Amit Prakash*, Doddabhadrappla Gowda Prakasha, and Pundikala Veeresha

A reliable algorithm for time-fractional Navier-Stokes equations via Laplace transform

Abstract: In this paper, numerical solution of fractional order Navier-Stokes equations in unsteady viscous fluid flow is found using q-homotopy analysis transform scheme. Fractional derivative is considered in Caputo sense. The proposed technique is a blend of q-homotopy analysis scheme and transform of Laplace. It executes well in efficiency and provides h-curves that show convergence range of series solution.

Keywords: Navier-Stokes (N-S) equations; q-homotopy analysis transform method (q-HATM); Caputo fractional derivative

1 Introduction

Every phenomenon in fields of Science and Engineering may be alternately modelled using fractional order derivatives. It is due to their non-local property, intrinsic to several complex systems. They are used as modelling tools in nanotechnology, viscoelasticity, anomalous transport, control theory, financial & biological modelling etc. The most important among such models are those described by arbitrary order PDEs. Adomian decomposition [1], homotopy analysis, residual power series, fractional reduced differential transform, fractional variational iteration method [2–4], Laplace homotopy technique [5], Laplace variational iteration method [6], homotopy perturbation transform method [7], q-homotopy analysis transform method [8–10], modified trial equation method [11], new iterative Sumudu transform method [12] and Laplace perturbation method [13] etc. are some impor-

tant methods which are applied to find numerical solution of these problems.

N-S equation define motion [14] of incompressible Newtonian fluid flow extending from enormous scale atmospheric motions to lubrication of bearings and express conservation of mass and momentum. Consider an unsteady, unidimensional viscous fluid motion in a long circular pipe. Fluid is initially at rest. Constant pressure gradient along pipe axis is abruptly imposed to set fluid in motion. Flow is taken as axially symmetric. It is given in [15] as,

$$\rho u_t = -p_z + \mu \left(D_r^2 u + \frac{1}{r} D_r u \right), \tag{1}$$

axis of pipe is z-axis and r is radial direction.

El-Shashed and Salem [16] generalized N-S Eq. by fractional N-S Eq. of order α, as

$$D_t^\alpha u = P + \mu \left(D_r^2 u + \frac{1}{r} D_r u \right), \quad 0 < \alpha \le 1, \tag{2}$$

where $P = -\frac{\partial p}{\rho \partial z}$, α is order of fractional derivative. If $\alpha = 1$, fractional N-S Eq. reduces to classical N-S equation. Here, efficiency of q-HATM is considered to numerically solve time-fractional N-S Eq. (2). It is an elegant union of q-HAM [17, 18] & transform of Laplace. The q-HAM is actually improvement of parameter $q \in [0, 1]$ in HAM to $q \in \left[0, \frac{1}{n} \right]$, $n \ge 1$. Its authority is its potential of adjusting two strong computational methodologies for probing FDEs. Eq. (2) has been handled by Momani and Odibat by ADM [19], Khan et al. by HPM and variation iteration method [20], Ragab et al. by HAM [21], Kumar et al. by fractional modified Laplace decomposition method [22] and HPTM [23], Wang and Liu by modified reduced differential transform method [24] etc. But it has not been solved by q-HATM.

2 Preliminaries

Definition 2.1. A real valued function $f(t)$, $t > 0$ is in space C_μ, $\mu \in R$ if $\exists$ a real number $p \, (> \mu)$ s.t. $f(t) = t^p f_1(t)$; $f_1 \in C[0, \infty)$, and is in space C_μ^m if $f^m \in C_\mu$, $M \in N \bigcup \{0\}$ [25, 26].

*Corresponding Author: Amit Prakash, Department of Mathematics, National Institute of Technology, Kurukshetra 136119, India, E-mail: amitmath@nitkkr.ac.in, amitmath0185@gmail.com
Doddabhadrappla Gowda Prakasha, Pundikala Veeresha, Department of Mathematics, Karnatak University, Dharwad-580003, India, E-mail: dgprakasha@kud.ac.in (D. G. Prakasha), viru0913@gmail.com (P. Veeresha)

Definition 2.2. Caputo fractional derivative [9] of f, $f \in C_{-1}^m$, $m \in N \bigcup \{0\}$ is

$$D_t^\alpha f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n \in N, \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) \, d\tau, \\ & n-1 < \alpha < n, n \in N. \end{cases}$$

Definition 2.3. Laplace transform (LT) of Caputo fractional derivative is

$$L\left[D_t^{n\alpha} u(r,t)\right] = s^{n\alpha} L\left[u(r,t)\right] - \sum_{k=0}^{n-1} s^{n\alpha-k-1} u^k(r,0),$$

$$n-1 < n\alpha \le n.$$

3 Implementation of q-HATM on Navier-Stokes equation

To illustrate its efficiency, we take a fractional nonlinear nonhomogeneous PDE,

$$D_t^\alpha u(r,t) + Ru(r,t) + Nu(r,t) = f(r,t), \quad n-1 < \alpha \le n, \tag{3}$$

where $D_t^\alpha u$ is Caputo fractional derivative, R, N are linear, nonlinear differential operators, $f(r,t)$ is source term.

Taking LT on Eq. (3) and applying its differentiation property, we get

$$L[u(r,t)] - \frac{1}{s^\alpha} \sum_{k=0}^{n-1} s^{\alpha-k-1} u^k(r,0)$$

$$+ \frac{1}{s^\alpha} \{L[Ru(r,t)] + L[Nu(r,t)] - L[f(r,t)]\} = 0. \tag{4}$$

Nonlinear operator N is

$$N[\varphi(r,t;q)] = L[\varphi(r,t;q)] - \frac{1}{s^\alpha} \sum_{k=0}^{n-1} s^{\alpha-k-1} \varphi^k(r,t;q)(0^+)$$

$$+ \frac{1}{s^\alpha} \{L[R\varphi(r,t;q)] + L[N\varphi(r,t;q)] - L[f(r,t)]\}, \tag{5}$$

here $q \in \left[0, \frac{1}{n}\right]$ and ϕ is real function.

Build homotopy as

$$(1 - nq)L[\varphi(r,t;q) - u_0(r,t)] = hqH(r,t)N[\varphi(r,t;q)],$$

$$n \ge 1 \tag{6}$$

q is embedding parameter, $H \ne 0$ is auxiliary function, $h \ne 0$ is auxiliary parameter, u_0 is initial guess.

For $= 0$, $\frac{1}{n}$; following results holds:

$$\varphi(r,t;0) = u_0(r,t), \quad \varphi\left(r,t;\frac{1}{n}\right) = u(r,t). \tag{7}$$

As q increases, φ varies from u_0 to solution $u(r,t)$. Expanding φ about q by Taylor's theorem, we get

$$\varphi(r,t;q) = u_0 + \sum_{m=1}^\infty u_m(r,t)q^m, \tag{8}$$

where

$$u_k(r,t) = \frac{1}{k!} \frac{\partial^k \varphi(r,t;q)}{\partial q^k}\Big|_{q=0}. \tag{9}$$

With suitable choice of u_0, n, $\hbar$, H, series (8) converges at $q = \frac{1}{n}$, we get

$$u(r,t) = u_0 + \sum_{m=1}^\infty u_m(r,t)\left(\frac{1}{n}\right)^m \tag{10}$$

Define vectors as

$$\vec{u}_m = \{u_0(r,t), u_1(r,t), \dots, u_m(r,t)\}. \tag{11}$$

Differentiating deformation Eq. (6) m-times w. r. t q, dividing by $m!$ & taking $q = 0$, we get

$$L[u_m(r,t) - k_m u_{m-1}(r,t)] = \hbar H(r,t)R_m(\vec{u}_{m-1}). \tag{12}$$

Using inverse transform, we get

$$u_m(r,t) = k_m u_{m-1}(r,t) + \hbar L^{-1}\left[H(r,t)R_m(\vec{u}_{m-1})\right], \tag{13}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(r,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{14}$$

and

$$k_s = \begin{cases} 0, s \le 1, \\ n, \ s > 1. \end{cases} \tag{15}$$

4 Numerical Experiments

Now we implement this method on few test problems.

Example 4.1. Consider a time-fractional N-S equation

$$D_t^\alpha u = D_r^2 u + \frac{1}{r} D_r u, \ 0 < \alpha \le 1, \tag{16}$$

with initial condition

$$u(r,0) = r. \tag{17}$$

Taking transform of Laplace on each side of Eq. (16) and simplifying, we get

$$L[u] - \frac{r}{s} - \frac{1}{s^\alpha} L\left[D_r^2 u + \frac{1}{r} D_r u\right] = 0. \tag{18}$$

N is defined as,

$$N\left[\varphi\left(r,t;q\right)\right] = L\left[\varphi\left(r,t;q\right)\right] - \left(1 - \frac{k_m}{n}\right)\frac{r}{s}$$
$$- \frac{1}{s^\alpha}L\left[\frac{\partial^2\varphi(r,t;q)}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi(r,t;q)}{\partial r}\right]. \tag{19}$$

The mth-order deformation eqn. for $H = 1$ is

$$L\left[u_m\left(r,t\right) - k_m u_{m-1}\left(r,t\right)\right] = \hbar R_m\left(\vec{u}_{m-1}\right), \tag{20}$$

where

$$R_m\left(\vec{u}_{m-1}\right) = L\left[u_{m-1}\right] - \left(1 - \frac{k_m}{n}\right)\frac{r}{s}$$
$$- \frac{1}{s^\alpha}L\left[\frac{\partial^2 u_{m-1}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{m-1}}{\partial r}\right]. \tag{21}$$

Taking inverse transform on Eq. (20), we find

$$u_m = k_m u_{m-1} + \hbar L^{-1}\left\{R_m\left(\vec{u}_{m-1}\right)\right\}. \tag{22}$$

Taking u_0 and solving, we get

$$u_1 = \frac{-\hbar t^\alpha}{r\,\Gamma(\alpha+1)},$$

$$u_2 = \frac{-(n+\hbar)\,\hbar t^\alpha}{r\,\Gamma(\alpha+1)} + \frac{\hbar^2 t^{2\alpha}}{r^3\,\Gamma(\alpha+1)},$$

$$u_3 = \frac{-(n+\hbar)^2\,\hbar t^\alpha}{r\,\Gamma(\alpha+1)} + \frac{2(n+\hbar)\,\hbar^2 t^{2\alpha}}{r^3\,\Gamma(2n+1)} - \frac{9\,\hbar^3 t^{3\alpha}}{r^5\,\Gamma(3\alpha+1)}, \cdots$$

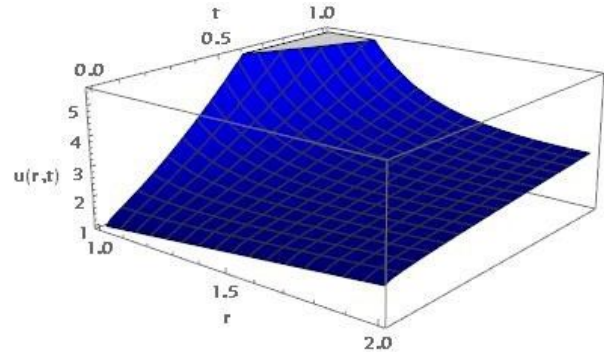Hence, next values may be got. The series solution of Eq. (16) is

$$u\left(r,t\right) = u_0 + \sum_{m=1}^{\infty} u_m(r,t)\left(\frac{1}{n}\right)^m. \tag{23}$$

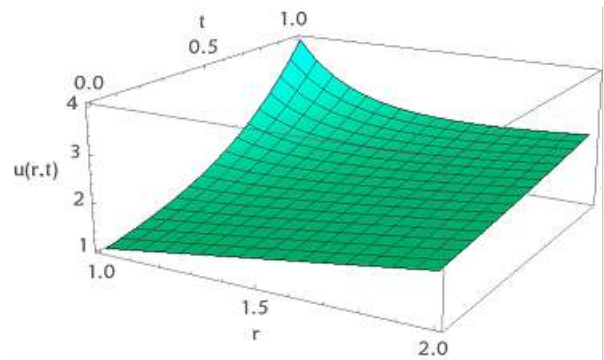If $n = 1$, $\hbar = -1$ in Eq. (23), we arrive at

$$u\left(r,t\right) = r + \sum_{n=1}^{\infty} \frac{1^2.3^2\ldots.(2n-3)^2}{r^{2n-1}}\frac{t^{n\alpha}}{n!}. \tag{24}$$

For $\alpha = 1$ in Eq. (24), we gain solution of standard N-S equation as

$$u\left(r,t\right) = r + \sum_{n=1}^{\infty} \frac{1^2.3^2\ldots.(2n-3)^2}{r^{2n-1}}\frac{t^n}{n!},$$

(a)

(b)

Fig. 1: Plots of q-HATM solution $u\left(r,t\right)$ Vs. r, t when $\hbar = -1$, $n = 1$, for (a) $\alpha = 0.5$ (b) $\alpha = 1$, for Ex. 4.1

Example 4.2. Now, consider time-fractional N-S equation

$$D_t^\alpha u = P + D_r^2 u + \frac{1}{r}D_r u, \, 0 < \alpha \le 1, \tag{25}$$

with initial condition

$$u\left(r,0\right) = 1 - r^2. \tag{26}$$

Taking LT on Eq. (25) and simplifying, we get

$$L\left[u\right] - \frac{\left(1-r^2\right)}{s} - \frac{P}{s^{\alpha+1}} - \frac{1}{s^\alpha}L\left[D_r^2 u + \frac{1}{r}D_r u\right] = 0. \tag{27}$$

Also,

$$N\left[\varphi\left(r,t;q\right)\right] = L\left[\varphi\left(r,t;q\right)\right] - \left(1 - \frac{k_m}{n}\right)\frac{\left(1-r^2\right)}{s}$$
$$- \frac{P}{s^{\alpha+1}} - \frac{1}{s^\alpha}L\left[\frac{\partial^2\varphi(r,t;q)}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi(r,t;q)}{\partial r}\right]. \tag{28}$$

For $H = 1$, deformation equation is

$$L\left[u_m\left(r,t\right) - k_m u_{m-1}\left(r,t\right)\right] = \hbar R_m\left(\vec{u}_{m-1}\right), \tag{29}$$

Table 1: Comparison of results by HAM, FMLDM, HPTM, q-HATM at different values of r and t when $\alpha = 1$, $\hbar = -1$, $n = 1$ for Ex. 4.1.

r	t	HAM[21]	FMLDM [22]	HPTM [23]	q-HATM
1.25		1.4736800000	1.4736800000	1.4736800000	1.4736800000
1.50	0.25	1.6790123457	1.6790123457	1.6790123457	1.6790123457
1.75		1.9001160231	1.9001160231	1.9001160231	1.9001160231
2.00		2.1296386719	2.1296386719	2.1296386719	2.1296386719
1.25		1.7754400000	1.7754400000	1.7754400000	1.7754400000
1.50	0.50	1.8950617284	1.8950617284	1.8950617284	1.8950617284
1.75		2.0704617124	2.0704617124	2.0704617124	2.0704617124
2.00		2.2714843750	2.2714843750	2.2714843750	2.2714843750
1.25		2.2013600000	2.2013600000	2.2013600000	2.2013600000
1.50	0.75	2.1666666667	2.1666666667	2.1666666667	2.1666666667
1.75		2.2696049265	2.2696049265	2.2696049265	2.2696049265
2.00		2.4299316406	2.4299316406	2.4299316406	2.4299316406
1.25		2.7975200000	2.7975200000	2.7975200000	2.7975200000
1.50	1	2.5123456790	2.5123456790	2.5123456790	2.5123456790
1.75		2.5061135241	2.5061135241	2.5061135241	2.5061135241
2.00		2.6093750000	2.6093750000	2.6093750000	2.6093750000
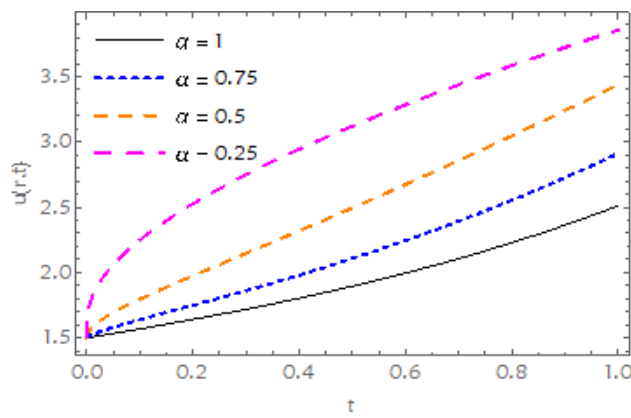
Fig. 2: α-curve of solution $u(r, t)$ Vs. t when $\hbar = -1$, $n = 1$, $r = 1.5$ for Ex. 4.1
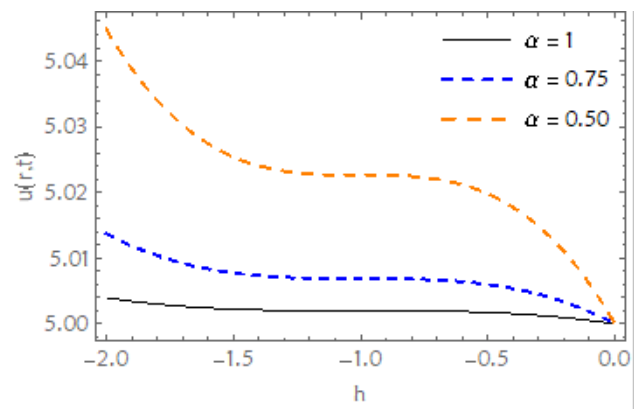
Fig. 4: $\hbar$-curve of q-HATM solution $u(r, t)$ Vs. h when $t = 0.01$, $r = 1$, $n = 1$ for Ex. 4.1
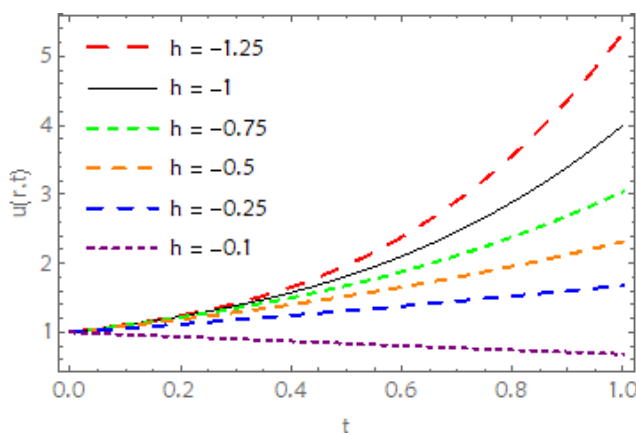
Fig. 3: Plot of solution $u(r, t)$ Vs. t when $n = 1 = \alpha = r$ for Ex. 4.1
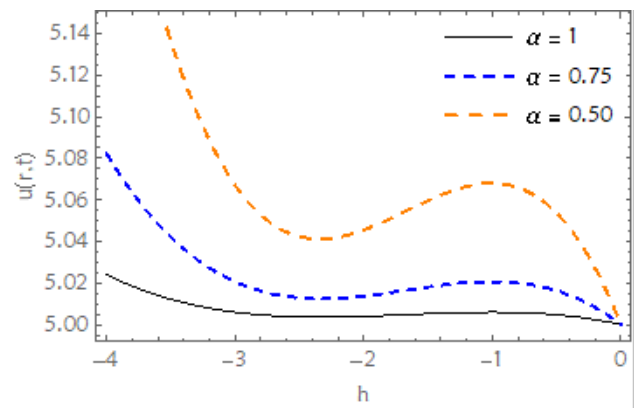
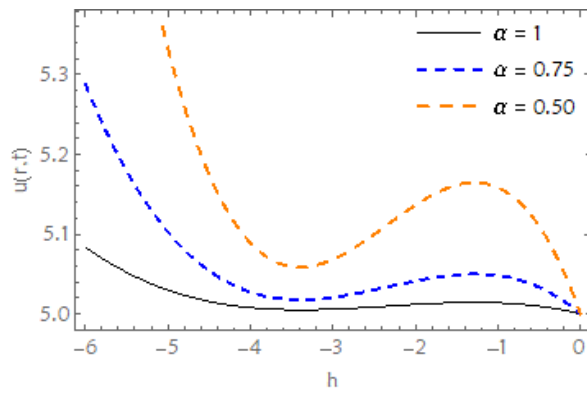**Fig. 5:** $\hbar$-curve of q-HATM solution $u(r, t)$ Vs. h when $t = 0.01$, $r = 1$, $n = 2$ for Ex. 4.1

Fig. 6: $\hbar$-curve of q-HATM solution $u(r, t)$ Vs. h when $t = 0.01$, $r = 1$, $n = 3$ for Ex. 4.1

Here

$$R_m\left(\vec{u}_{m-1}\right) = L\left[u_{m-1}\right] - \left(1 - \frac{k_m}{n}\right)\frac{\left(1 - r^2\right)}{s}$$

$$- \frac{P}{s^{\alpha+1}} - \frac{1}{s^{\alpha}}L\left[\frac{\partial^2 u_{m-1}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{m-1}}{\partial r}\right]. \tag{30}$$

By inverse transform,

$$u_m = k_m u_{m-1} + \hbar L^{-1}\left\{R_m\left(\vec{u}_{m-1}\right)\right\}. \tag{31}$$

Taking u_0 and solving, we get

$$u_1 = \frac{\hbar(4 - P)t^{\alpha}}{\Gamma(\alpha + 1)},$$

$$u_2 = \frac{\left[(n + \hbar)(4 - P) - P\right]\hbar t^{\alpha}}{\Gamma(\alpha + 1)},$$

$$u_3 = \frac{\left[(n + \hbar)^2(4 - P) - (n + \hbar)P - P\right]\hbar t^{\alpha}}{\Gamma(\alpha + 1)},$$
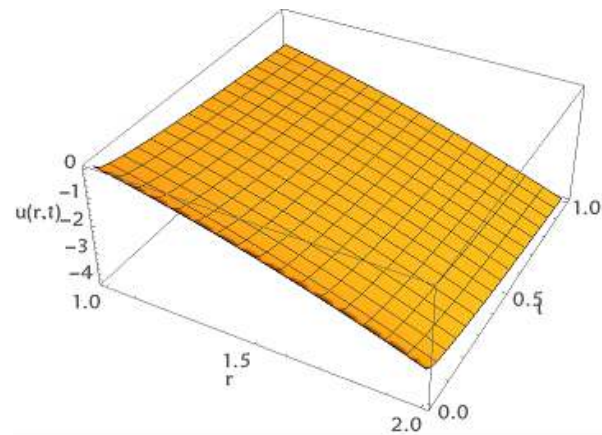
and so on ....

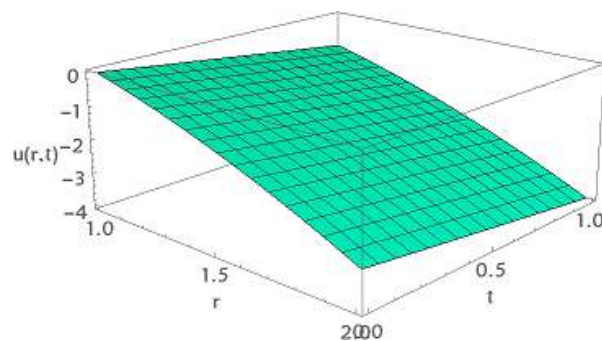Series solution of Eq. (25) is

$$u(r, t) = u_0 + \sum_{m=1}^{\infty} u_m(r, t)\left(\frac{1}{n}\right)^m.$$

5 Numerical simulations and discussion

Figs. 1 and 7 show plots of numerical results of Eqs. (16) and (25) respectively when $\alpha = 0.5$ and 1 for Ex. 4.1 and 4.2. Fig. 2 displays behaviour of solution for arbitrary order α and standard case $\alpha = 1$ at $r = 0.5$, $\hbar = -1$, $n = 1$ for Ex. 4.1. Fig. 8 shows behaviour of solution for arbitrary order

(a)

(b)

Fig. 7: Plots of q-HATM solution $u(r, t)$ Vs. r, t when $P = 1$, $\hbar = -1$, $n = 1$, (a) $\alpha = 0.5$, (b) $\alpha = 1$, for Ex. 4.2.

α and standard case $\alpha = 1$ at $r = 1.5$, $\hbar = -1$, $n = 1 = P$ for Ex. 4.2. It is clear from Fig. 2 and 8 that as α tends to 1, the q-HATM solution converges. Fig. 3 and 9 represent convergence control parameter curves for Ex. 4.1 and 4.2. The value of $\hbar$ should be negative. From Fig. 3 and 9, it is clear that, as $\hbar$ decreases, rate of covergence increases. Figs. 4–6 and 10–12, show $\hbar$-curves at distinct order of fractional derivative at $n = 1$, 2, 3 for Ex. 4.1 and 4.2. In h-curves, horizontal line exhibits convergence range of solution. We observe that as arbitrary order of derivative increases, range of convergence increases. Also, from Figs. 4–6, 10–12, it is clear that range of convergence depends positively on n. Table 1 shows comparison of results by HAM, FMLDM, HPTM, q-HATM at different values of r and t when $\alpha = 1 = n$, $\hbar = -1$ for Ex. 4.1. It can be observed from Table 1 that our results are accurate and agree with existing methods.
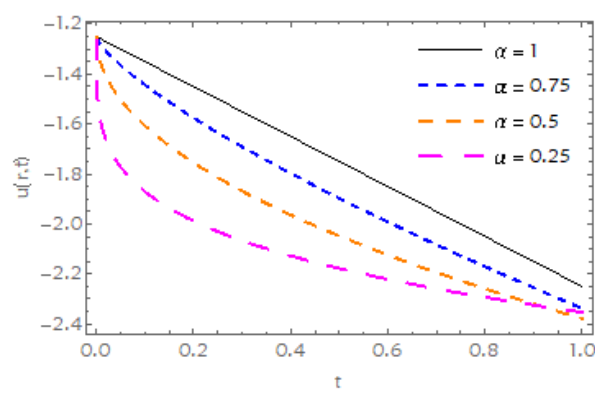
Fig. 8: α-curve of solution $u(r, t)$ Vs. t when $P = 1$, $\hbar = -1$, $n = 1$, $r = 1.5$ for Ex. 4.2.
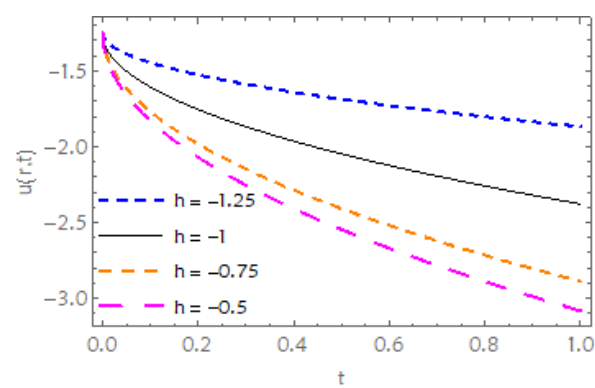

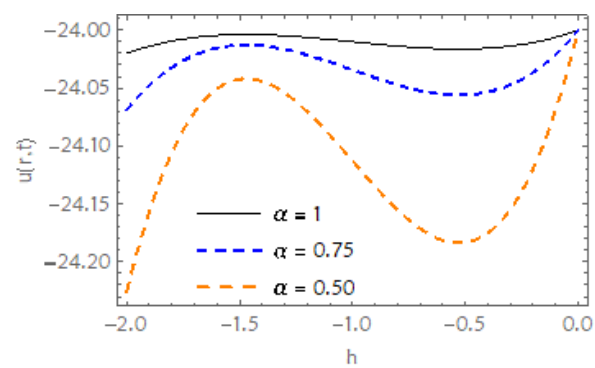**Fig. 9:** Plot of solution $u(r, t)$ Vs. t when $P = 1$, $n = 1$, $\alpha = 1$, $r = 1$, for Ex. 4.2.

Fig. 10: h-curve of solution $u(r, t)$ Vs. $\hbar$ obtained when $t = 0.01$, $r = 1$, $n = 1$, for Ex. 4.2

Fig. 11: h-curve of solution $u(r, t)$ Vs. $\hbar$ obtained when $t = 0.01$, $r = 1$, $n = 2$ for Ex. 4.2
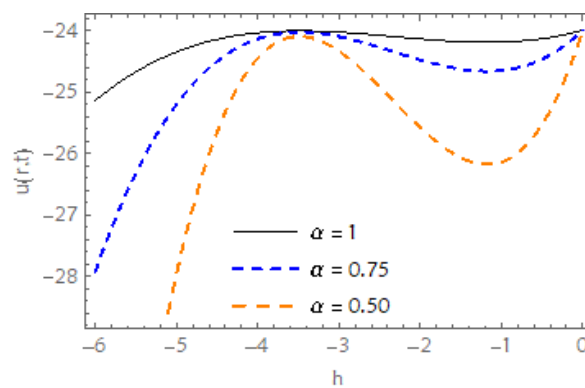
Fig. 12: h-curve of solution $u(r, t)$ Vs. $\hbar$ when $t = 0.01$, $r = 1$, $n = 3$ for Ex. 4.2

q-HATM contains parameters $\hbar$ and n, that can be adopted to manage convergence of solution. It makes this scheme more powerful and exciting.

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6 Conclusion

In this paper, approximate analytic solution of time-fractional N-S equation is gained by the q-HATM. It is worth mentioning that q-HATM is capable of reducing time and work of computation in comparison to existing numerical methods keeping higher accuracy of results intact. The
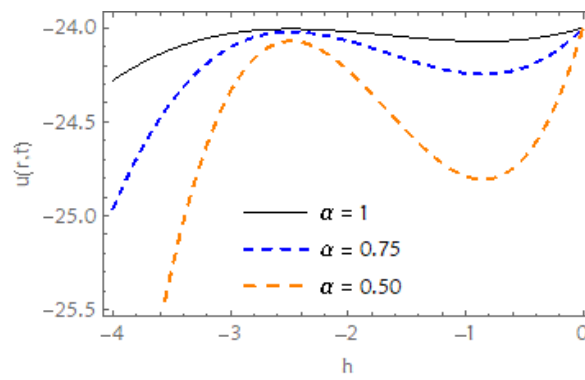
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