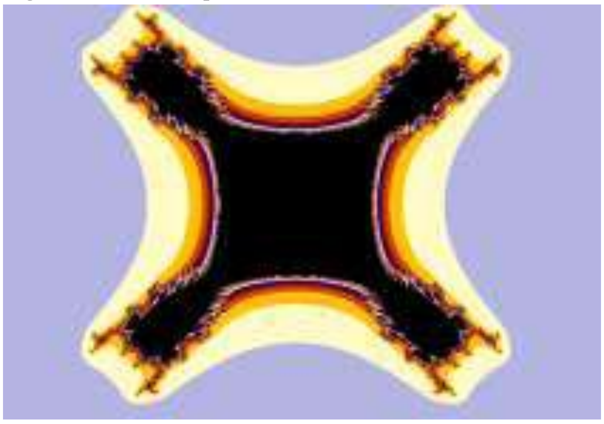
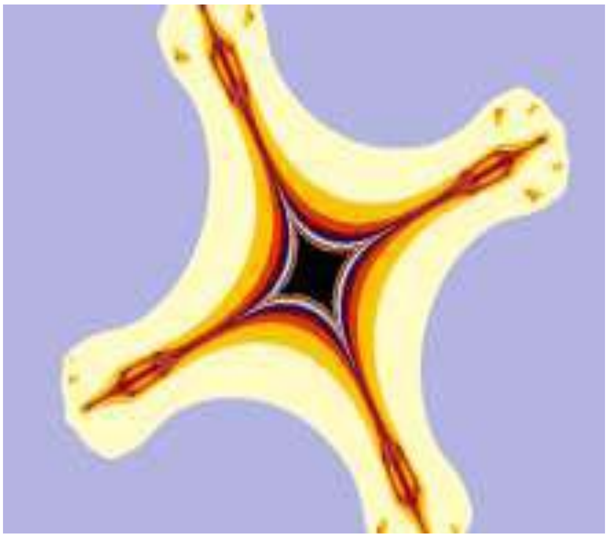


Fig. 16 : Relative Superior Multicorns for s=0.5, s'=0.4



Relative Superior Multicorns for Bi-quadratic function

Fig. 17 : Relative Superior Multicorns for s=s'=1

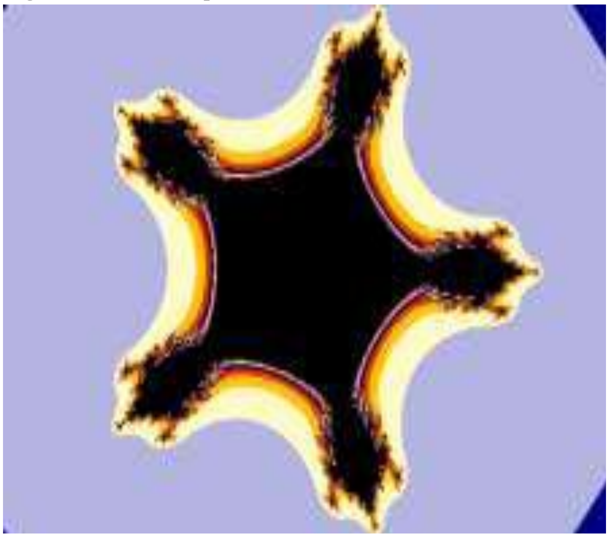
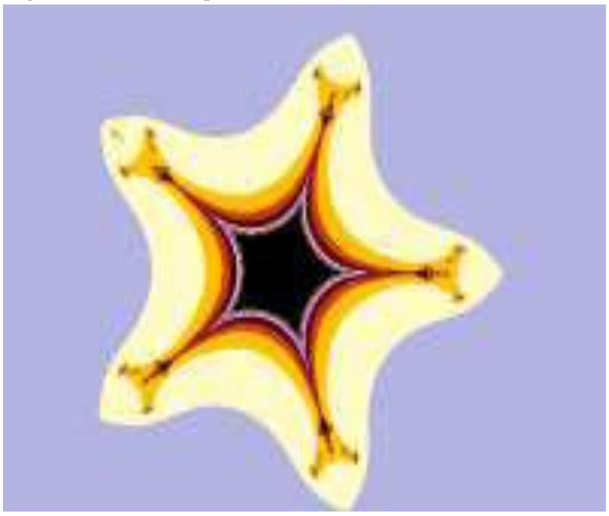


Fig. 18 : Relative Superior Multicorns for s=0.8, s'=0.2



Generalization of Relative Superior Multicorns

Fig. 19 : Relative Superior Multicorns for s=0.1, s' = 0.3, n=19

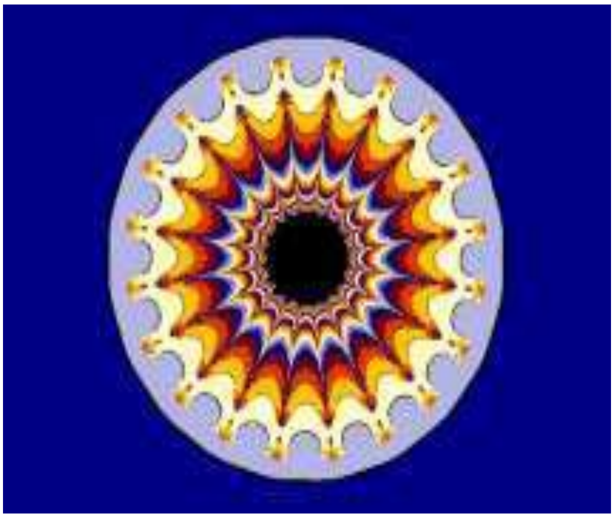
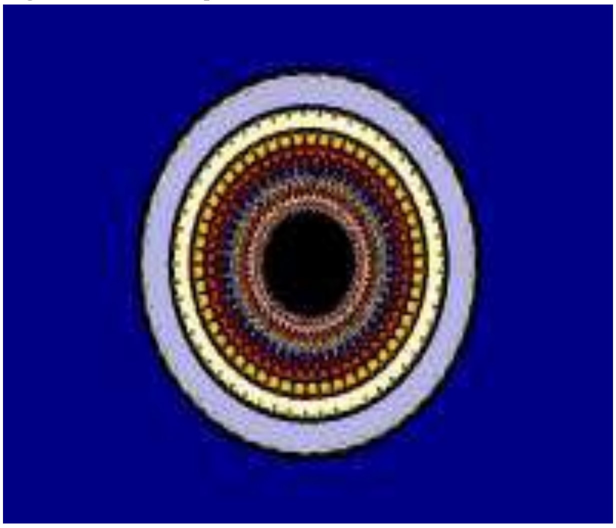


Fig. 20 : Relative Superior Multicorns for s=0.1, s'=0.3, n=52



6. CONCLUSION

- The geometry of Relative Superior Julia sets and Mandelbrot set are explored for Ishikawa iteration and corresponding fractal images are generated. Different types of orbit traps are generated for Ishikawa iteration procedure.
- The study shows that these sets are exclusively elite and effectively different from other existing Mandelbrot sets. In dynamics of antipolonomial of complex polynomial z'^n+c, where $n \geq 2$, there exist many Tricorns and Multicorns antifractals for a value of n with respect to Relative Superior orbit.
- Further, for the odd value of n, all the Relative Superior Multicorns are symmetrical objects, and for even values of n, all the Relative superior Multicorns (including Relative Superior Tricorns) are symmetrical about x-axis.

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