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A RELATIVISTIC VERSION OF THE GAUSS-BONNET FORMULA

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Introduction

The Gauss-Bonnet formula relates the sum of the exterior angles of a geodesic polygon on a surface to the total Gaussian curvature which the polygon encloses. Thus one obtains such statements as: the sum of the interior angles of a geodesic triangle is π if and only if the total curvature enclosed by the triangle is zero.

To develop a version of the formula which applies to surfaces with an indefinite metric requires only a careful definition of a quantity to replace "angle" and a check that the arguments of the definite case remain valid. This is done in §§ 1 and 2.

In § 3 an example is given to indicate the kind of physical quantity which the total Gaussian curvature might measure.

1. The flat case

In this section the "pseudo-angle" or "proper velocity" between two vectors in a plane with indefinite metric is defined and some elementary properties listed.

Let M^2 denote the space of pairs of real numbers with inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = -a_1a_2 + b_1b_2 . \tag{1}$$

Take the positive orientation of M^2 to be that given by the vector space basis $\{e_1 = (1,0), e_2 = (0,1)\}$.

Let $\alpha\colon I \to M^2$ be a continuously differentiable curve parametrized with respect to proper time, i.e.,

$$\langle \alpha'(s), \alpha'(s) \rangle = -1, 1, 0 . \tag{2}$$

The curve α is called timelike, spacelike or null respectively.

Next define a moving frame $\{T(s), N(s)\}$ on α as follows. Let $\{u_1, u_2\}$ be an orthonormal frame at $\alpha(s)$, and set

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$$T(s) = \alpha'(s) = x_1u_1 + x_2u_2 \ ,$$

$$(3) \quad N(s) = \begin{cases} x_2u_1 + x_1u_2 & \text{if } \{u_1, u_2\} \text{ has positive orientation ,} \\ -(x_2u_1 + x_1u_2) & \text{if } \{u_1, u_2\} \text{ has negative orientation .} \end{cases}$$

The definition of N is independent of the choice of $\{u_1, u_2\}$ since N is simply T reflected in one piece of the light cone.

Lastly define a real valued function ϕ with domain I by

$$(4) \qquad \phi(s) = \begin{cases} \ln|a + b| & \text{if } a + b \neq 0 \ , \\ -\ln|a - b| & \text{if } a + b = 0 \ , \end{cases}$$

where $T(s) = ae_1 + be_2$. Since $|a + b||a - b| = 1$ or 0, the two functions on the right hand side of (4) are equal where they are both defined.

Theorem 1. *There is a unique function g defined on I for which*

$$T'(s) = g(s)N(s) \ , \qquad N'(s) = g(s)T(s) \ .$$

In fact $g = \phi'(s)$.

Proof. Since α is parameterized with respect to proper time, using the logarithmic forms of the inverse hyperbolic functions one sees that T may be written in one of the forms:

$$\pm(e_1 \cosh\phi + e_2 \sinh\phi) \ , \quad \pm(e_1 \sinh\phi + e_2 \cosh\phi) \ , \quad \pm a(e_1 \pm e_2) \ .$$

Direct calculation now gives the theorem. q.e.d.

The Euclidean version of Theorem 1 is the starting point of the theory of plane curves. There s is the arc length and ϕ is the angle which T makes with the x-axis. Here ϕ is the "pseudo-angle" which T makes with e_1, i.e., with the time axis. The functions T, N, g are invariants in the sense that their definition does not depend on the choice of basis $\{e_1, e_2\}$. On the other hand if one changes, the basis ϕ will change by an additive constant and its sign depends on the orientation of the basis. As in the Euclidean theory, it may be shown that g determines α up to a Lorentz transformation (translations included).

Suppose a particle is constrained to move in one spatial dimension, say the e_2 axis where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of the Minkowski space of special relativity ($c = 1$). Then by suppressing the irrelevant directions e_3, e_4 we may consider the space time trace of the particle to be the curve α above. In this case $g(s)$ is the acceleration at time s as measured by an observer at rest with respect to the particle and since $\phi' = g$, one might call ϕ a "proper velocity."

For an observer at rest with respect to the frame $\{e_1, e_2\}$ the expression

$$\alpha(t) = te_1 + x(t)e_2$$

describes the motion of the particle. Taking $\{e_1, e_2\}$ oriented so that $T = e_1 \cosh \phi + e_2 \sinh \phi$ we see that $t'(s) = \cosh \phi$ and hence the speed of the particle measured by this observer is

$$v = \frac{dx}{dt} = \frac{ds}{dt}\frac{dx}{ds} = \tanh \phi \ ,$$

and so $\phi = \tanh^{-1} v = v + \frac{1}{3}v^3 + \cdots$. Thus for $v \ll 1$, ϕ is indistinguishable from v. The sum formula for the hyperbolic tangent shows that composing velocities corresponds to adding ϕ's.

For a particle moving with the speed of light, ϕ is the logarithm of twice the energy ($= e_1$ component of T). This reduces to

$$\phi = \log \nu + \text{const} \ , \tag{5}$$

where ν is the frequency and hence $g = \phi' = \nu'/\nu$. If α is spacelike, ϕ gives the relative velocity of the orthonormal frame $\{N, T\}$ with respect to $\{e_1, e_2\}$.

To define an "angle" between any two unit or null vectors proceed as follows. If $\mathcal{O} = \{u_1, u_2\}$ is an orthonormal basis, and $u = au_1 + bu_2$ is a unit or null vector, then define $\phi_{\mathcal{O}}(u)$ by (4). If $\mathcal{O}' = \{u_1', u_2'\}$, it is not difficult to verify the formulas

$$\begin{array}{ll} \phi_{\mathcal{O}'}(u) = \phi_{\mathcal{O}}(u) + \phi_{\mathcal{O}'}(u_1) & \text{if } \mathcal{O} \text{ and } \mathcal{O}' \text{ similarly oriented ,} \\ -\phi_{\mathcal{O}'}(u) = \phi_{\mathcal{O}}(u) - \phi_{\mathcal{O}'}(u_1) & \text{if } \mathcal{O} \text{ and } \mathcal{O}' \text{ oppositely oriented .} \end{array} \tag{6}$$

If u, v are unit or null vectors, and $\mathcal{O}$ is an orthonormal set, define $\phi_{\mathcal{O}}(u, v) = \phi_{\mathcal{O}}(u) - \phi_{\mathcal{O}}(v)$. It follows from (6) that $\phi_{\mathcal{O}}(u, v)$ depends only on the orientation of $\mathcal{O}$. Thus define $\phi(u, v) = \phi_{\mathcal{O}}(u, v)$ where $\mathcal{O}$ is any positively oriented orthonormal basis of M^2. If $u_1, \cdots, u_n$ are unit or null vectors we have the following formulas

$$\phi(u_1, u_2) = -\phi(u_2, u_1) \ , \tag{7}$$

$$\phi(u_1, u_2) + \phi(u_2, u_3) = \phi(u_1, u_3) \ , \tag{8}$$

$$\phi(u_1, u_2) + \cdots + \phi(u_{n-1}, u_n) + \phi(u_n, u_1) = 0 \ . \tag{9}$$

Formula (9) is the simplest case of the Gauss-Bonnet theorem. The corresponding statement in the Euclidean plane is that the exterior angles of a polygon sum to 2π.

2. General case

Throughout this section M will denote a Minkowski surface, i.e., an abstract surface with each tangent plane a Minkowski plane. Attention will be restricted to a region of M oriented by a frame field $\{E_1, E_2\}$. The following notation will be used. A general reference for the Euclidean case is [2, Chapter 7].

The dual 1-forms θ_1, θ_2 are defined by $\theta_i(E_j) = \langle E_i, E_j \rangle$. The connection forms ω_{ij} are defined by the equations

$$d\theta_1 = \omega_{12} \wedge \theta_2 , \quad d\theta_2 = \omega_{21} \wedge \theta_1 , \quad \omega_{12} = \omega_{21} ,$$

where d denotes the exterior derivative, and $\wedge$ the wedge or exterior product. The "area form" is "dM" $= \theta_1 \wedge \theta_2$; this form depends only on the orientation of $\{E_1, E_2\}$. The Gaussian curvature K is defined by the formula

$$d\omega_{12} = -K\theta_1 \wedge \theta_2 .$$

The covariant derivative in the direction of the tangent vector v is denoted by ∇_v. Recall that its action on a vector field $Y = y_1E_1 + y_2E_2$ is given by

$$\nabla_v Y = (v[y_1] + y_2\omega_{21}(v))E_1 + (v[y_2] + y_1\omega_{12}(v))E_2 , \tag{10}$$

where $v[f]$ denotes the directional derivative of the function f in the direction v.

Let $\alpha\colon I \to M$ be a continuously differentiable curve parameterized with respect to proper time. Let $T(s) = \alpha'(s)$. Then T is a unit or null vector field along α. If $T(s) = a(s)E_1(\alpha(s)) + b(s)E_2(\alpha(s))$, set $N(s) = bE_2 + aE_1$ along α, and define $g(s)$ by

$$\nabla_{\alpha'} T = gN .$$

At each point of $\alpha(s)$ define $\phi(s)$ by

$$\phi(s) = \phi_{\{E_1, E_2\}}(T(s)) .$$

Using (10) we immediately generalize Theorem 1 to

Theorem 2. *Let $\alpha\colon I \to M$ be a differentiable curve parameterized with respect to proper time and having image contained in a region oriented by the frame field $\{E_1, E_2\}$. Then*

$$g(s) = d\phi/ds + \omega_{12}(\alpha'(s)) .$$

Thus the acceleration measured by an observer riding with α breaks into two parts. The term $\phi'(s)$ is due to motion relative to the frame field $\{E_1, E_2\}$, and the term $\omega_{12}(\alpha')$ is due to the acceleration in the frame field itself. Notice that α is a geodesic of M if and only if $g \equiv 0$. If M is a submanifold of a higher dimensional space, then g gives that component of acceleration in the larger space which is tangent to M. The corresponding Euclidean concept is that of *geodesic curvature*.

Theorem 3 (*Gauss-Bonnet formula*). *Let R be a region in the plane, and $X\colon R \to M$ a restriction of a coordinate patch mapping. Let $X[R]$ lie in a region oriented by the frame field $\{E_1, E_2\}$, and the boundary of $X[R]$ be given by $\partial X = \sum_1^n \alpha_i$ where $\alpha_i\colon [a_i, b_i] \to M$ is a continuously differentiable curve parameterized with respect to proper time. Assume $\alpha_{i+1}(a_{i+1}) = \alpha_i(b_i)$ for $i \leq n-1$, and $\alpha_1(a_1) = \alpha_n(b_n)$. Set $\phi_{i,i+1} = \phi(\alpha'_{i+1}(a_{i+1}), \alpha'_i(b_i))$ for $i \leq n-1$, and $\phi_{n,1} = \phi(\alpha'_1(a_1), \alpha'_n(b_n))$. Then*

$$\iint_X KdM + \int_{\partial X} g + \phi_{12} + \cdots + \phi_{n-1,n} + \phi_{n,1} = 0 \,.$$

Proof. By Stokes theorem

$$\iint_X d\omega_{12} = \int_{\partial X} \omega_{12} \,.$$

Since $d\omega_{12} = -K\theta_1 \wedge \theta_2 = -KdM$ it is sufficient to evaluate

$$\int_{\partial X} \omega_{12} = \textstyle\sum_1^n \displaystyle\int_{\alpha_i} \omega_{12} \,.$$

To evaluate a typical term of this integral, apply Theorem 2 to get

$$\begin{aligned}\int_{\alpha_i} \omega_{12} &= \int_{a_i}^{b_i} \omega_{12}(\alpha'_i(s)ds = \int_{a_i}^{b_i} g(s)ds - \int_{a_i}^{b_i} \frac{d\phi}{ds} ds \\ &= \int_{\alpha_i} g + \phi(a_i) - \phi(b_i) \,.\end{aligned}$$

Since by definition we have $\phi_{i,i+1} = \phi(a_{i+1}) - \phi(b_i)$ for $i \leq n-1$ and $\phi_{n,1} = \phi(a_1) - \phi(b_n)$, summing the last formula gives the desired result.

Remark. In the notation of § 1, $\phi_e(u) = \phi_e(-u)$ for any unit or null vector u. This means that the direction in which each boundary curve is traced affects only the integral of g in Theorem 3. If the boundary curves are geodesics of M, then $g \equiv 0$ and this integral drops out.

3. Example-a Doppler formula

Suppose that a photon is emitted at a point A in the space-time of general relativity and observed at a point B. Let α_1 be the space-time trace of the photon from A to B, which is assumed to be a geodesic. Let α_2 be the space-time geodesic which the source would follow if unaccelerated. Let β be the space-time trace of the observer. Let α_3 be a spacelike geodesic which

i) cuts β orthogonally at B and ii) intersects α_2 at some point C.

The curve α_3 will exist if the region under consideration lies in a sufficiently small geodesic neighborhood of A. It is the intersection of two geodesic submanifolds, the first of which is the Euclidean 3-manifold of all geodesics through B orthogonal to β, i.e., that portion of space-time which the observer calls space at the instant when he observes the photon. The second submanifold is Minkowski surface of all geodesics emanating from A and tangent to the plane of tangent vectors spanned by α_1' and α_2' at A. Let Δ denote the section of this latter manifold bounded by the curves α_i. Then Theorem 3 gives the formula

$$\iint_{\Delta} KdM + \phi_{12} + \phi_{23} + \phi_{31} = 0 . \tag{11}$$

By (7), $\phi_{12} = -\phi_{21}$ and so $\phi_{12} = -(\ln \nu_e + a)$ by the remarks following Theorem 1 where ν_e is the frequency of the emitted photon. By (8), we have

$$\phi_{31} = \phi(\alpha_3', \alpha_1') = \phi(\alpha_3', \beta') + \phi(\beta', \alpha_1') = \phi(\beta', \alpha') = \ln \nu_a + a ,$$

where ν_a is the frequency of the photon observed at B. (Since α_3 is orthogonal to β one finds $\phi(\beta', \alpha_3') = \ln 1 = 0$.)

The pair β', α_3' is a frame at B. Parallelly translate this frame along α_3 to C. Since α_3 is a geodesic, the resulting frame is of the form $\{u, \alpha_3'\}$ and it is the frame at C which is "at rest" with respect to the observer at the event B. Thus by (7) and (8)

$$\phi_{23} = \phi(\alpha_2', \alpha_3') = -\phi(u, \alpha_2') + \phi(u, \alpha_3') = -\tanh^{-1} v = \frac{1}{2} \ln \left\{\frac{1-v}{1+v}\right\} ,$$

where v is the velocity of the unaccelerated source with respect to the observer at the moment when the photon is observed.

Substituting these values for the ϕ_i into (11) gives

$$\nu_e = \nu_a \sqrt{\frac{1-v}{1+v}} \exp \left\{\iint_{\Delta} KdM\right\} .$$

In the case where Δ is flat ($K \equiv 0$), this reduces to the usual formula from special relativity.

References

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