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A Reliability Model Applied to Emergency Service Vehicle Location

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A RELIABILITY MODEL

APPLIED TO EMERGENCY SERVICE VEHICLE LOCATION *

By

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ABSTRACT

This article proposes a reliability model for the emergency service vehicle location problem. Emergency services planners must solve the strategic problem of where to locate emergency services stations and the tactical problem of the number of vehicles to place in each station. We view the problem as one of optimizing the reliability of a system, where system failure is interpreted as the inability of a vehicle to respond to a demand call within an acceptable amount of time. Our model handles the stochastic problem aspects in a more explicit way than previous models in the literature. Based on a reliability bound on the probability of system failure, we derive a 0-1 integer programming (IP) optimization model. To solve it, we propose valid inequalities as a preprocessing technique to augment the IP and solve the IP using a branch-and-bound procedure. Our computational results show that the preprocessing techniques are highly effective. We feel that the reliability model should have applications beyond this context and hope that it will lead to ideas for similar optimization models for designing other systems.

1. Introduction

Emergency services, such as emergency medical service (EMS) and fire rescue, are a major concern of most regional and urban planners. Emergency services planners must solve the strategic problem of where to locate emergency services stations and the tactical problem of the number of vehicles to place in each station. Significant research attention has been directed toward these problems; see, for example, Toregas et al. (1971), Church and ReVelle (1974), Chapman and White (1974), Daskin (1983), and ReVelle and Hogan (1989). Furthermore, many models have been applied in practice and have led to practical successes; see, for example, Plane and Hendrick (1977), Schilling et al. (1979), and Eaton et al. (1985). In this paper we present an optimization model for determining the location of stations and the number of vehicles to place at each station. Our model handles the stochastic problem aspects in a more explicit way than previous models. In fact, the perspective we view the problem is one of optimizing the reliability of a system, where system failure is interpreted as the inability of a vehicle to respond to a demand call within an acceptable amount of time. This reliability model is robust in the sense that the reliability constraints are established independent of the vehicle dispatching rules. We feel that the reliability perspective should have applications beyond this context and hope that it will lead to ideas for similar optimization models in the context of designing reliable systems. A 0-1 integer programming optimization problem is derived from the reliability model. To solve it, we propose valid inequalities as a preprocessing technique to augment the IP and solve the IP using a branch-and-bound procedure. Our computational results show that the preprocessing techniques are highly effective.

Nearly all previous work in this area has involved the variants or extensions of the set covering model by Toregas et al. (1971). The basic inputs to these models are:

1.) A set of demand points: each demand point represents a geographic area to which service must be provided.

2.) A set of potential vehicle locations: associated with each location is the set of demand points that can be "covered" from that location. In this context a demand point is covered by the vehicle location if it is within a specified distance, and hence response time, of the location.

One of the early models was the set covering location model (SCLM), proposed in 1971 by Toregas et al. (1971). The objective function is to minimize the number of facility stations (and/or vehicles) required; and the constraints, one for each demand point, stipulate that each demand point be covered by at least one chosen station that is within the specified response time of the demand point. It was typically formulated as follows,

$$\text{Min} \quad \Sigma_{j \in J} X_j$$

s.t.

$$\Sigma_{j \in cov(i)} X_j \geq 1 \qquad \text{for all } i \in I,$$

$$X_j \in \{0,1\} \qquad \text{for all } j \in J,$$

where

$$X_j = \begin{cases} 1 & \text{if facility station j is chosen,} \\ 0 & \text{otherwise;} \end{cases}$$

J = the set of eligible facility stations;

I = the set of demand points;

COV(i) = the set of stations that are within the specified response time of demand point i.

It had been contended by some that the constraints in SCLM were too conservative in that it might not be necessary to require coverage of all demand points, especially considering the common existence of peripheral demand points with diminutive demand. To circumvent this alleged flaw, Church and ReVelle (1974) pioneered another line of modeling, the maximum covering location model (MCLM), in which the objective function maximizes the sum of covered demands, and in which the constraints limit the total number of stations to a fixed value while allowing some demand points not to be covered. Variants and applications of the model followed since then.

MCLM has been employed extensively in analyzing locations for emergency service facilities. MCLM has been successfully applied in practice by Schilling et al. (1979) and Eaton et al. (1985). These successes have established SCLM together with its variants and extensions as a widely accepted framework for the emergency facility location. The long list of variants and extensions of the covering models is described by Daskin et al. (1988).

With both SCLM and MCLM, the issue of system congestion remains open. That is, when eligible vehicles are all out for services elsewhere, an arriving demand call must be put on hold until a vehicle becomes available. As a result of this, both types of covering models, deterministic in nature, tend to oversimplify and thus overestimate the performance by a given number of stations and their vehicles. In light of this weakness, Chapman and White (1974) derived a stochastic version of the set covering model, while Daskin (1983) explored a stochastic version of the maximum covering model, which he named as the maximum expected covering location model (MECLM). Both models were built on the assumption that the probabilities that vehicles or stations were busy could be determined in advance.

As a variant of MECLM, ReVelle and Hogan (1989) proposed another stochastic model, the Maximum Availability Location Model (MALM), in which each constraint guarantees that the probability that demand point i receives service within an acceptable time be no less than a required value. Assuming independence among eligible vehicles and estimating the average busy fraction of the service vehicles, they defined the following constraints:

$$1 - \Pi_{j \in cov(i)} r_j^{X_j} \geq \alpha \text{ , for all demand points i,} \qquad (1)$$

where r_j denotes the busy fraction of a vehicle at site j, X_j equals 1 if there is a facility at site j and 0 if otherwise, and α is the required probability. After simple algebra and taking logarithms, these constraints become linear and lead to a 0-1 integer linear program. MALM bears strong similarities to our model. However, we do not start with estimates of the r_j but rather directly model the source of randomness, namely the service calls originating from each demand point. We then show how a constraint similar to (1) can be derived based on bounding arguments. Specifically, the left hand side of our version of (1) is an upper bound on the probability that demand point i will not receive an immediate response when a call arises. In effect, our constraints guarantee that each demand point will receive at least the required level of service. By deriving these constraints from more fundamental information we allow for the r_j's to be derived from natural problem data which is especially important for proposed sites. Furthermore, this approach opens up the possibility for the derivation and use of tighter upper bounds on the required probabilities.

This paper is organized as follows. Section 2 describes how the reliability concepts may be used to model the location problem. Section 3 constructs the reliability constraints using a product form bound which then leads to an integer linear program. Section 4 gives the preprocessing methods we use to strengthen the integer program. Section 5 presents our computational results.

2. A Reliability Model for the Emergency Services Location Problem

Our model starts with a set of demand points to which service must be delivered by the emergency vehicles and a set of location candidates to be used as the vehicle stations. For an illustration of the location problem, see Figure 1. It is assumed that the location candidates have been predetermined due to some previous studies. (From now on, we use the terms "location" and "station" interchangeably in referring to a site where vehicles are located.) In addition, we assume that each potential location has an associated coverage area. Consider, for example, the EMS Act of 1973, which stipulated that 95% of service requests be met within 30 minutes for rural areas and 10 minutes for urban areas. If a restricted travel distance or response time is specified, a station is associated with some demand points it can feasibly service or "cover" within the restriction. The area which is within the restricted traveling distance from a station is called the <u>coverage area</u> of the station and vehicles located at a station are called the <u>feasible vehicles</u> to its associated demand points. We assume that one vehicle trip serves only one demand call, as is the case for most emergency services. In a general service trip, a vehicle travels to a demand point, provides the on-scene services, possibly followed by a tour to a nearby hospital in case of medical services, and returns to its home station.

We now present our reliability model. While presenting this model we will assume that locations and vehicles have already been chosen and address the issue of how reliable is a particular solution. In the next section we address how to embed this model into an optimization model. Unlike previous models, we model directly the inherent source of randomness, that is, the service requests. Specifically, we assume service requests are randomly generated from the demand points according to a certain probability distribution. When a demand call occurs it is assigned to a feasible vehicle, if one is available. The vehicle assigned to the

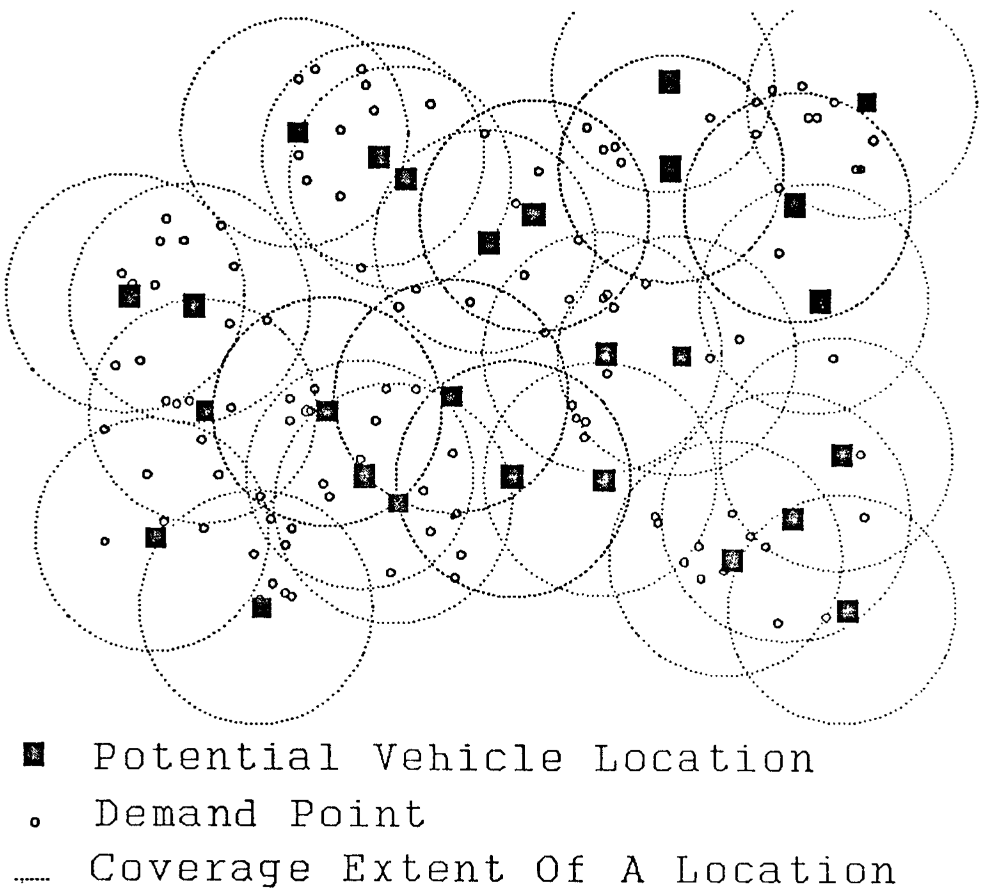


Figure 1.

A Geographical Illustration Of The Location Problem

service call remains busy and cannot service other calls for the (random) length of time needed to service the call. After the call is completed the vehicle once again is available at its home station. Any operating system will employ specific rules for assigning a feasible vehicle to each new call. Our analysis is independent of the particular rules employed. We note that we do not explicitly model what is done and how the system performs when no feasible vehicles are available to service a call.

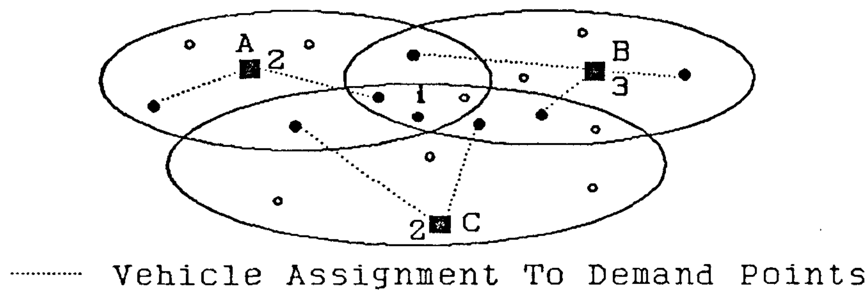


Figure 2.A.

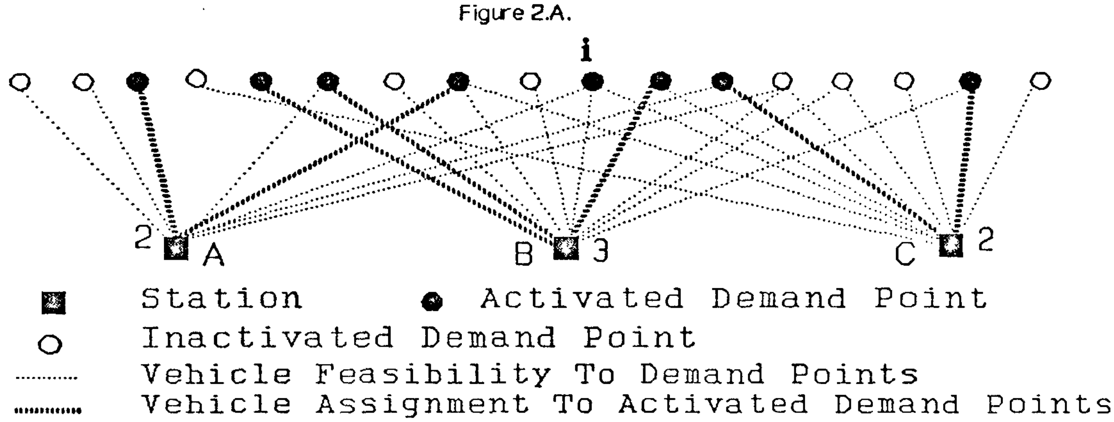


Figure 2.B.

Figure 2.

An Illustration For A Three-Station Coverage

Our model is built upon the requirement that each individual demand point achieve a desired level of service. **In order to assess the level of service a demand point attains, an individual reliability system will be defined for each demand point.** A global reliability system could have been defined for the entire geographical area, as is done in some of the stochastic models in the literature. However, a solution satisfying only a global system performance criterion may balance a lower service level in one area with a higher service level in another. By defining an individual system for each demand point we give the decision maker more control and eliminate certain solutions that could easily be judged unfair.

The performance criterion we use to assess an individual system is based on the primary concern of the system planner: if there is a possibility that a demand call may not receive an immediate response, then what is the probability of this event occurring? In other words, what is the fraction of demand calls arising at a particular demand point that cannot be immediately assigned to a feasible vehicle? The formal definition of system operation is:

Definition 1. An individual reliability system at a demand point i is said to operate with respect to a particular demand call if, when the demand call occurs, there is a feasible vehicle available to service it.

If we denote as F_i the event that a demand call arises from a demand point i, and if we let E_i be the event that a feasible vehicle is available for its service, then the conditional event $(E_i | F_i)$ characterizes the system operation. The conditional probability $Pr[E_i | F_i]$ is hence the probability that the system operates. We call this probability the reliability of the demand point i, denoted as r_i. If when a demand call occurs there are no feasible vehicles available for its service, the system does not perform its required function, and, thus, we say that the system fails. We call the probability that the system fails as the failure probability of the demand point, denoted as q_i.

The system performance standard we wish to enforce is that the failure probability be within an upper limit. Indeed, imposing an upper limit for the failure probability is equivalent to guaranteeing a lower limit for the reliability. For example, the failure being less than or equal to 0.05 would insure that 95% of the calls have at least one feasible vehicle available for immediate response to their service requests. From now on, we intend to dwell on the upper limit. Due to the difficulty of computing the exact failure probability, we wish to develop an upper bound for it, and impose an upper limit on the upper bound. This upper limit will in turn be valid for the failure probability itself.

To develop the upper bound, we first derive a necessary condition for the failure of an individual system. We will base our discussion on an individual system as illustrated in Figure 2.A where the demand point i is covered by three stations A, B, and C. Suppose that 2 vehicles are housed at station A, 3 vehicles at station B, and 2 vehicles at station C. If a call which occurred from demand point i found no feasible vehicles, then the 7 feasible vehicles must have been servicing other demand calls arising from the combined coverage area of the three stations. Specifically, there must be feasible assignments from seven uncompleted service calls arising in the combined coverage areas to vehicles located at the three stations. More generally, if COV(i) is a set of stations feasible to demand point i and K_j is the number of vehicles housed at station j, then a demand call arising at demand point i could not be serviced if the $\Sigma_{j \in cov(i)} K_j$ vehicles were assigned to the uncompleted calls arising in the combined coverage area of the locations in COV(i). As Figure 2.B illustrates, we can view the possible assignments in terms of a bipartite graph. One node set consists of the demand points in $U_{j \in cov(i)}$ PT(j), where PT(j) is the demand points covered by station j, and the other consists of the stations in COV(i). An arc is drawn between demand point k and station j if $k \in PT(j)$. Our bound on the probability that no vehicles are available for demand point i will be based on a necessary condition that such a demand point to vehicle assignment exists. The required necessary condition is actually equivalent to a necessary and sufficient condition due to Hall (1935), for the existence of a feasible solution to a certain transportation problem:

> We are given a bipartite graph (N_1,N_2,A) and associated supplies s_k for each $k \in N_1$ and demands d_j for each $j \in N_2$. For any $S \subseteq N_2$, define $SUP(S) = \Sigma_{k:(k,j)\in A, j\in S}\, s_k$. Then, there exists a feasible solution to the associated transportation problem if and only if for all $S \subseteq N_2$, $SUP(S) \geq \Sigma_{j\in S}\, d_j$.

Before giving our necessary condition we define,

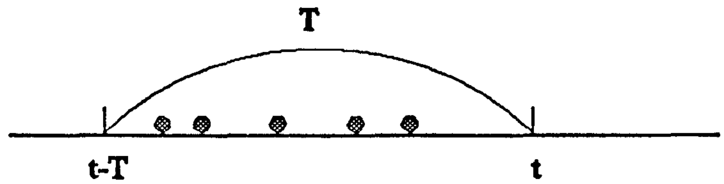
A(S) = the number of demand calls that arose in $\cup_{j\in S}$ PT(j) and that are still active at a point in time, say, t.

The necessary condition is,

Proposition 1: If there is no vehicle available to service a call arising from demand point i at time t, then $A(S) \geq \Sigma_{j\in S}\, K_j$ for all $S \subseteq COV(i)$.

proof: The proposition follows from Hall's Theorem and the fact that the assignment illustrated in Figure 2.B must exist. ||

In order to make use of this condition we require information on the duration of calls so that we can associate probabilities with A(j) values. The simplest model would be one in which vehicle service trips require a constant time duration, say, T. Thus, A(j) would simply be the number of demand calls that arose in PT(j) during time interval (t-T,t] as is illustrated in Figure 3.



t The time when a demand call just arises at the demand point i

T A time interval of fixed length

⊗ Demand calls that arose between t and t-T

Figure 3.

An Illustration Of The Time Frame

Our ultimate objective is to derive an upper bound on the probability that a call does not receive immediate service and to embed this upper bound into an integer program. Such a simple bound can be obtained even when service times are not constant when we set T equal to an upper bound on service time. We contend that since there are stringent requirements to complete service calls quickly, such an upper bound can usually be derived in practice. We can now state our bound. Following the discussion above, the bound is stated in terms of the number of calls that have occurred in the previous interval of size T rather than in terms of active calls. We define,

D(S) = the number of calls that arose in $\cup_{j \in S}$ PT(j) during the time interval (t-T,t], INT(i) = the random event: $\cap_{S \subseteq cov(i)} D(S) \geq \Sigma_{j \in S} K_j$.

Proposition 2 gives our bound.

Proposition 2: If T is an upper bound on the duration of all service trips, then $q_i \leq Pr[INT(i)]$.

Proof: If T is an upper bound on the duration of a service trip, then $D(j) \leq A(j)$. The result now follows from Proposition 1. ||

Here, we emphasize that this bound holds irrespective of any specific vehicle assignment policy. In the next section, we will provide an integer optimization model which is constructed using a product form upper bound for Pr[INT(i)].

3. The Integer Program With Reliability Constraints

Having defined an appropriate reliability system, we now wish to use it to determine the ultimate emergency service locations and the number of vehicles for each chosen location. In particular, we will define an optimization problem which includes constraints that address the requirement of $q_i \leq 1-\beta$, which is equivalent to $r_i \geq \beta$, for each demand point $i \in I$. That is to say, we intend that each r_i has to achieve the same specified level β, as each demand point is equally important. In particular, we wish to derive a computationally efficient upper bound for Pr[INT(i)] in order to achieve a model that we can expect to solve in a reasonable amount of computing time. The upper bound we will establish is in a product form:

$$PROD(i) = \Pi_{j \in cov(i)} Pr[\, D(j) \geq K_j \,].$$

The essential chain of implication we will have is:

$$\Pr[\text{system fails}] = q_i \leq \Pr[\text{INT}(i)] \leq \text{PROD}(i). \tag{2}$$

For our model, we feel that Poisson distribution, in particular, is the most reasonable to use for the distribution of demand calls. This is based on the common supposition that the interarrival times between consecutive service calls arising at a demand point are independent memoryless random variables. As a result of the independence among demand points, the demand calls arising from a coverage area is also Poisson with arrival rate equal to the sum of the arrival rates of the respective demand points in the coverage area.

In establishing the last inequality of (2), we find that the new better than used (NBU) property in probability theory (see, for example, Barlow and Proschan 1975) plays a significant role. Informally, the NBU property says that when a stochastic system has already survived s time units, the probability that it will survive t additional time units is no larger than the probability that it will survive t time units if it has not survived any. The following definition is given.

Definition 2. A nonnegative random variable X has the new better than used (NBU) property if

$$\Pr[X \geq s+t \mid X \geq s] \leq \Pr[X \geq t] \qquad \text{for } s>0,\ t>0. \tag{3}$$

It has been well known in the literature that the equality of (3) holds if and only if X is the exponential random variable in continuous case, or the geometric random variable in discrete case. When the equality holds, it is what has been called the memoryless property.

Traditionally, the NBU property has been particularly identified for the continuous distributions. In the case of discrete distribution, however, the discrete NBU property can be stated as follows. When a stochastic system has already survived i arrivals (e.g. shocks or years of aging), the probability that it will survive j additional arrivals is no larger than the probability that it will survive j arrivals if it has not survived any. Namely, $\Pr[X \geq i+j \mid X \geq i] \leq \Pr[X \geq j]$ for nonnegative integers i and j. In the Appendix we establish:

Proposition 3. A Poisson random variable $N(t)$ for a fixed $t > 0$ satisfies the discrete NBU property.

Although this result has been implicit in the literature, we prove it in the Appendix since it does not seem to have been explicitly stated elsewhere. The importance of this result for our model results from the following Theorem.

Theorem 4. If the demand calls are generated according to a NBU distribution, then $\Pr[INT(i)] \leq PROD(i)$ for a demand point i.

Proof: See Ball and Shanthikumar (1992). ||

Corollary 5. If calls arise from each demand point according to a Poisson distribution, then $\Pr[INT(i)] \leq PROD(i)$ for a demand point i.

Proof: The result follows from Proposition 3 and Theorem 4. ||

Almost all earlier models in the literature simply assumed that neighboring stations function independently. In spite of the fact that interactions among neighboring stations always exist, they hardly offered justification for the assumption. With the independence assumption, they dealt directly with product forms of probabilities based on the randomness of stations, and used the estimates of these station-level probabilities as it is difficult to compute them exactly. While our PROD(i) amounts to the effect that the PT(j)'s for $j \in COV(i)$ do not overlap, that is, the demand calls arising from within every PT(j) are independent, we do not assume the independence in arriving at PROD(i). Although both approaches lead to the use of product forms, Theorem 4 and Corollary 5 yield justifications for the use of PROD(i) and in addition provides a way of computing it from basic problem data.

Based on the reliability analysis given in the above, we now provide an integer programming formulation, denoted as Rel-P, as a way to solve the emergency service vehicle location problem. Rel-P uses the decision variables defined as follows.

$$X_{jk} = \begin{cases} 1 & \text{if k vehicles are stationed at location j,} \\ 0 & \text{otherwise,} \quad \text{for } j \in J \text{ and } k=1,2,\ldots, M_j, \end{cases}$$

where M_j is the maximum number of vehicles that can be located at j. For each demand point i, we wish that the restriction PROD(i) $\leq 1-\beta$ holds. To enforce this restriction we require that

$$\Pi_{j \in cov(i)} \Pi_{1 \leq k \leq Mj} [P(D(j) \geq k)]^{Xjk} \leq 1-\beta \quad \text{for all } i \in I, \tag{4}$$

$$\Sigma_{1 \leq k \leq Mj} X_{jk} \leq 1 \quad \text{for all } j \in J. \tag{5}$$

Constraints (4) describe that an upper limit, $1-\beta$, is imposed on the failure probability of each demand point by placing the limit on the upper bound of its failure probability. (5), sometimes called the special order constraints, states that each location j either is not chosen or is decided to house exactly one number, k, of vehicles.

By taking the logarithm function on both sides and changing the signs, constraint (4) can be transformed into a linear constraint as follow:

$$\Sigma_{j\in cov(i)} \Sigma_{1\leq k\leq Mj} \; a_{jk} X_{jk} \quad \geq \quad b_i \qquad \text{for all } i\in I \tag{6}$$

where

a_{jk} = - log[P(D(j) ≥ k)],

b_i = - log (1-β),

and both a_{jk} and b_i are positive numbers. The complete IP formulation is:

(Rel-P) Min $\Sigma_{j\in J} \Sigma_{1\leq k\leq Mj} W_{jk}X_{jk}$

s.t.

(5), (6)

$X_{jk} \in \{0,1\}$ for all j,k,

where the costs W_{jk} indicate the cost of housing k vehicles at station j, which would include both the "fixed" costs of opening a new station (or renovating an existing station) and the variable costs associated with the number of vehicles.

4. IP Preprocessing Techniques

We propose to solve the model given in the previous section using an "off the shelf", linear programming (LP) based branch-and-bound code. It is well known that the performance of such branch and bound algorithms depends critically on the quality of the bounds produced by the LP relaxation. In this section we present a set of preprocessing techniques for improving the quality of the bounds. Our motivation for this approach comes from the highly successful application of similar techniques by Crowder, Johnson and Padberg (1983).

In general, when one drops the integer restrictions from an integer programming (IP) formulation, the polyhedron that results strictly contains the convex hull of all the integer points. If one could write down a linear description of the convex hull of integer points, then a solution to the associated LP would solve the IP. The objective of preprocessing techniques is to "shrink" the LP feasible region so that it is closer to the convex hull of integer points. By doing this the LP bound is improved and there is a greater possibility that the LP will produce an integer solution. Valid inequalities are typically generated dynamically by solving a separation problem; see Nemhauser and Wolsey (1988). That is, an LP would be solved and a valid inequality would be identified and added to the LP. The LP would be resolved and the process would iterate several times in this manner. Our approach is to selectively define a small number of valid inequalities that can be added to the original formulation. While this approach limits the potential impact of the use of valid inequalities, it does not require developing an algorithm to solve the separation problem, which is in general a difficult problem. Thus, it is probably most accurately to view our approach as a process of formulation strengthening. This preprocessing approach, however, has its theoretical basis. Some recent developments in integer programming indicated that the successes of the general polyhedral approaches to solving integer programming problems depend critically on the strong valid inequalities that are found and added to the problems. This somehow suggests that strong valid inequalities need not be generated dynamically as done in the general cutting plane algorithms. As will be shown by our computational experience, the valid inequalities added to Rel-P provide decisive efficiencies for the branch-and-bound procedure, even though they are generated at the beginning step, rather than during the course, of the algorithms. As the efficiency of LP optimizers have constantly improved, solving a large LP as a result of adding many valid inequalities simultaneously is hardly the computational bottleneck.

We employ two classes of preprocessing techniques. The first is coefficient round-down. This method involves changing the coefficients of existing constraints. The second class involves defining valid inequalities which are new inequalities that are added to the problem. In general, a method which alters an existing inequality or produces a new one can be shown to be correct if it can be shown that the resultant inequality must be satisfied by all integer solutions.

4.1. Coefficient Round-Down

Often, the fact that variables take on either 0 or 1 values renders the possibility of rounding down the coefficients in the constraints.

For the following constraint:

$$\Sigma_j \Sigma_k a_{jk} X_{jk} \geq b_i \text{ , where } X_{jk} \in \{0,1\}, \text{ and all } a_{jk} \geq 0,$$

if any $a_{jk} > b_i$, then replacing a_{jk} by b_i in the constraint will not affect the set of integer feasible solution, whereas in general some non-integer solution may be eliminated. For instance, $2X_1 + 3X_2 + 7X_3 + 9X_4 \geq 6$ can be rounded down to $2X_1 + 3X_2 + 6X_3 + 6X_4 \geq 6$. Any feasible integer point satisfying one constraint must also satisfy the other constraint.

4.2. Valid Inequalities

Formulation-induced valid inequalities usually are capable of augmenting the IP in the sense of trimming off unnecessary portion of the underlying LP polytope. We now propose classes of valid inequalities which are identified from each individual constraint (6).

4.2.1. Cover Inequality

In order to satisfy a constraint

$$\Sigma_{j\in J}\, a_j\, Y_j \geq b \tag{7}$$

where $Y_j = 0$ or 1, and b, $a_j > 0$, there must be at least one Y_j that receives the value 1. Furthermore, if there exists a subset J' such that $\Sigma_{j\in J'}\, a_j < b$ then it is clear that at least one Y_j with $j \in J\text{-}J'$ must be 1. In other words, the following is a valid inequality.

$$\Sigma_{j\in J\text{-}J'}\, Y_j \geq 1 \tag{8}$$

In fact, this idea is a simple reinterpretation of the corresponding idea of the well known <u>cover inequality</u> for the knapsack problem. For more valid inequalities along this line, see, for instance, Padberg (1979) and Nemhauser and Wolsey (1988).

The constraint $\Sigma_{j\in cov(i)}\Sigma_{1\leq k\leq Mj}\, a_{jk}\, X_{jk} \geq b_i$ from our model has the same form as (7), however, the additional constraint $\Sigma_{k=1,Mj}\, X_{jk} \leq 1$ allows us to derive an inequality stronger than (8). One way of doing that is illustrated in Proposition 6.

Proposition 6. Let a" = Max {a': $b_i > \Sigma_{j\in cov(i)}\, \mathrm{Max}_k\{a_{jk} : a_{jk} \leq a'\}$}. Then

$$\Sigma_{jk:\, ajk>a''}\, X_{jk} \geq 1 \tag{9}$$

is a valid inequality.

proof: a" is defined so that if the largest a_{jk} less than or equal to a" is chosen for each j then the sum obtained is less than b_i. Thus, since only one X_{jk} can be set to one for each j, some other X_{jk} must be set to one. This is exactly what the valid inequality implies. ||

To illustrate the difference between (8) and (9) consider the following constraint:

$$2X_{11} + 5X_{12} + 6X_{13} + 8X_{14} + 3X_{21} + 4X_{22} + 7X_{23} + 10X_{24} \geq 13 .$$

If J' is defined as $\{(j,k): a_{jk} \leq 4\}$, then (6) yields a cover inequality (8):

$$X_{12} + X_{13} + X_{14} + X_{23} + x_{24} \geq 1 ;$$

whereas Proposition 6 with a" = 6 produces a stronger cover inequality (9):

$$X_{14} + X_{23} + X_{24} \geq 1 .$$

Note that (9) is a special cover inequality stronger than (8).

In our computational testings, we used (9) as the cover inequalities added to the test problems.

4.2.2. Clip Inequality

Note that in both real problems and in randomly generated problems it is not unusual to encounter demand points i with $|COV(i)|=2$. With such a 2-station reliability constraint, it is thus possible to derive a special class of valid inequalities.

Proposition 7. Let i' be such that $COV(i') = \{j_1,j_2\}$ and suppose $b_{i'} > a'_{j1} = Max_k\{a_{j1,k}\}$. Then,

$$\Sigma_{k\in S'} X_{j2,k} \geq 1$$

is a valid inequality where $S' = \{k : a_{j2,k}+a'_{j1} \geq b_{i'}\}$.

Proof: If $a'_{j1} < b_{i'}$ then it is clear that at least one $X_{j2,k}$ must be one. Furthermore, the $X_{j2,k}$ that is one must have an $a_{j2,k}$ value at least large enough to achieve that $b_{i'}$ value when combined with a'_{j1}. ||

We call this type of valid inequality: $\Sigma_{k\in S} X_{j2,k} \geq 1$, a clip inequality. In fact, together with a constraint of (5): $\Sigma_{1\leq k\leq Mj} X_{jk} \leq 1$ for $j=j_2$, a clip inequality can be represented as $\Sigma_{k\in S} X_{j2,k} = 1$. This in turn implies that in Rel-P a constraint of (5) for $j=j_2$ is indeed satisfied at equality and thus can be replaced by $\Sigma_{1\leq k\leq Mj} X_{jk} = 1$ for $j=j_2$.

As an example, let us consider the 2-station constraint:

$$2X_{11} + 5X_{12} + 6X_{13} + 8X_{14} + 3X_{21} + 4X_{22} + 7X_{23} + 10X_{24} \geq 13.$$

The fact that the respective largest coefficients of the X_{1k}'s and X_{2k}'s are less than the right hand side gives rise to two clip inequalities: $X_{23} + X_{24} = 1$ and $X_{12} + X_{13} + X_{14} = 1$, as Proposition 7 affirms.

4.2.3. Constraint Substitution

We developed one final technique that replaces constraints (6) for a given i with a completely new set of constraints. As such, this technique makes the methods described in Sections 4.2.1 and 4.2.2 unnecessary. This technique, however, is only practical for (5) and (6) when $|COV(i)|$ is small. Thus, in general, an approach that uses this technique for some demand points and the previous techniques for others is appropriate. The method employs concepts from blocking polyhedra.

The replacement inequalities are identified by what we define as "cutsets" for a reliability constraint (6). The cutsets are conceptually similar to the cutsets defined in reliability literature in that if all elements of a cutset are removed or "fail" to contribute, then it is impossible to satisfy the constraint. Our cutsets will always be defined to be minimal with respect to set inclusion.

Definition 4. Given i, a set H of a_{jk}'s is called a <u>cutset</u> relative to constraint (6) if it is a minimal set satisfying $\Sigma_{j\in COV(i)} \text{Max}_k\{a_{jk}: a_{jk} \notin H\} < b_i$.

To illustrate the cutset as defined, consider the following example.

Example 1. Consider the following instance of constraint (6) and its associated special ordered constraints (5).

$$2X_1 + 3X_2 + 5X_3 + 7X_4 + 3Y_1 + 4Y_2 + 8Y_3 + 9Y_4 + Z_1 + 5Z_2 + 7Z_3 + 10Z_4 \geq 18$$
$$X_1 + X_2 + X_3 + X_4 \leq 1$$
$$Y_1 + Y_2 + Y_3 + Y_4 \leq 1$$
$$Z_1 + Z_2 + Z_3 + Z_4 \leq 1$$

Let H = the coefficient set associated with X_3, X_4, Y_4, Z_3, and Z_4

= {5,7,9,7,10}.

Then, H is a cutset, since $\Sigma_j \text{Max}_k\{a_{jk}: a_{jk} \notin H\} = 3 + 8 + 5 = 16 < 18$.

Given i, a $j \in COV(i)$ and a cutset H we define $cp(H,j) = \text{Max}\{k: a_{jk} \notin H\}$. It follows that a cutset H is completely determined by $\{cp(H,j): j \in COV(i)\}$. We call the cp(H,j) cutting points. Clearly, any a_{jk} which is greater than or equal to b_i is always in every cutset, namely, is never considered as a cutting point. We adopt the convention that if a cutset H is such that for some $j' \in COV(i)$, $\text{Min}\{k: a_{j'k} \in H\} = 1$, then $cp(H,j') = 0$. Suppose that all the cutsets of (6) are enumerated for a given i. Then (6) can be replaced by the set of valid inequalities formed by the cutting points. These ideas are summarized by Proposition 8. For detailed analyses, see Ball and Lin (1992).

Proposition 8. For any i, the set of feasible integer solutions to (5) and (6) is the same as the set of feasible solutions to (5) and

$$\Sigma_{j \in cov(i)} \Sigma_{k:\, k > cp(H,j)} X_{jk} \geq 1 \quad \text{for all cutsets H derived from (6).} \qquad (10)$$

Proof: See Ball and Lin (1992). ||

Note that Proposition 8 holds for a general reliability constraint (6), regardless of the |COV(i)|. Ball and Lin (1992) also show that when |COV(i)|=2 the set of inequalities given in Proposition 8 actually define the convex hull of integer solutions to (5) and (6). Consequently, for this case we call the inequalities Convex Inequalities. When |COV(i)|≥3, they are called Blocking Inequalities due to their use of concepts of blocking polyhedra.

5. Computational Results

This section presents the computational results based on our IP model Rel-P. The purpose of this computational study is twofold: (1) to verify the effectiveness of the various preprocessing techniques proposed for solving the Rel-P, and (2) to explore the sensitivity of the model to changes in key data characteristics. The results show that the preprocessing techniques dramatically reduce the computation time required by branch and bound solution algorithms. The overall impact is that our model is very practical for problems whose constraint matrices are relatively "sparse" in terms of nonzero coefficients. Sparsity here means that each demand point is covered by a relatively small number of location candidates. Our sensitivity studies show that the planner can produce a variety of different desired solution outcomes by appropriate altering parameters. We also give experience with the application of our model to a frequently used test problem found in the literature.

Our computational study used two branch-and-bound codes: the MILP88 (Eastern Software Products, Inc. 1988) on an 8 MHZ Intel-286 DOS machine with no math co-processor and the MPSX/MIP/370 on an IBM 3081 VM/SP/CMS mainframe. The formulation strengthening techniques proposed earlier were applied to preprocess the problems. As a result, the branch-and-bound codes are able to solve the augmented IPs very efficiently. When incorporated with all the proposed valid inequalities, the problems which were put to the MPSX code were all solved in less than 5 CPU minutes (with most of them solved in less than 3 CPU minutes) while those which were run on the MILP88 code in the 8 MHZ IBM/PC were solved within 10 minutes. Throughout our computations, we used the MILP88 to solve problems 1, 2, 3, and 5 due to their smaller sizes. For larger problems, we used MPSX.

We report on the solution of 9 test problems. Eight of them are random problems whose sizes are commensurate with models in the literature. In a 100.0x100.0 geographical region all the demand points and potential locations were randomly generated according to a uniform distribution. The coverage area of each location candidate is assumed to be a circle. To avoid overly simplistic problems we require that each demand point be covered by at least two potential locations. Besides the eight random problems, our study includes a 55-node problem (each node being a demand point as well as a location candidate), which had been used for testing in several articles. See Daskin (1983) and Hogan and ReVelle (1984), for example.

For all the 9 problems, the maximum number of vehicles planned at each station is set to four; the unreliability level for each demand point is fixed for the same problem; and the "fixed" cost associated with a location (e.g., renovation or expansion cost for an existing station) is assumed to be the product of a constant and a vehicle's cost. Also, the problem parameters are experimentally selected so that the IP is feasible and difficult to solve. As noted earlier, each demand point generates calls independently according to a Poisson process. The Poisson arrival rate λ_i for demand point i for $i \in I$ was selected. Thus, D(j) for $j \in J$ also forms a Poisson process with arrival rate equal to $\Sigma_{i \in PT(j)} \lambda_i$, where PT(j) represents all the demand points covered by station j for $j \in J$.

In describing the data of a test problem, besides the problem size and the basic problem parameters, we also provide information regarding the "distribution" of the demand points in terms of the number of their covering location candidates. Given a Rel-P problem, it is important to know, for example, how many demand points are covered by 4 location candidates, 3 location candidates, and so on. This distribution is significantly related to the difficulty in solving the Rel-P. When a demand point is covered by, say, 4 candidate locations, we call it a 4-location demand point. In general, the more demand points which are covered by high number of locations, the more difficult it is to solve the Rel-P.

5.1. Preprocessing and Branch-And-Bound Computations

In verifying the effectiveness of the various preprocessing techniques proposed for solving the Rel-P, we used problems 1 - 7 as the test problems. We assess the branch-and-bound efficiencies based on the number of branches or nodes created during the branch-and-bound procedure and their CPU time on the respective machines. For each problem tested in this section, seven IP instances were input into the same branch-and-bound procedure: (1) plain IP, noted as **P**; (2) IP with the coefficient round-down, noted as **R**; (3) IP with Cover Inequalities generated and added to **R**, denoted as **RV**; (4) IP with Clip Inequalities generated and added to **R**, indicated as RL; (5) IP with both the Cover and Clip Inequalities added to **R**, i.e., **RVL**; (6) IP as a result of generating Cover, Clip, and the Convex Inequalities on **R**, marked as **RVLX**; and (7) IP obtained by including Cover, Clip, Convex, and the Blocking Inequalities onto **R**, represented as **RVLXB**.

The outcomes, reporting the additional efficiency gained as a result of adding the various valid inequalities in solving the problem optimally, were summarized in Table 1 - 7. In each table, "v(IP)" is the optimal objective value of the Rel-P, and the "C/W" represents the ratio between the "fixed" cost associated with a location and the cost of one vehicle unit. For each problem, we made two runs with each run being with a different C/W ratio in the objective function. The intention is to see to what degree the effectiveness of the various valid inequalities is affected by the C/W ratio. The first two columns, respectively, give the number of branches or nodes created and the amount of time used, during the course of the branch-and-bound procedure. The "v(LP)" column indicates the optimal objective values of the LP relaxations augmented by the various valid inequalities. Defining the "duality gap" as the quantity v(IP)-v(LP), we calculated the percentage reduction of the duality gap, noted as %RDG, by a preprocessing step i as follows: $\%RDG_i = 100\%*[v(LP_i)-v(LP_0)]/[v(IP)-v(LP_0)]$, where LP_0 is the LP relaxation of the plain IP formulation. As such, the "%RDG" column shows the percentage reduction of the duality gap resultant from adding the various inequalities.

Clearly, throughout the seven problems tested, the solution efficiency keeps improving as the variety of valid inequalities are added progressively. For problems 2, 3, and 4, where the 2-location demand points are the significant majority among all the demand points, the Convex Inequalities generated on the 2-location reliability constraints outstandingly enhance the branch-and-bound efficiency in terms of the number of branches, the solution time, and the LP relaxation. On the other hand, the remaining test problems, where the number of 2-location demand points is not significant, relate the remarkable effectiveness of the other type of inequalities, i.e., the Blocking Inequalities. Indeed, problem 7 needed the Blocking Inequalities to obtain the optimal solution. And, problems 1, 5, and 6 showed the significant efficiency gained by the Blocking Inequalities. Occasionally, the number of nodes created by the branch-and-bound exceeded the allocated memory and the branch-and-bound terminated with a "FULL" indication, as can be seen in Table 4, 6, and 7.

As a result of the seven problems tested here, it is evident that our preprocessing techniques are very effective for problems whose constraint matrix is not too "dense". As a matter of fact, our 9 test problems all have a constraint matrix with a "density" as high as 16% or more, which are not "sparse" at all. Typically, according to the literature, for instance, Crowder, Johnson, and Padberg (1983), a constraint matrix is considered "sparse" if the total number of nonzero coefficients divided by the product of the number of rows and the number of columns is less than 5%.

	PROBLEM 1			
	Demand=40 Location=15 %2-Loc-Demand=42.5%			
	C/W = 1/3 v(IP) = 139.0			
	Branch	Minute	v(LP)	%RDG
P	62	45	116.3	----
R	54	31	118.8	11.31
RV	31	29	119.9	15.82
RL	6	0.8	135.1	82.87
RVL	3	1	137.3	92.69
RVLX	3	2.2	137.3	92.69
RVLXB	1	1.8	139.0	100.00
	C/W = 10/1 v(IP) = 172.0			
	Branch	Minute	v(LP)	%RDG
P	----	----	125.2	----
R	296	271.5	131.1	12.61
RV	69	44.5	135.3	21.58
RL	12	4.5	166.2	87.61
RVL	4	5.2	166.9	89.10
RVLX	4	8.1	166.9	89.10
RVLXB	1	1.7	172.0	100.00

Table 1.

Problem 1: Effectiveness of the Various Valid Inequalities

# of demand points = 40, # of locations=15
β=0.97, λ=0.025, coverage radius=20.0

# of 2-location demand points	: 17
# of 3-location demand points	: 16
# of 4-location demand points	: 3
# of 5-or-more-location demand points	: 4

	PROBLEM 2			
	Demand=60 Location=15 %2-Loc-Demand=55.00%			
	C/W = 1/3 v(IP) = 138.0			
	Branch	Minute	v(LP)	%RDG
P	143	184	110.9	----
R	72	62	114.1	11.69
RV	18	37	117.8	25.52
RL	9	1	134.3	86.23
RVL	7	3.5	134.4	86.58
RVLX	1	2.6	138.0	100.00
RVLXB	1	0.85	138.0	100.00
	C/W = 10/1 v(IP) = 162.0			
	Branch	Minute	v(LP)	%RDG
P	----	----	119.4	----
R	65	18.3	126.0	15.44
RV	15	5	134.7	35.83
RL	8	4.5	160.9	97.44
RVL	6	1.4	161.0	97.53
RVLX	1	1.2	162.0	100.00
RVLXB	1	0.85	162.0	100.00

Table 2.

Problem 2: Effectiveness of the Various Valid Inequalities

# of demand points=60, # of locations=15
β=0.97, λ=0.018, coverage radius=20.0

# of 2-location demand points	: 33
# of 3-location demand points	: 24
# of 4-location demand points	: 3
# of 5-or-more-location demand points	: 0

	PROBLEM 3			
	Demand=57 Location=15 %2-Loc-Demand=54.39%			
	C/W = 1/3 v(IP) = 132.0			
	Branch	Minute	v(LP)	%RDG
P	295	272	106.3	----
R	87	104	113.2	26.93
RV	30	82	118.6	47.97
RL	14	12	126.7	79.45
RVL	5	4	129.2	89.12
RVLX	1	5.6	132.0	100.00
RVLXB	1	1.9	132.0	100.00
	C/W = 10/1 v(IP) = 160.0			
	Branch	Minute	v(LP)	%RDG
P	----	----	114.5	----
R	314	103.7	126.5	26.37
RV	43	34.9	134.1	43.08
RL	10	10.5	158.5	96.70
RVL	2	1.5	159.4	98.68
RVLX	1	2.3	160.0	100.00
RVLXB	1	1.9	160.0	100.00

Table 3.

Problem 3: Effectiveness of the Various Valid Inequalities

# of demand points=57, # of locations=15
β=0.98, λ=0.015, coverage radius=20.0

# of 2-location demand points	: 31
# of 3-location demand points	: 20
# of 4-location demand points	: 6
# of 5-or-more-location demand points	: 0

	PROBLEM 4			
	Demand=183 Location=40 %2-Loc-Demand=51.37%			
	C/W = 1/3 v(IP) = 337.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	----	262.7	----
R	FULL	----	274.3	15.63
RV	FULL	----	283.1	27.51
RL	FULL	2.05	322.0	79.87
RVL	358	1.01	328.8	88.94
RVLX	1	0.06	337.0	100.00
RVLXB	1	0.05	337.0	100.00
	C/W = 10/1 v(IP) = 414.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	----	282.9	-----
R	FULL	----	304.5	16.48
RV	FULL	----	318.1	26.85
RL	265	0.32	410.8	97.56
RVL	6	0.05	412.3	98.70
RVLX	1	0.05	414.0	100.00
RVLXB	1	0.05	414.0	100.00

Table 4.

Problem 4: Effectiveness of the Various Valid Inequalities

# of demand points=183, # of locations=40
β=0.962, λ=0.019, coverage radius=15.0

# of 2-location demand points	: 94
# of 3-location demand points	: 68
# of 4-location demand points	: 16
# of 5-or-more-location demand points	: 5

	PROBLEM 5			
	Demand=30 Location=15 %2-Loc-Demand=23.33%			
	C/W = 1/3 v(IP) = 96.0			
	Branch	Minute	v(LP)	%RDG
P	----	----	73.9	----
R	157	85.7	78.4	20.36
RV	125	103.6	78.4	20.36
RL	188	89.3	82.9	40.72
RVL	98	50.2	84.7	48.87
RVLX	19	12.2	86.7	57.92
RVLXB	4	8.7	93.8	90.04
	C/W = 10/1 v(IP) = 119.0			
	Branch	Minute	v(LP)	%RDG
P	----	----	79.55	----
R	98	47.7	91.04	29.13
RV	78	60.6	91.04	29.13
RL	143	19.5	99.47	50.49
RVL	27	9.3	100.05	51.96
RVLX	13	6.6	100.05	51.96
RVLXB	4	9.4	118.25	98.09

Table 5.

Problem 5: Effectiveness of the Various Valid Inequalities

# of demand points=30, # of locations=15
β=0.97, λ=0.025, coverage radius=20.0

# of 2-location demand points	:	7
# of 3-location demand points	:	16
# of 4-location demand points	:	3
# of 5-or-more-location demand points	:	4

	PROBLEM 6			
	Demand=119 Location=30 %2-Loc-Demand=10.92%			
	C/W = 1/3 v(IP) = 185.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	----	142.7	-----
R	FULL	----	158.0	36.17
RV	FULL	----	158.0	36.17
RL	FULL	----	163.9	50.12
RVL	FULL	----	165.2	53.19
RVLX	1932	2.45	166.1	55.32
RVLXB	26	0.13	173.8	73.52
	C/W = 10/1 v(IP) = 226.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	----	153.7	----
R	FULL	----	182.5	39.83
RV	FULL	----	182.5	39.83
RL	FULL	----	195.7	58.09
RVL	FULL	----	196.2	58.78
RVLX	2019	2.30	196.2	58.78
RVLXB	10	0.08	217.5	88.24

Table 6.

Problem 6: Effectiveness of the Various Valid Inequalities

# of demand points=119, # of locations=30
β=0.962, λ=0.016, coverage radius=15.0

# of 2-location demand points	: 13
# of 3-location demand points	: 66
# of 4-location demand points	: 31
# of 5-or-more-location demand points	: 9

	PROBLEM 7			
	Demand=201 Location=44 %2-Loc-Demand=12.43%			
	C/W = 1/3 Best v(IP) Found = 332.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	-----	254.5	-----
R	FULL	-----	268.1	17.55
RV	FULL	-----	272.7	23.48
RL	FULL	-----	290.1	45.93
RVL	FULL	-----	291.0	47.09
RVLX	FULL	-----	294.9	52.13
RVLXB	FULL	-----	314.3	77.16
	C/W = 10/1 v(IP) = 403.0			
	Nodes	Minute	v(LP)	%RDG
P	FULL	----	274.1	----
R	FULL	----	298.1	18.62
RV	FULL	----	309.4	27.39
RL	FULL	----	342.5	53.06
RVL	FULL	----	342.7	53.22
RVLX	FULL	----	343.6	53.92
RVLXB	375	1.19	382.3	83.94

Table 7.

Problem 7: Effectiveness of the Various Valid Inequalities

# of demand points=201, # of locations=44
β=0.962, λ=0.020, coverage radius=12.7

# of 2-location demand points	: 25
# of 3-location demand points	:122
# of 4-location demand points	: 38
# of 5-or-more-location demand points	: 16

5.2. Solution Sensitivity Of The Reliability Model

In this section we report on three studies concerning the sensitivity of our reliability model to changes in data characteristics. We used problems 8 and 9 for our sensitivity studies. In each study, sensitivity results were summarized in tables and illustrated by figures, including the number of locations and the number of vehicles chosen.

Unfortunately, problem 8, the 55-node problem, poses a stiff challenge to the valid inequalities we have proposed in that most of the nodes are so close to one another that there is a heavy coverage overlapping among the locations. As a result, 43 out of the 55 demand points have more than 4 location candidates to choose from; indeed, 27 demand points are associated with 10 or more location candidates. Thus, the constraint matrix is highly "dense" in the nonzero coefficients, and our preprocessing techniques are not expected to be effective for this problem. In fact, after we generated the valid inequalities for only those 7-or-less-location reliability constraints, the augmented formulation had become very close to our machine's memory limit, and little efficiency was gained. We thus used a two-stage strategy to tackle this problem. For Stage 1, we included additional 0-1 location variables and did not require that the X_{jk} be integer. Stage 1 is intended to reduce the set of location candidates that can be used in stage 2. We note that all the preprocessing techniques remain valid for both stages. Our computational experience with the stage 1 problem indicated that its LP relaxation has a highly degenerate solution, where only 13 out of 55 Y_j's (i.e., location variables) received positive value. As a result of using these 13 locations, we obtained from stage 2 a solution (feasible to Rel-P) that ultimately choose 11 out of the 13 candidate locations and determined the appropriate number of vehicles to house at each of the 11 chosen stations. By comparing the final solution value with the value of the stage 1 solution, which gives an overall lower bound we found that the final (approximate) solution was within 14% of the optimal solution to Rel-P. This two-stage strategy is the approach we used to conduct the sensitivity study on problem 8.

5.2.1. Location/Vehicle Sensitivity To C/W Ratio

In this part of the sensitivity study we intend to examine how the C/W ratio affects the locations/vehicles chosen by our reliability model. Everything else being fixed, two C/W ratios, 1/3 and 20/1, were assumed. The results are shown in Table 8 with Figure 4 and Figure 5 for problem 8; and Table 9 with Figure 6 and Figure 7 for problem 9. As indicated in the results, when the "fixed" cost of a station is much higher than the cost of a vehicle unit, our model responds with choosing fewer locations but more vehicles, though not significantly. Since the coverage radius of the locations is predetermined, and since the distances between locations and demand points are neglected in our objective function, we are little surprised that the C/W ratio had insignificant effect on the locations and vehicles chosen.

T	C/W	β	PROBLEM 8	
			Demand=55 Location=55 (Demand Point = Location Candidate)	
T	C/W	β	1 * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 4 0 0 0 0 0 0 0 0 2 0	
10.0	20/1	0.990	0 0 0 0 4 0 0 0 0 0 0 0 0 2 0	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 3 3 3 0 0 0 0 1 0 2	
10.0	20/1	0.990	0 0 0 0 0 3 4 4 0 0 0 0 0 0 2	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 4 0 0 0 0 0 0 0 0 0	
10.0	20/1	0.990	0 0 0 0 0 4 0 0 0 0 0 0 0 0 0	
T	C/W	β	* * * * * * * * * 55	(Locs. Vehs.)
10.0	1/3	0.990	0 2 0 0 3 0 0 0 0 4	(11 31)
10.0	20/1	0.990	0 2 0 0 3 0 0 0 0 4	(10 32)

Table 8.

Problem 8: Location/Vehicle Sensitivity to C/W Ratio

# of demand points=55, # of locations=55
β=0.99, λ=0.01, coverage radius=10.0

# of 2-location demand points	:	2
# of 3-location demand points	:	4
# of 4-location demand points	:	6
# of 5-to-9-location demand points	:	16
# of 10-or-more-location demand points	:	27

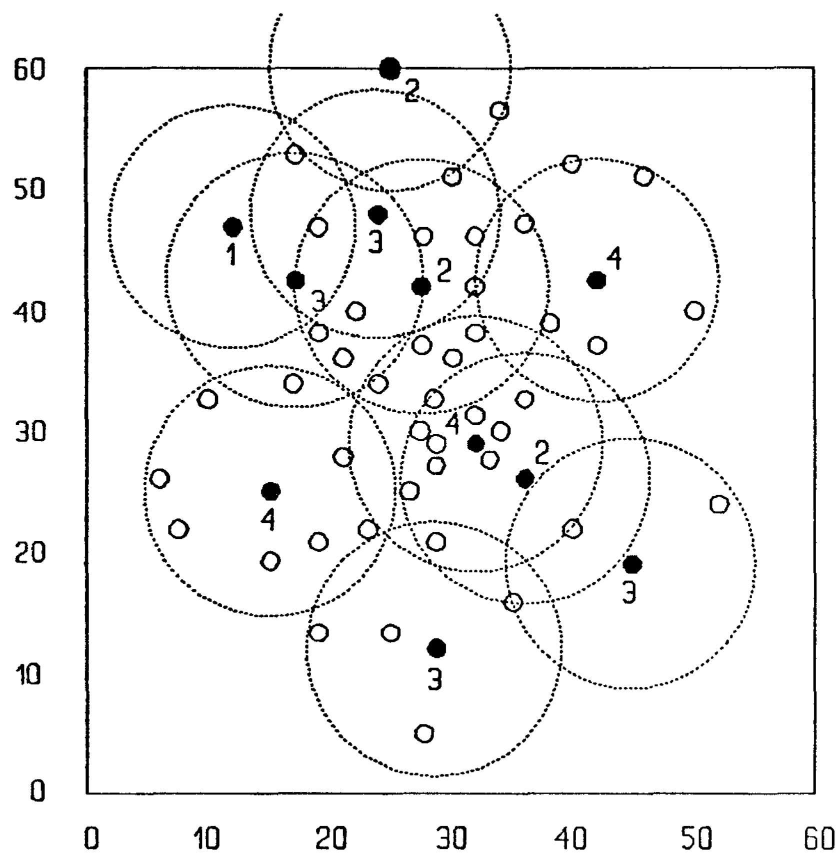


Figure 4.

Problem 8: T=10.0 C/W=1/3 β=0.990
Result: 11 Locations 31 Vehicles

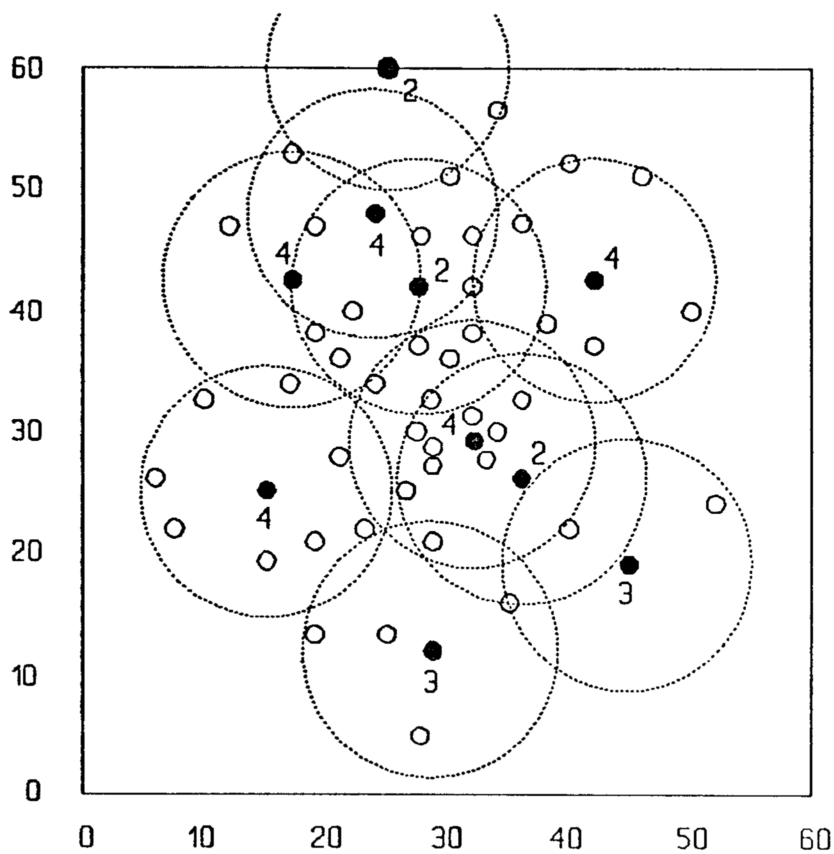


Figure 5.

Problem 8: T=10.0 C/W=20/1 β=0.990
Result: 10 Locations 32 Vehicles
(In Comparison With Figure 4.)

	PROBLEM 9	
	Demand=183 Location=47	
T C/W β	1 * * * * * * * * * * * * * *	
15.0 1/3 0.962 15.0 20/1 0.962	0 0 2 4 4 4 4 4 3 4 4 2 0 0 0 0 0 3 1 4 4 4 4 3 4 0 4 0 0 0	
T C/W β	* * * * * * * * * * * * * * *	
15.0 1/3 0.962 15.0 20/1 0.962	2 0 1 0 1 4 4 0 0 1 2 4 3 1 0 2 0 4 0 4 0 4 0 0 4 1 4 3 1 0	
T C/W β	* * * * * * * * * * * * * * *	
15.0 1/3 0.962 15.0 20/1 0.962	1 3 4 4 2 3 4 4 1 4 4 4 3 0 0 4 3 4 4 4 3 0 4 4 0 4 4 3 0 0	
T C/W β	* 47	(Locs. Vehs.)
15.0 1/3 0.962 15.0 20/1 0.962	0 0 0 4	(33 99) (30 103)

Table 9.

Problem 9: Location/Vehicle Sensitivity to C/W Ratio

# of demand points=183, # of locations=47
β=0.962, λ=0.019, coverage radius=15.0

# of 2-location demand points	: 53
# of 3-location demand points	: 75
# of 4-location demand points	: 38
# of 5-or-more-location demand points	: 17

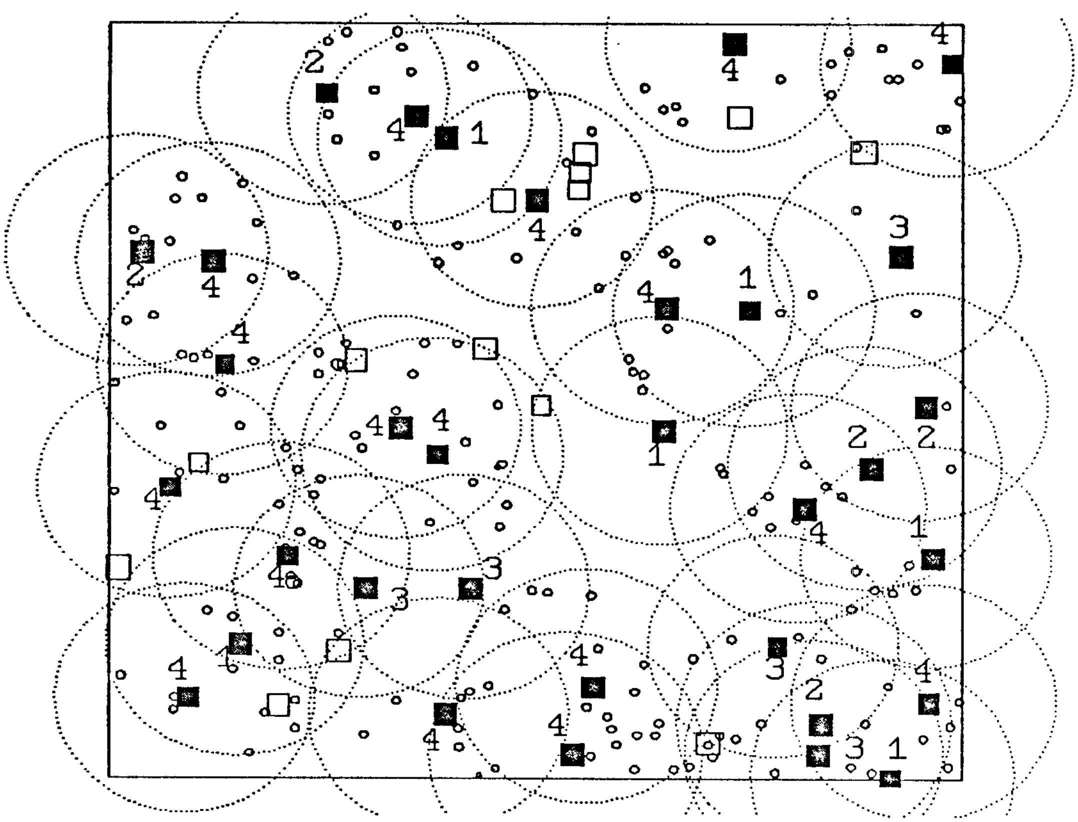


Figure 6.

Problem 9: T=15.0 C/W=1/3 β=0.962
Result: 33 Locations 99 Vehicles

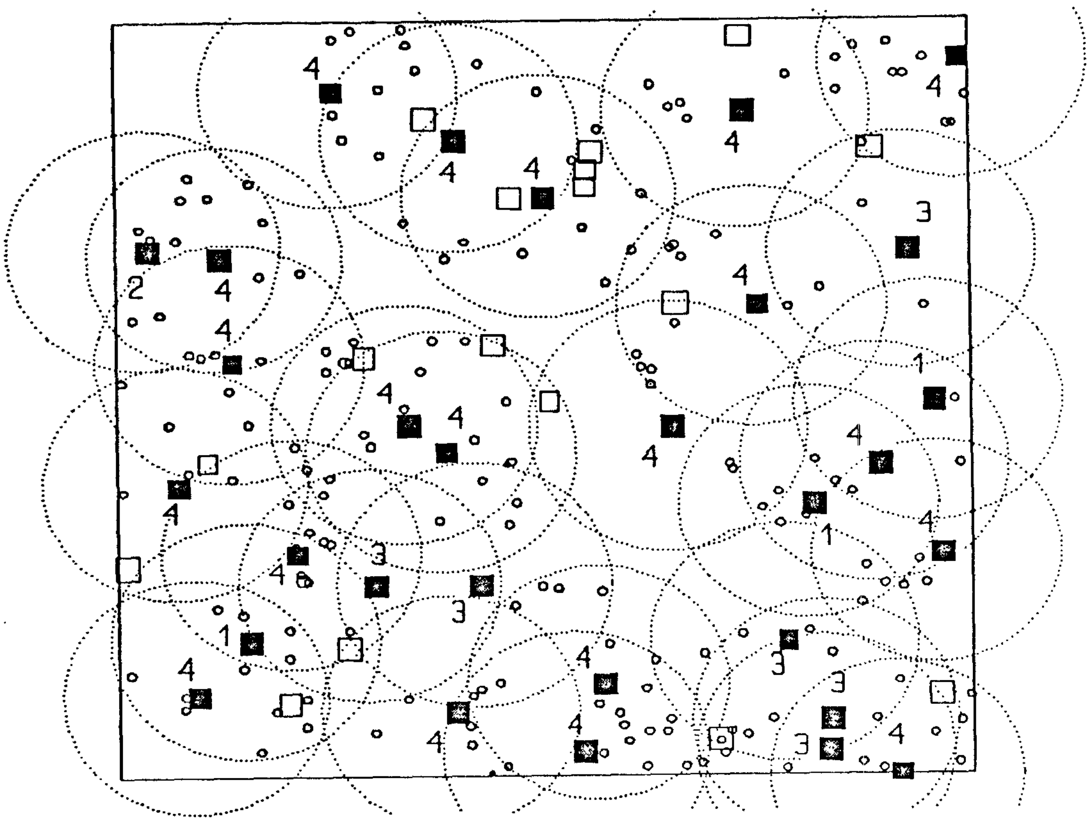


Figure 7.

Problem 9: T=15.0 C/W=20/1 β=0.962
Result: 30 Locations 103 Vehicles
(In Comparison With Figure 6.)

5.2.2. Location/Vehicle Sensitivity To Reliability Level

The locations/vehicles chosen also varied with different reliability levels as everything else was fixed. The sensitivity results for problem 8 are shown in Table 10 with Figures 8 - 10, and the results for problem 9 in Table 11 with Figure 11 and Figure 12. As anticipated, the model is sensitive to the reliability level. The total locations and/or vehicles chosen decreased with the reliability level, as expected.

			PROBLEM 8	
			Demand=55 Location=55 (Demand Point = Location Candidate)	
T	C/W	β	1 * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 4 0 0 0 0 0 0 0 0 2 0	
10.0	1/3	0.975	0 0 0 0 0 0 0 0 0 0 0 0 0 2 0	
10.0	1/3	0.950	0 4 0 0 0 0 0 0 0 0 0 0 0 2 0	
10.0	1/3	0.900	0 0 0 0 3 0 0 0 0 0 0 0 0 2 0	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 3 3 3 0 0 0 0 1 0 2	
10.0	1/3	0.975	0 3 0 0 0 3 0 0 0 0 0 0 0 0 4	
10.0	1/3	0.950	0 0 0 0 0 3 0 0 0 0 0 0 2 0 4	
10.0	1/3	0.900	0 0 0 0 0 2 0 0 0 0 0 0 2 0 3	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 4 0 0 0 0 0 0 0 0 0	
10.0	1/3	0.975	0 0 0 0 0 3 0 0 0 0 0 0 0 0 0	
10.0	1/3	0.950	0 0 0 0 0 3 0 0 0 0 0 0 0 0 0	
10.0	1/3	0.900	0 0 0 0 0 3 0 0 0 0 0 0 0 0 0	
T	C/W	β	* * * * * * * * * 55	(Locs. Vehs.)
10.0	1/3	0.990	0 2 0 0 3 0 0 0 0 4	(11 31)
10.0	1/3	0.975	0 4 0 0 2 0 0 0 0 3	(8 24)
10.0	1/3	0.950	0 0 0 0 2 0 0 0 0 3	(8 23)
10.0	1/3	0.900	0 0 0 0 2 0 0 0 0 2	(8 19)

Table 10.

Problem 8: Location/Vehicle Sensitivity to Reliability Level

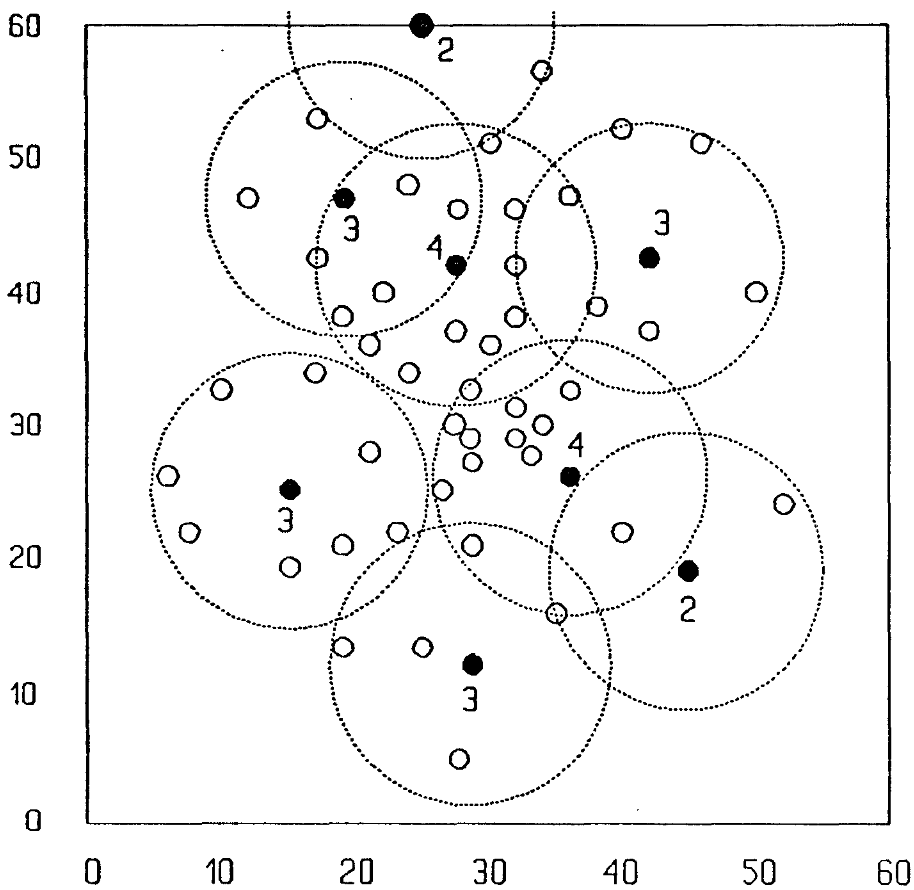


Figure 8.

Problem 8: T=10.0 C/W=1/3 β=0.975
Result: 8 Locations 24 Vehicles
(In Comparison With Figure 4.)

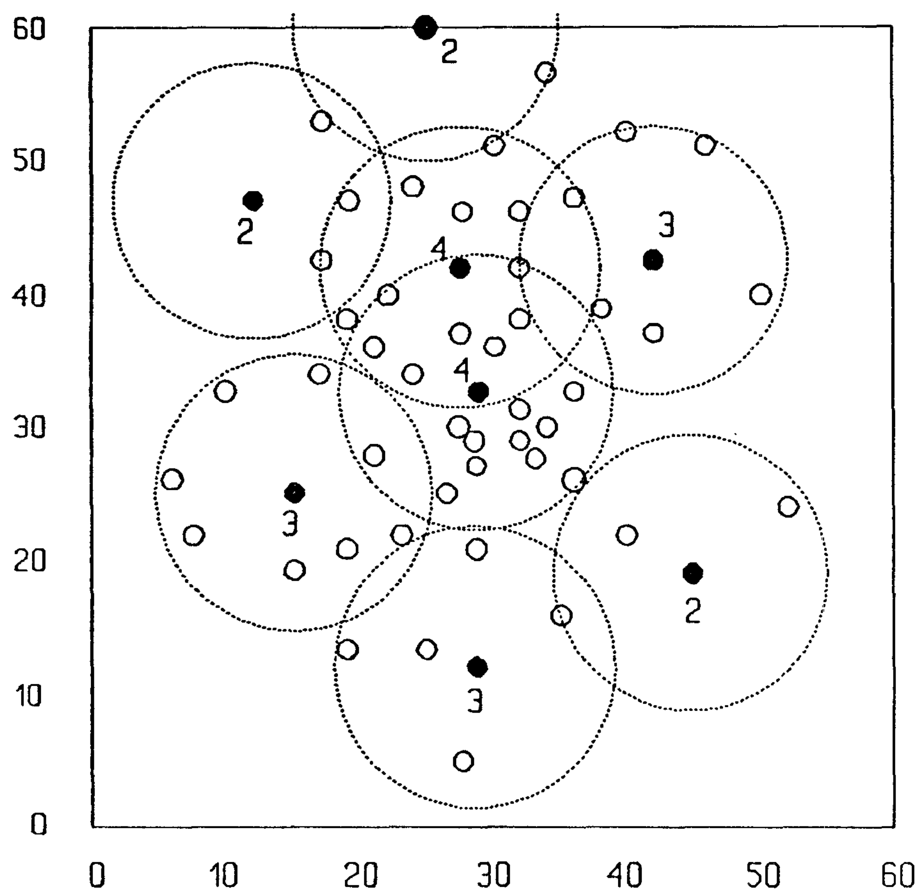


Figure 9.

Problem 8: T=10.0 C/W=1/3 β=0.950
Result: 8 Locations 23 Vehicles
(In Comparison With Figure 4.)

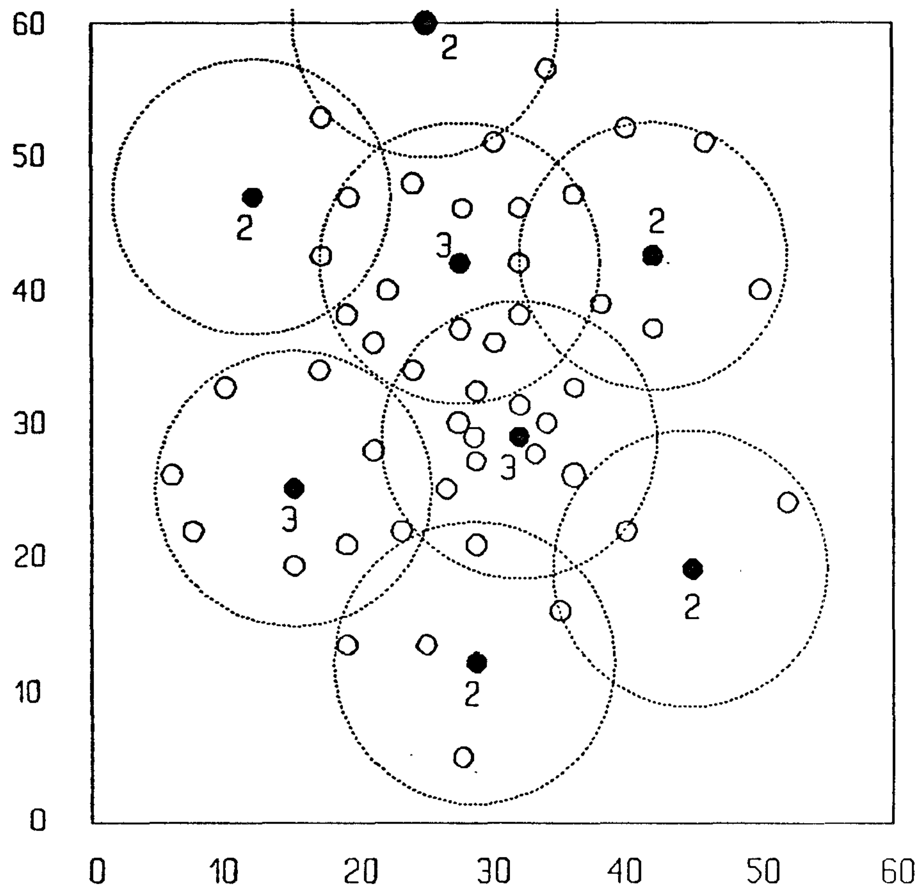


Figure 10.

Problem 8: T=10.0 C/W=1/3 β=0.900
Result: 8 Locations 19 Vehicles
(In Comparison With Figure 4.)

			PROBLEM 9	
			Demand=183 Location=47	
T	C/W	β	1 * * * * * * * * * * * * * *	
15.0	1/3	0.962	0 0 2 4 4 4 4 4 3 4 4 2 0 0 0	
15.0	1/3	0.950	0 0 2 0 4 4 4 2 0 4 4 4 0 0 4	
15.0	1/3	0.925	0 4 4 4 4 1 4 3 4 0 4 0 0 4 0	
T	C/W	β	* * * * * * * * * * * * * * *	
15.0	1/3	0.962	2 0 1 0 1 4 4 0 0 1 2 4 3 1 0	
15.0	1/3	0.950	2 0 0 3 0 0 4 0 0 3 0 4 3 0 2	
15.0	1/3	0.925	1 0 0 4 0 4 4 0 0 0 2 4 2 0 4	
T	C/W	β	* * * * * * * * * * * * * * *	
15.0	1/3	0.962	1 3 4 4 2 3 4 4 1 4 4 4 3 0 0	
15.0	1/3	0.950	0 4 3 3 2 4 4 4 3 0 0 3 4 0 0	
15.0	1/3	0.925	0 0 1 4 1 2 4 4 0 3 0 0 0 0 0	
T	C/W	β	* 47	(Locs. Vehs.)
15.0	1/3	0.962	0 0	(33 99)
15.0	1/3	0.950	0 3	(27 90)
15.0	1/3	0.925	0 0	(25 80)

Table 11.

Problem 9: Location/Vehicle Sensitivity to Reliability Level

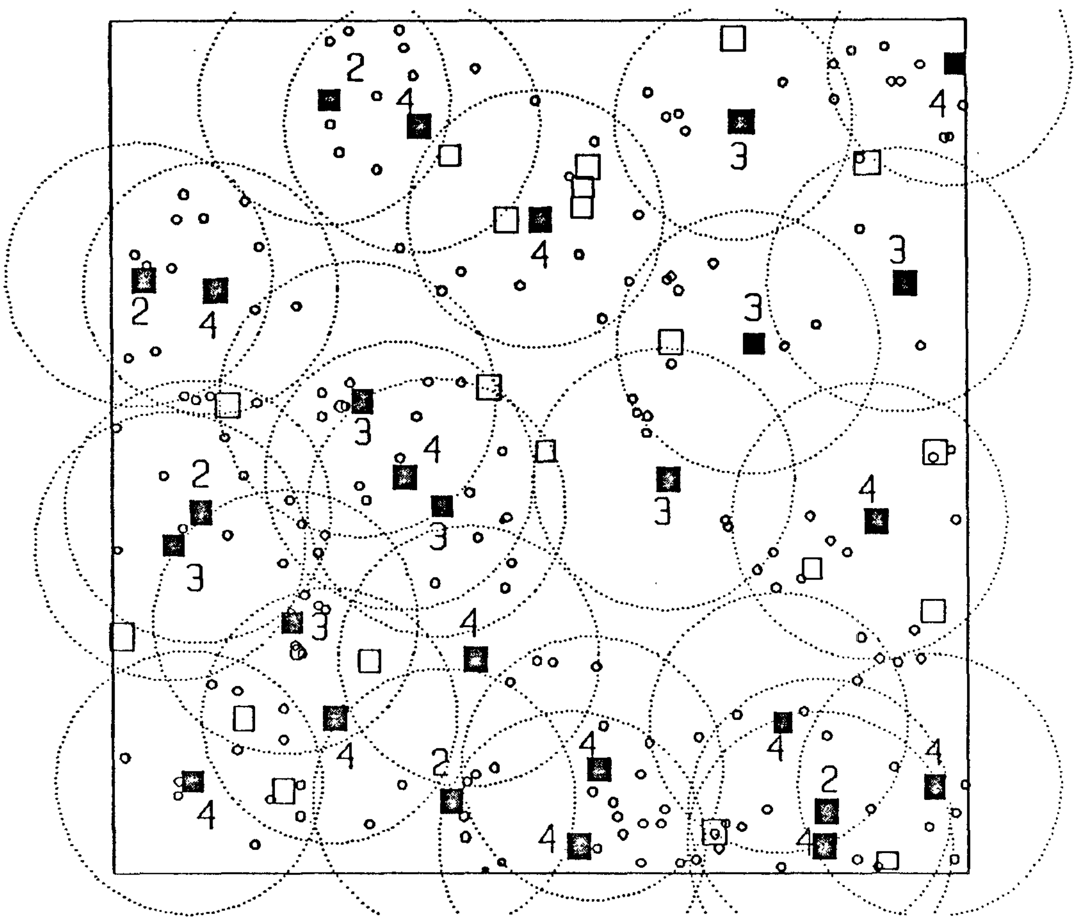


Figure 11.

Problem 9: T=15.0 C/W=1/3 β=0.950
Result: 27 Locations 90 Vehicles
(In Comparison With Figure 6.)

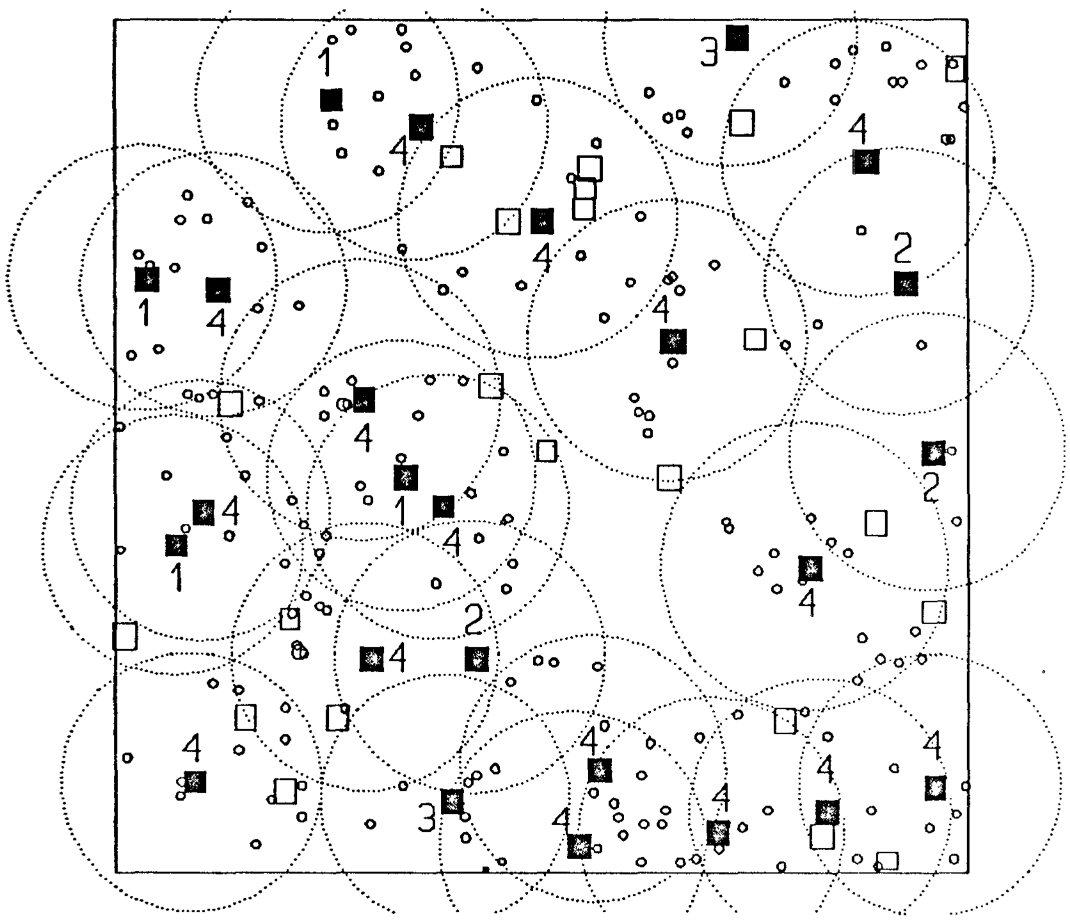


Figure 12.

Problem 9: T=15.0 C/W=1/3 β=0.925
Result: 25 Locations 80 Vehicles
(In Comparison With Figure 6.)

5.2.3. Location/Vehicle Sensitivity To Location Coverage

Here, we used problem 8 to briefly demonstrate the locations and vehicles chosen as a result of different coverage extents. When the location coverage radius decreased from 10 to 5, the results were shown in Table 12 and illustrated by Figure 13. Obviously, more locations and vehicles must be used when vehicles were allowed to travel up to only a half of the original limit.

			PROBLEM 8	
			Demand=55 Location=55 (Demand Point = Location Candidate)	
T	C/W	β	1 * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 4 0 0 0 0 0 0 0 0 2 0	
5.0	1/3	0.990	2 4 4 0 0 2 0 0 4 4 0 3 4 3 3	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 3 3 3 0 0 0 0 1 0 2	
5.0	1/3	0.990	3 4 0 0 4 0 4 0 4 4 3 3 3 4 0	
T	C/W	β	* * * * * * * * * * * * * * *	
10.0	1/3	0.990	0 0 0 0 0 4 0 0 0 0 0 0 0 0 0	
5.0	1/3	0.990	4 0 0 3 0 3 3 4 3 3 0 4 4 3 4	
T	C/W	β	* * * * * * * * * 55	(Locs. Vehs.)
10.0	1/3	0.990	0 2 0 0 3 0 0 0 0 4	(11 31)
5.0	1/3	0.990	0 4 1 3 3 3 3 3 3 3	(40 133)

Table 12.

Problem 8: Location/Vehicle Sensitivity to Coverage

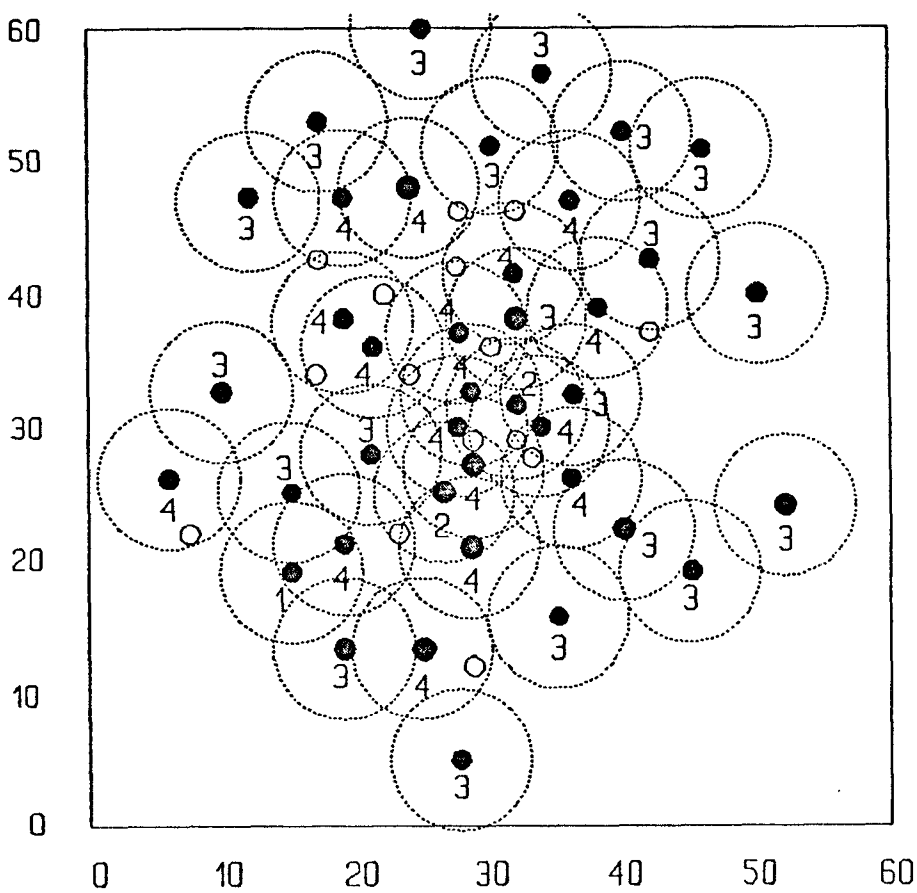


Figure 13.

Problem 8: T=5.0 C/W=1/3 β=0.990
Result: 40 Locations 133 Vehicles
(In Comparison With Figure 4.)

APPENDIX

To establish Proposition 3, we describe two of the properties: increasing failure rate (IFR) and increasing failure rate average (IFRA), that identify important classes of probability distributions. As given in the literature, their definitions are provided as follow. Note that these definitions can be applied to both continuous and discrete distributions.

Definition A1. A nonnegative random variable X or its distribution function F(x) is said to have the increasing failure rate (IFR) property if $-\log[1-F(x)]$ is a convex function in $x \geq 0$.

Definition A2. A distribution F(x) has the increasing failure rate average (IFRA) property if $-(1/x)\log[1-F(x)]$ is increasing in $x \geq 0$.

It has been shown that, particularly in the context of general continuous distributions, IFR implies IFRA and IFRA implies NBU. It is easy to see that the IFRA property is equivalent to that $[1-F(x)]^{1/x}$ is decreasing for $x \geq 0$. As a result, for renewal processes, Barlow and Proschan (1975) provided Proposition A1.

Proposition A1. Let N(t) denote the number of renewals during [0,t] in a renewal process. Then for fixed $t>0$, N(t) satisfies the discrete IFRA property: $P^{1/k}[N(t) \geq k]$ is decreasing in $k=1,2,\ldots$.

Proof: See Barlow and Proschan (1975), page 177. | |

The proof of Proposition A2 is developed similarly to the proof for continuous distributions, which can be found from, for example, Gertsbakh (1989), page 60.

Proposition A2. Discrete IFRA implies discrete NBU.

Proof: Discrete IFRA states that $(1/k)\cdot\log P[N(t)\geq k]$ is decreasing in $k=1,2,\ldots$. Let $g_1 = 1/(i+j)\cdot\log P[N(t)\geq i+j]$, $g_2 = 1/i\cdot\log P[N(t)\geq i]$, $g_3 = 1/j\cdot\log P[N(t)\geq j]$, for $i,j\geq 1$. IFRA implies that $g_1 \leq \min(g_2,g_3)$. Hence, we have

$$\log P[N(t)\geq i+j] = g_1(i+j) \leq \min(g_2,g_3)\cdot(i+j) \leq g_2 i + g_3 j$$

$$= \log P[N(t)\geq i] + \log P[N(t)\geq j] = \log\{P[N(t)\geq i]\cdot P[N(t)\geq j]\}.$$

Therefore, $P[N(t)\geq i+j] \leq P[N(t)\geq i]\cdot P[N(t)\geq j]$, which is equivalent to that $P[N(t)\geq i+j \mid N(t)\geq i] \leq P[N(t)\geq j]$, and thus the discrete NBU property. | |

Proposition 3. A Poisson random variable N(t) for a fixed $t>0$ satisfies the discrete NBU property.

Proof: A Poisson process is a renewal process. By Propositions A1 and A2, Proposition 3 is proved. | |

REFERENCES

M. Ball and J. Provan (1988). "Disjoint Products and Efficient Computation of Reliability", Operations Research, 36, No.5, 703-715.

M. Ball and F. Lin (1992). "Solving A Special Type of Covering IP Using The Concepts Of Blocking Clutter," manuscript in preparation.

M. Ball and J. Shanthikumar (1992). "Bounding A Chance Constraint Over A Polymatroid," manuscript in preparation.

R. Barlow and F. Proschan (1975). Statistical Theory Of Reliability And Life Testing, Holt, Rinehart and Winston, Inc..

S. Chapman and J. White (1974). "Probabilistic Formulations Of Emergency Service Facilities Location Problems," paper presented at the 1974 ORSA/TIMS Conference, San Juan, Puerto Rico.

R. Church and C. ReVelle (1974). "The Maximal Covering Location Problem," Papers of Regional Science Association, 32, 101-118.

H. Crowder, E. Johnson, and M. Padberg (1983). "Solving Large-Scale Zero-One Linear Programming Problems," Operations Research, 31, 803-834.

M. Daskin (1982). "An Expected Covering Model To Emergency Medical Service System Design," Decision Sciences, 13, 416-439.

M. Daskin (1983). "The Maximal Expected Covering Location Model: Formulation, Properties, and Heuristic Solution," Transportation Science, 17, 48-70.

M. Daskin, K. Hogan, and C. ReVelle (1988). "Integration Of Multiple, Excess, Backup, and Expected Covering Models,"Environment and Planning B: Planning and Design, 15, 15-35.

D. Eaton, M. Daskin, D. Simmons, B. Bulloch, and G. Jansma (1985). "Determining Emergency Medical Service Vehicle Deployment in Austin, Texas," Interfaces, 15:1, 96-108.

O. Fujiwara, T. Makjamroen, and K. Gupta (1986). "Ambulance Deployment Analysis: A Case Study of Bangkok."

I. B. Gertsbakh (1989). Statistical Reliability Theory, Marcel Dekker, Inc., New York.

P. Hall (1935). "On Representatives of Subsets," Journal of London Math. Society, **10**, 26-30.

F. Lin (1992). "Integer Programming Models With Reliability Constraints," Ph.D. Thesis, University of Maryland.

G. Nemhauser and L. Wolsey (1988). "Integer and Combinatorial Optimization," John Woley & Sons.

M. Padberg (1979). "Covering, Packing And Knapsack Problems," Annals of Discrete Mathematics, **4**, 265-287.

D. Plane and T. Hendrick (1977). "Mathematical Programming and the Location of Fire Companies for the Denver Fire Department," Operations Research, **25**, 563-578.

C. ReVelle and K. Hogan (1989). "The Maximum Availability Location Problem," Transportation Science, **23**, 192-200.

C. ReVelle and K. Hogan (1988). "A Reliability-Constrained Siting Model With Local Estimates of Busy Fractions," Environment and Planning B: Planning and Design, **15**, 143-152.

D. Schilling, D. Elzinga, J. Cohon, R. Church, and C. ReVelle (1979). "The Team/Fleet Models For Simultaneous Facility and Equipment Siting," Transportation Science, **13**, 163-175.

C. Toregas, R. Swain, C. ReVelle, and L. Bergman (1971). "The Location Of Emergency Service Facilities," Operations Research, **19**, 1363-1373.