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A REMAINDER FORMULA AND LIMITS OF CARDINAL SPLINE INTERPOLANTS

BY

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ABSTRACT. A Peano-type remainder formula

$$f(x) - S_n(f; x) = \int_{-\infty}^{\infty} K_n(x, t) f^{(n+1)}(t)\, dt$$

for a class of symmetric cardinal interpolation problems C.I.P. $(E, F, \mathbf{x})$ is obtained, from which we deduce the estimate $\| f - S_{n,r}(f;\ )\|_\infty \leqslant K \| f^{(n+1)}\|_\infty$. It is found that the best constant K is obtained when $\mathbf{x}$ comprises the zeros of the Euler-Chebyshev spline function. The remainder formula is also used to study the convergence of spline interpolants for a class of entire functions of exponential type and a class of almost periodic functions.

1. Introduction. As in [5], for $\mathbf{x} = (x_0, x_1, \ldots, x_m)$, $0 = x_0 < x_1 < \cdots < x_m = 1$ and incidence matrices $E = \| E_{ij} \|_{i=0}^{m}{}_{j=0}^{n}$, $F = \| F_{ij} \|_{i=0}^{m}{}_{j=0}^{n}$ with $E_{0j} = E_{mj}$ and $F_{0j} = F_{mj}$, $j = 0, \ldots, n$, let $\mathcal{C}(F, \mathbf{x}) := \{f\colon\ \mathbf{R} \to \mathbf{C};\ \forall \nu \in \mathbf{Z},\ f|(\nu + x_i, \nu + x_{i+1}) \in \pi_n$, $i = 0, \ldots, m-1$, and $f^{(n-j)}(\nu + x_i^-) = f^{(n-j)}(\nu + x_i^+)\ \forall (i, j)$ with $F_{ij} = 0\}$, and refer to the following 'cardinal' interpolation problem as the C.I.P. $(E, F, \mathbf{x})$:

For sequences of numbers $\{y^{(i,j)}\} = \{y_\nu^{(i,j)};\ 0 \leqslant i < m$ and $E_{ij} = 1\}$, find $S \in \mathcal{C}(F, x)$ satisfying $S^{(j)}(\nu + x_i) = y_\nu^{(i,j)}$.

Sufficient conditions for C.I.P. $(E, F, \mathbf{x})$ to be poised, i.e. existence of a unique $S \in \mathcal{C}(F, \mathbf{x})$, $S(x) = O(|x|^\gamma)$ as $x \to \pm\infty$ satisfying $S^{(j)}(\nu + x_i) = y_\nu^{(i,j)}$ when $y_\nu^{(i,j)} = O(|\nu|^\gamma)$, are given in [5].

Suppose that the C.I.P. $(E, F, \mathbf{x})$ is poised; then given a sufficiently smooth function f of power growth $\exists$ a unique $S_n(f;\ ) \in \mathcal{C}(F, \mathbf{x})$ of power growth which interpolates f in the sense that

$$S_n^{(j)}(f; \nu + x_i) = f^{(j)}(\nu + x_i), \qquad \nu \in \mathbf{Z}, E_{ij} = 1. \tag{1.1}$$

The following problem then arises.

Problem. Find necessary and sufficient conditions so that $S_n(f;\ ) \to f$ uniformly as $n \to \infty$.

This question was first raised by Schoenberg [9] who also found a sufficient condition for the convergence of $S_n(f;\ )$ for the case where $m = 1$, n is odd, $E = F$ and

$$E_{0j} = E_{1j} = \begin{cases} 0, & j = 1, 2, \ldots, n, \\ 1, & j = 0. \end{cases} \tag{1.2}$$

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In this case $\mathcal{C}(F, \mathbf{x})$ comprises odd degree cardinal splines with integer knots. Schoenberg [**9**] proves the following

THEOREM A. *Let*

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iux}\, d\alpha(u) \tag{1.3}$$

where $\alpha(u)$ *is a function of bounded variation in* $[-\pi, \pi]$ *and let* $A = \alpha(-\pi + 0) - \alpha(-\pi)$, $B = \alpha(\pi) - \alpha(\pi - 0)$. *Then*

$$\lim_{k\to\infty} S_{2k-1}(f; x) = f(x) + \frac{i(A-B)}{2\pi}\sin \pi x$$

uniformly on $\mathbf{R}$.

A partial converse of Theorem A was given by Richards and Schoenberg [**8**]. Subsequently, Marsden and Riemenschneider [**6**] generalised Theorem A to the case of cardinal Hermite interpolation which corresponds to $m = 1$, $E = F$ and, for some $1 \leqslant r \leqslant \frac{1}{2}(n+1)$,

$$E_{0j} = E_{1j} = \begin{cases} 1, & j = 0, 1, \ldots, r-1, \\ 0, & j = r, \ldots, n. \end{cases}$$

In this case, a sufficient condition for convergence of $S_n(f;\ )$ analogous to that of Richards and Schoenberg was given by Goodman [**4**].

Recently, in an attempt to obtain more information on the convergence problem, I. J. Schoenberg [**10**] obtained a Peano type remainder formula

$$f(x) - S_{2k-1}(f; x) = \int_{-\infty}^{\infty} K_{2k-1}(x; t) f^{(2k)}(t)\, dt \qquad (x \in \mathbf{R}) \tag{1.4}$$

for the case $m = 1$, $n = 2k - 1$, $E = F$ and E satisfies (1.2), where $K_{2k-1}(x, t) := (x-t)_+^{2k-1} - S_{2k-1}((\cdot - t)_+^{2k-1}; x)$. From (1.4), Schoenberg [**10**] deduced Theorem A and also obtained a convergence result for a class of almost periodic functions.

In this paper we shall consider the symmetric C.I.P. $(E, F, \mathbf{x})$

$$\begin{aligned} &x_i = 1 - x_{m-i}, \qquad i = 0, \ldots, m, \\ &F_{ij} = 1 \quad \text{iff } i = 0 \text{ or } m \text{ and } j = 0, \ldots, r-1, \end{aligned} \tag{1.5}$$

for some $1 \leqslant r \leqslant n + 1$, and either

(a) $n + r$ is even, $m = r$ and $E_{ij} = 1$ iff $j = 0$ and $i = 0, \ldots, m$, or

(b) $n + r$ is odd, $m = r + 1$ and $E_{ij} = 1$ iff $j = 0$ and $i = 1, \ldots, m - 1$.

That this problem is poised follows from Corollary 4.3 of [**5**] and was earlier shown by Micchelli [**7**]. In this case the class of cardinal spline functions $\mathcal{C}(F, \mathbf{x})$ is usually denoted by $\mathcal{S}_{n,r}$, and clearly

$$\mathcal{S}_{n,r} \equiv \mathcal{C}(F, \mathbf{x}) = \left\{ S \in C^{n-r}(\mathbf{R});\ S|_{(\nu,\nu+1)} \in \pi_n, \forall \nu \in \mathbf{Z} \right\}.$$

For r, n as above, we define $\mathcal{F}_{n,r} := \{ f \in C^{n+1-r}(\mathbf{R});\ f|_{(\nu,\nu+1)} \in C^n[(\nu, \nu+1)]$ and $f^{(n)}$ bounded and absolutely continuous on $(\nu, \nu + 1)$, $\forall \nu \in \mathbf{Z}\}$.

For $f \in \mathfrak{F}_{n,r}$ of power growth, we let $S_{n,r}(f;\ )$ denote the unique function of power growth that interpolates f as in (1.1). Following the approach of Schoenberg [**10**] we derive a formula for the remainder $f - S_{n,r}(f;\ )$ and deduce the following result.

THEOREM 1. *For fixed* n, r *and* $\mathbf{x}$, $\exists K$ *such that for any* $f \in \mathfrak{F}_{n,r}$ *with* $f^{(n+1-r)}$ *of power growth and* $\| f^{(n+1)} \|_\infty < \infty$,

$$\| f - S_{n,r}(f;\ ) \|_\infty \leqslant K \| f^{(n+1)} \|_\infty, \tag{1.6}$$

and equality is attained for some $f \in \mathfrak{F}_{n,r}$. *For fixed* n, r, K *is a minimum when* $\mathbf{x}$ *comprises the zeros of the Euler-Chebyshev spline* $\mathcal{E}_{n+1,r}$ (*see* [**1**] *and* [**4**]) *and in this case equality is attained for* $f = \mathcal{E}_{n+1,r}$.

For $r = n + 1$, this result reduces to a classical result on optimal constants in the remainder for Lagrange interpolation by polynomials (see [**3**, p. 64]).

Now let $B_\sigma = \{f; f$ is the restriction to $\mathbf{R}$ of an entire function of exponential type $\leqslant \sigma$ and $\| f \|_\infty < \infty\}$.

By deriving bounds on the best constants K in (1.6), we prove the following results, all of which are generalisations of results of Schoenberg [**10**].

THEOREM 2. *For fixed* $\mathbf{x}$ *and* $r \geqslant 1, \exists K_r$ *such that for all* $n \geqslant r - 1$ *and* $f \in B_\sigma$,

$$\| f - S_{n,r}(f;\ ) \|_\infty \leqslant K_r (\sigma / r\pi)^{n+1} \| f \|_\infty.$$

THEOREM 3. *If* $f(x) = \int_{-r\pi}^{r\pi} e^{iux}\, d\alpha(u)$, *where* $\alpha(u)$ *is a function of bounded variation in* $[-r\pi, r\pi]$, *then*

$$\lim_{n \to \infty} S_{n,r}(f; x) = f(x) + C 2^{r-1} \prod_{i=1}^{r} \sin \pi(x - x_i)$$

uniformly, *where*

$$C = \begin{cases} (-1)^r i\{\alpha(r\pi) - \alpha(r\pi - 0) + \alpha(-r\pi) - \alpha(-r\pi + 0)\} & \text{if } n + r \text{ is even}, \\ (-1)^{r-1}\{\alpha(r\pi) - \alpha(r\pi - 0) - \alpha(-r\pi) + \alpha(-r\pi + 0)\} & \text{if } n + r \text{ is odd}. \end{cases}$$

THEOREM 4. *If* $f \in B_{r\pi}$ *is almost periodic in the sense of Bohr*, *then*

$$\lim_{n \to \infty} S_{n,r}(f; x) = f(x) + C 2^{r} \prod_{i=1}^{r} \sin \pi(x - x_i)$$

uniformly, *where*

$$C = \begin{cases} (-1)^r \lim\limits_{T \to \infty} \dfrac{1}{T} \displaystyle\int_0^T f(x) \sin r\pi x\, dx & \text{if } n + r \text{ is even}, \\ (-1)^{r-1} \lim\limits_{T \to \infty} \dfrac{1}{T} \displaystyle\int_0^T f(x) \cos r\pi x\, dx & \text{if } n + r \text{ is odd}. \end{cases}$$

In §2 we apply the results of our preceding paper [**5**] to the study of the sign structure of the kernel, from which the corresponding remainder formula is derived. The proof of Theorem 1 is given in §3. Theorem 1 is then used in §4 to derive convergence results.

2. The remainder formula. Take $1 \leqslant r \leqslant n+1$ and consider the poised, symmetric C.I.P. $(E, F, \mathbf{x})$ defined in §1. In this case the relation (1.5) and the conditions (a) and (b) imply that $0 < x_1 < \cdots < x_r \leqslant 1$ with equality iff $n+r$ is even. We shall also write $\mathcal{S}_{n,r}$ for $\mathcal{C}(F, \mathbf{x})$ and $\tilde{\mathcal{S}}_{n,r}$ for $\mathcal{C}(E, \mathbf{x})$ which corresponds to the dual C.I.P. $(F, E, \mathbf{x})$.

Specialising the results of [**5**] to this special case it is easy to see that the corresponding null space

$$\mathcal{C}^0 \equiv \mathcal{C}^0(F, \mathbf{x}) := \{S \in \mathcal{C}(F, \mathbf{x});\ S(\nu + x_i) = 0\ \forall \nu \in \mathbf{Z},\ E_{i0} = 1\}$$

has dimension d, where

$$d = \begin{cases} n - r & \text{if } n + r \text{ is even,} \\ n - r + 1 & \text{if } n + r \text{ is odd,} \end{cases}$$

and is spanned by d eigensplines S_j, $j = 1, \ldots, d$, satisfying the functional relation

$$S_j(x+1) = \lambda_j S_j(x) \qquad \forall x \in \mathbf{R}. \tag{2.1}$$

The eigenvalues λ_j, $j = 1, \ldots, d$, of the C.I.P. $(E, F, \mathbf{x})$ are real, distinct, of sign $(-1)^r$ and are precisely the eigenvalues of the matrix $C = (C_{\mu\nu})_{d\times d}$ where $S^{\mu}(1) = C_{\mu\nu} S^{(\nu)}(0)$, $\mu, \nu = 0, \ldots, d$.

Now for $i = 1, \ldots, r$, we let L_i denote the unique element of $\mathcal{C}(F, \mathbf{x})$ of power growth (actually of exponential decay) satisfying

$$L_i(\nu + x_j) = \delta_{\nu 0}\delta_{ij}, \qquad \forall \nu \in \mathbf{Z}, j = 1, \ldots, r.$$

Then, for $f \in \mathcal{F}_{n,r}$ of power growth, the unique function $S_{n,r}(f;\ ) \in \mathcal{C}(F, \mathbf{x})$ of power growth that interpolates f for the C.I.P. $(E, F, \mathbf{x})$ is given by

$$S_{n,r}(f; x) = \sum_{i=1}^{r} \sum_{\nu=-\infty}^{\infty} f(\nu + x_i) L_i(x - \nu). \tag{2.2}$$

We let $\tilde{S}_{n,r}(f;\ ) \in \tilde{\mathcal{S}}_{n,r}$ denote the unique spline function of power growth that interpolates f for the dual C.I.P. $(F, E, \mathbf{x})$.

For $x, t \in \mathbf{R}$ we define

$$g_t(x) \equiv \tilde{g}_x(t) := (1/n!)(x-t)_+^n$$

and

$$K(x,t) \equiv K_t(x) \equiv \tilde{K}_x(t) := g_t(x) - S_{n,r}(g_t; x). \tag{2.3}$$

Clearly, from (2.1), we have

$$K(x+1, t+1) = K(x,t), \qquad \forall x, t \in \mathbf{R}, \tag{2.4}$$

and

$$K_t(\nu + x_i) = 0, \qquad \forall \nu \in \mathbf{R} \text{ and } i = 1, \ldots, r. \tag{2.5}$$

Now we see from (2.2) and (2.3) that, for $\rho = 0, \ldots, n$,

$$\tilde{K}_x^{(\rho)}(t) = (-1)^\rho g_t^{(\rho)}(x) - S_{n,r}\big((-1)^\rho g_t^{(\rho)}; x\big).$$

But for $\nu \in \mathbf{Z}$ and $\rho = 0, \ldots, r-1$, $g_\nu^{(\rho)} \in \mathcal{C}(F, \mathbf{x})$ and so

$$\tilde{K}_x^{(\rho)}(\nu) = 0, \qquad \forall x \in \mathbf{R}. \tag{2.6}$$

Now we see from (2.2) that, for fixed x, $S_{n,r}(g_t;\, x)$ as a function of t lies in $\mathcal{C}(E, \mathbf{x})$ and has power growth. By (2.6) it interpolates $\tilde{g}_x$ for the C.I.P. $(F, E, \mathbf{x})$. Hence $S_{n,r}(g_t;\, x) = \tilde{S}_{n,r}(\tilde{g}_x;\, t)$, $\forall x, t \in \mathbf{R}$, and so

$$K(x, t) = \tilde{g}_x(t) - \tilde{S}_{n,r}(\tilde{g}_x;\, t). \tag{2.7}$$

LEMMA 2.1. *For $1 < r < n$, there is a constant $\beta > 0$ such that the following hold for $\rho = 0, \ldots, n$.*

(a) *For $t \in \mathbf{R}$, $K_t^{(\rho)}(x) = O(e^{-\beta|x|})$ as $x \to \pm\infty$ and the only zeros of K_t ($t \notin \mathbf{Z}$) are simple zeros at $\nu + x_i$, $\nu \in \mathbf{Z}$, $i = 1, \ldots, r$.*

(b) *For $x \in \mathbf{R}$, $\tilde{K}_x^{(\rho)}(t) = O(e^{-\beta|t|})$ as $t \to \pm\infty$ and the only zeros of $\tilde{K}_x$ ($x \notin \mathbf{Z} + x_i$) are isolated zeros of multiplicity r at the integers.*

PROOF. We shall prove only (a) as (b) follows similarly by duality, and only for even $n + r$ as the result for odd $n + r$ follows similarly. By (2.4) and (2.6) we may suppose $0 < t < 1$.

Now, for $n + r$ even the C.I.P. $(E, F, \mathbf{x})$ has $n - r$ distinct eigenvalues $\lambda_1, \ldots, \lambda_{n-r}$ of sign $(-1)^r$ with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_{n-r}| > 0$ and $\lambda_{n-r-i+1} = \lambda_i^{-1}$, $i = 1, \ldots, n - r$.

We let $S_1, \ldots, S_{n-r}$ denote the corresponding eigensplines, which span $\mathcal{C}^0$. The eigenvalues $\lambda_1, \ldots, \lambda_{n-r}$ are precisely the eigenvalues of the matrix C where $(-1)^r C$ is an oscillation matrix with corresponding eigenvectors $(S_i'(0), \ldots, S_i^{(n-r)}(0))$, $i = 1, \ldots, n - r$. So by a theorem of Gantmacher and Krein we have

$$S\big(S_i'(0), \ldots, S_i^{(n-r)}(0)\big) = i - 1, \qquad i = 1, \ldots, n - r$$

(see also Micchelli [7]).

Now $K_t \in C^{n-r}(\mathbf{R})$ and for $\nu = 1, 2, \ldots$, $K_t|_{(\nu,\nu+1)} \in \pi_n$. Thus, by (2.5), $K_t|_{[1,\infty)}$ can be extended to an element of $\mathcal{C}^0$. Since K_t is of power growth, we therefore have

$$K_t(x) = \sum_{i \geq p} c_i S_i(x), \qquad \forall x \geq 1,$$

where $c_p \neq 0$ and $p > \frac{1}{2}(n - r)$. Similarly $K_t(x) = \sum_{i \leq q} c_i S_i(x)$, $\forall x \leq 0$, where $c_q \neq 0$ and $q < \frac{1}{2}(n - r)$.

The first part of (a) follows.

Now for $\rho = 0, \ldots, n - r$ and $\nu = 1, 2, \ldots$,

$$K_t^{(\rho)}(\nu) = \sum_{i \geq p} c_i \lambda_i^\nu S_i^{(\rho)}(0) \quad \text{and} \quad K_t^{(\rho)}(-\nu) = \sum_{i \leq q} c_i \lambda_i^{-\nu} S_i^{(\rho)}(0).$$

So $K_t^{(\rho)}(\nu) = c_p \lambda_p^\nu S_p^{(\rho)}(0) + O(\lambda_p^\nu)$ as $\nu \to \infty$ and $K_t^{(\rho)}(-\nu) = c_q \lambda_q^{-\nu} S_q^{(\rho)}(0) + O(\nu_q^{-\nu})$ as $\nu \to \infty$. Thus for large enough N,

$$S\big(K_t'(N), \ldots, K_t^{(n-r)}(N)\big) = S\big(S_p'(0), \ldots, S_p^{(n-r)}(0)\big) = p - 1 \tag{2.8}$$

and

$$S\big(K_t'(-N), \ldots, K_t^{(n-r)}(-N)\big) = S\big(S_q'(0), \ldots, S_q^{(n-r)}(0)\big) = q - 1. \tag{2.9}$$

We now show that K_t is oscillating in $(-N, N)$. For suppose $K_t = 0$ on some interval $(a, b) \in (-N, N)$. Since $K_t \in C^{n-r}(\mathbf{R})$ and is a piecewise polynomial with knots at $\mathbf{Z} \cup \{t\}$, $a, b \in \mathbf{Z} \cup \{t\}$. Furthermore, either $a < t$ or $b > t$. Suppose $a < t$. Then $K_t^{(\rho)}(a) = 0$, $\rho = 0, \ldots, n - r$, and $K_t(a - 1 + x_i) = 0$, $i = 0, \ldots, r - 1$, implies that $K_t = 0$ on $(a - 1, a)$. Hence $K_t^{(\rho)}(a - 1) = 0$, $\rho = 0, \ldots, n - r$. By induction we have $K_t^{(\rho)}(-N) = 0$, $\rho = 0, \ldots, n - r$, which contradicts (2.9). Similarly $b > t$ leads to $K_t^{(\rho)}(N) = 0$, $\rho = 0, \ldots, n - r$, which contradicts (2.8).

Now K_t has exact degree n in $(0, 1)$ and so we may apply Theorem 2.1 of **[5]** to $g := K_t|_{(-N,N)}$ to give

$$\begin{aligned} Z(g) &\leqslant (2N - 1)r + 1 + S^- \left(g(-N^+), \ldots, g^{(n)}(-N^+)\right) \\ &\quad - S^+ \left(g(N^-), \ldots, g^{(n)}(N^-)\right) \\ &\leqslant (2N - 1)r + 1 + (q - 1 + r) - p \quad (\text{since } g(-N^+) = g(N^-) = 0) \\ &= 2Nr + q - p \leqslant 2Nr - 1. \end{aligned}$$

But K_t has $2Nr - 1$ zeros in $(-N, N)$ at points as in (2.5) and hence these are the only zeros of K_t in $(-N, N)$. Since N can be arbitrarily large, (a) follows. □

We must also consider the special cases $r = n$ or $n + 1$, for which it is easy to see the following. If $t \in (\nu, \nu + 1)$, then K_t vanishes outside $[\nu, \nu + 1]$ and vanishes in $(\nu, \nu + 1)$ only at $\nu + x_i$, $i = 1, \ldots, r$. Similarly if $x \in (\nu, \nu + 1)$, then $\tilde{K}_x$ vanishes outside $[\nu, \nu + 1]$ and vanishes nowhere in $(\nu, \nu + 1)$.

THEOREM 2.1. *If $f \in \mathfrak{F}_{n,r}$ and $f^{(n+1-r)}$ is of power growth, then*

$$f(x) - S_{n,r}(f; x) = \int_{-\infty}^{\infty} K(x, t) f^{(n+1)}(t)\, dt. \tag{2.10}$$

PROOF. By (2.4) we may assume $0 < x \leqslant 1$. We may also assume $x \neq x_i$, $i = 1, \ldots, r$, since otherwise (2.10) is trivially satisfied.

Now integrating by parts and applying (2.6) gives, for any $\nu \in \mathbf{Z}$,

$$\int_{\nu}^{\nu+1} K(x, t) f^{(n+1)}(t)\, dt = (-1)^r \int_{\nu}^{\nu+1} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t)\, dt.$$

So $\int_{-\infty}^{\infty} K(x, t) f^{(n+1)}(t)\, dt = (-1)^r \int_{-\infty}^{\infty} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t)\, dt$ which converges since $f^{(n+1-r)}$ is of power growth and $K_x^{(r)}$ decays exponentially. Also $f^{(\rho)}$ is of power growth for $0 \leqslant \rho \leqslant n + 1 - r$ and so we may integrate by parts to give

$$(-1)^r \int_{-\infty}^{\infty} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t)\, dt = (-1)^n \int_{-\infty}^{\infty} \tilde{K}_x^{(n)}(t) f'(t)\, dt.$$

Now $(-1)^n \tilde{K}_x^{(n)}(t) = (x - t)_+^0 - \sum_{i=1}^r \sum_{\nu=-\infty}^{\infty} (\nu + x_i - t)_+^0 L_i(x - \nu)$ and so

$$\begin{aligned} (-1)^n \int_{-\infty}^{\infty} \tilde{K}_x^{(n)}(t) f'(t)\, dt &= f(x) - \sum_{i=1}^{r} \sum_{\nu=-\infty}^{\infty} f(\nu + x_i) L_i(x - \nu) \\ &= f(x) - S_{n,r}(f; x). \quad \square \end{aligned}$$

3. Proof of Theorem 1. We see immediately from Theorem 2.1 that for $f \in \mathfrak{F}_{n,r}$ with $f^{(n+1-r)}$ of power growth and $\| f^{(n+1)} \|_\infty < \infty$

$$\| f - S_{n,r}(f;\ ) \|_\infty \leqslant \sup_{x \in \mathbf{R}} \left\{ \int_{-\infty}^{\infty} | K(x, t) |\, dt \right\} \| f^{(n+1)} \|_\infty.$$

Now there is an eigenspline $E_{n+1,r} \in \mathbb{S}_{n+1,r}$ with eigenvalue $(-1)^r$ which vanishes at $\nu + x_i$, $\forall \nu \in \mathbf{Z}$ and $i = 1,\dots,r$. We assume it is normalised so that $|E_{n+1,r}^{(n+1)}| = 1$.

Putting $f = E_{n+1,r}$ in (2.10), we see from Lemma 2.1(b) that

$$|E_{n+1,r}(x)| = \int_{-\infty}^{\infty} |K(x,t)|\,dt, \qquad \forall x \in \mathbf{R},$$

so that the constant K in (1.6) is given by

$$K = \|E_{n+1,r}\|_\infty. \tag{3.1}$$

Following the definitions of Cavaretta [**1**] and Goodman [**4**], we let $\mathcal{E}_{n+1,r} \in \mathbb{S}_{n+1,r}$ denote the Euler-Chebyshev spline, normalised so that $|\mathcal{E}_{n+1,r}^{(n+1)}| = 1$. Now the zeros of $\mathcal{E}_{n+1,r}$ are points $\nu + \beta_i$, $\nu \in \mathbf{Z}$, where β_i, $i = 1,\dots,r$, are symmetric about $x = \frac{1}{2}$, and $0 < \beta_1 < \cdots < \beta_r \leqslant 1$, with equality iff $n + r$ is even. Furthermore

$$\mathcal{E}_{n+1,r}(x+1) = (-1)^r \mathcal{E}_{n+1,r}(x), \qquad \forall x \in \mathbf{R},$$

and $f = \mathcal{E}_{n+1,r}$ minimises $\|f\|_\infty$ over all $f \in \mathbb{S}_{n+1,r}$ with $\|f^{(n+1)}\|_\infty = 1$. The result follows. $\square$

4. Convergence of $S_{n,r}(f;\ )$. Henceforth we shall examine the behaviour of $S_{n,r}(f,x)$ as $n \to \infty$. Analogous problems were studied in [**6**, **9** and **10**]. We first derive an estimate for $\|E_{n+1,r}\|_\infty$.

LEMMA 4.1. *For fixed* **x** *and* $r \geqslant 1$, $\exists K_r$ *such that*

$$\|E_{n+1,r}\|_\infty \leqslant K_r/(r\pi)^{n+1}, \qquad \forall n \geqslant r-1. \tag{4.1}$$

PROOF. First, assume that r is odd. For $x \in [0,1]$, let

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \cdots + a_n E_1(x) + a_{n+1} E_0(x),$$

where $E_k(x)$, $k = 0,1,\dots,n+1$, are the Euler polynomials. The conditions $E_{n+1,r}^{(\rho)}(1) = -E_{n+1,r}^{(\rho)}(0)$ for $\rho = 0,1,\dots,n-r+1$ imply that $a_{n-k+1} = 0$ $\forall k = 0,1,\dots,n-r+1$. Hence for $x \in [0,1]$,

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \cdots + a_{r-1} E_{n-r+2}(x).$$

Now $E_{n+1,r}(x_i) = 0$, $i = 1,2,\dots,r$, gives a homogeneous system of equations $a_0 E_{n+1}(x_i) + a_1 E_n(x_i) + \cdots + a_{r-1} E_{n-r+2}(x_i) = 0$, $i = 1,2,\dots,r$, whose determinant must be zero. Hence we can write

$$E_{n+1,r}(x) = \frac{\det(E_{n-m+2}(\beta_l))_{l,m=1}^r}{\det(E_{n-m+2}(x_l))_{l,m=2}^r} \qquad \forall x \in [0,1]$$

where

$$\beta_l = \begin{cases} x & \text{if } l = 1, \\ x_l & \text{if } l \neq 1. \end{cases}$$

Using the Fourier expansions of the Euler polynomials $E_k(x)$ we obtain

$$|E_{n+1,r}(x)| = \frac{2}{\pi^{n+2}}\left|\frac{\det\left(\sum_{-\infty}^{\infty} e^{2k\pi i\beta_l}/(2k+1)^{n-m+3}\right)_{l,m=1}^{r}}{\det\left(\sum_{-\infty}^{\infty} e^{2k\pi i x_l}/(2k+1)^{n-m+3}\right)_{l,m=2}^{r}}\right|$$

$$= \left(\frac{2^r}{\pi^{n+2}}\right)\left|\frac{\sum_{k_1,k_2,\dots,k_r} V(k_1,k_2,\dots,k_r)\prod_{j=1}^{r} e^{2k_j\pi i\beta_j}/(2k_j+1)^{n+1}}{\sum_{k_2,k_3,\dots,k_r} V(k_2,k_3,\dots,k_r)\prod_{j=2}^{r} e^{2k_j\pi i x_j}/(2k_j+1)^{n+1}}\right|$$

where $V(a_1, a_2, \dots, a_r)$ denotes the Vandermonde determinant $\det(a_m^{l-1})_{l,m=1}^r = \prod_{1\le j<k\le r}(a_k - a_j)$. A straightforward computation gives

$$|E_{n+1,r}(x)|$$

$$(4.2)\quad \left(\frac{2^r}{\pi^{n+2}}\right)\left|\frac{\sum_{k_1<\cdots<k_r} V(k_1,k_2,\dots,k_r)\det(e^{2k_m\pi i\beta_l})_{l,m=1}^{r}\sum_{j=1}^{r}(2k_j+1)^{-(n+2)}}{\sum_{k_2<\cdots<k_r} V(k_2,k_3,\dots,k_r)\det(e^{2k_m\pi i x_l})_{l,m=2}^{r}\sum_{j=2}^{r}(2k_j+1)^{-(n+1)}}\right|.$$

The dominant term in the numerator of (4.2) is

$$\prod_{j=1}^{r}(2j-r)^{-(n+2)}\left\{V\left(-\frac{(r-1)}{2}, -\frac{(r-3)}{2}, \dots, \frac{r-1}{2}\right)(e^{(2m-r-1)\pi i\beta_l})_{l,m=1}^{r}\right.$$

$$(4.3)\qquad \left. - V\left(-\frac{(r+1)}{2}, -\frac{(r-1)}{2}, \dots, \frac{r-3}{2}\right)(e^{(2m-r-3)\pi i\beta_l})_{l,m=1}^{r}\right\}.$$

The dominant term in the denominator is

$$(4.4)\qquad \prod_{j=2}^{r}(2j-r-2)^{-(n+1)}V\left(-\frac{(r-1)}{2}, -\frac{(r-3)}{2}, \dots, \frac{(r-3)}{2}\right)$$

$$\times\det(e^{(2m-r-3)\pi i x_l})_{l,m=2}^{r}.$$

It follows from (4.2), (4.3) and (4.4) that $\forall x \in [0, 1]$

$$|E_{n+1,r}(x)| = \left(\frac{2^r}{\pi^{n+2}}\right)\left|\frac{\prod_{j=2}^{r}(2j-r-2)^{n+1}}{\prod_{j=1}^{r}(2j-r)^{n+2}}\right| O(1),$$

and (4.1) follows for odd r.

If r is even, a similar argument shows that for $x \in [0, 1]$ the eigensplines $E_{n+1,r}(x)$ may be expressed in terms of Bernoulli polynomials $B_k(x)$ as follows:

$$E_{n+1,r}(x) = \frac{\det^*(B_{n-m+3}(\beta_l))_{l=1;m=2}^{r}}{\det^*(B_{n-m+3}(x_l))_{l=2;m=3}^{r}},$$

where det* means that all the entries in the last row of the determinant are 1. Expanding each determinant along the last row and applying a similar method to each term gives the inequality (4.1) for r even. □

Now recall the definition of B_σ in §1. We shall need Bernstein's theorem that if $f \in B_\sigma$ then, for each integer n, $f^{(n)} \in B_\sigma$ and

$$\|f^{(n)}\|_\infty \leqslant \sigma^n \|f\|_\infty. \tag{4.5}$$

From (1.6), (3.1), (4.1) and (4.5), we can immediately deduce Theorem 2.

COROLLARY 4.1. *If $f \in B_\sigma$ and $\sigma < r\pi$, then*

$$\lim_{n\to\infty} S_{n,r}(f; x) = f(x) \quad \textit{uniformly on } \mathbf{R}.$$

COROLLARY 4.2. *If $f \in B_{r\pi}$, then for all $n \geqslant r - 1$*

$$\|f - S_{n,r}(f)\|_\infty \leqslant K_r \|f\|_\infty.$$

We now follow a similar approach to that of Schoenberg [**10**] in proving Theorem 3. First we introduce the class $B^*_{r\pi}$ of functions which are uniform limits on $\mathbf{R}$ of functions belonging to B_p for $p < r\pi$, i.e. $f \in B^*_{r\pi}$ if and only if $\exists f_j \in B_{p_j}$, $p_j < r\pi$, $j = 1, 2, 3, \ldots,$ such that $\|f - f_j\|_\infty \to 0$ as $j \to \infty$.

LEMMA 4.2. *If $f \in B^*_{r\pi}$ then*

$$\lim_{n\to\infty} S_{n,r}(f; x) = f(x) \quad \textit{uniformly}.$$

PROOF. Suppose $f_j \in B_{p_j}$, $p_j < r\pi$, $j = 1, 2, 3, \ldots,$ and $\|f - f_j\|_\infty \to 0$ as $j \to \infty$. Then $f - S_{n,r}(f;\ ) = f - f_j - S_{n,r}(f - f_j;\ ) + f_j - S_{n,r}(f_j;\ )$ and using Corollaries 4.1 and 4.2, the result follows as in [**10**]. □

Before we prove Theorems 3 and 4 we first study the behaviour of the spline functions $S_{n,r}(\cos r\pi x)$ and $S_{n,r}(\sin r\pi x)$ that interpolate $\cos r\pi x$ and $\sin r\pi x$ respectively at $\nu + x_i$, $i = 1, 2, \ldots, r$, $\nu \in \mathbf{Z}$. In order to simplify writing, we define $\alpha_1, \ldots, \alpha_r$ by

$$\alpha_i = x_i, \qquad i = 1, \ldots, r, \text{ if } n + r \text{ is odd},$$
$$\alpha_1 = 0 \quad \text{and} \quad \alpha_i = x_{i-1}, \qquad i = 1, \ldots, r - 1, \text{ if } n + r \text{ is even}.$$

We first introduce the exponential Euler splines

$$S_{n,r}(x; u) = \sum_{s=1}^{r} e^{iux}\Omega_s(x, u) \qquad \forall x \in \mathbf{R}, \tag{4.6}$$

where

$$\Omega_s(x, u) = \frac{\det\left(\sum_{k=-\infty}^{\infty} e^{2k\pi i\beta_l}/(u + 2k\pi)^{n-m+2}\right)_{l,m=1}^{r}}{\det\left(\sum_{k=-\infty}^{\infty} e^{2k\pi i\alpha_l}/(u + 2k\pi)^{n-m+2}\right)_{l,m=1}^{r}},$$

$(r - 2)\pi < u \leqslant r\pi$, and

$$\beta_l = \begin{cases} x & \text{if } l = s, \\ \alpha_l & \text{if } l \neq s. \end{cases}$$

Clearly $S_{n,r}(\nu+\alpha_i;\ u)=e^{iu(\nu+\alpha_i)}$ $\forall i=1,2,\ldots,r$, $\nu\in\mathbf{Z}$, and $S_{n,r}(x,r\pi)=S_{n,r}(\cos r\pi x)+iS_{n,r}(\sin r\pi x)$. Therefore we are interested in the limit of $S_{n,r}(x;r\pi)$ as $n\to\infty$. First we prove

LEMMA 4.3. *For $s=1,2,\ldots,r$,*

$$\lim_{n\to\infty}\Omega_s(x,r\pi)=e^{\pi i(\alpha_s-x)}\cos\pi(\alpha_s-x)\frac{V(e^{-2\pi i\beta_1},e^{-2\pi i\beta_2},\ldots,e^{-2\pi i\beta_r})}{V(e^{-2\pi i\alpha_1},e^{-2\pi i\alpha_2},\ldots,e^{-2\pi i\alpha_r})}\tag{4.7}$$

uniformly on $\mathbf{R}$.

PROOF. Assume $(r-1)\pi<u<r\pi$. A straightforward computation shows that

$$\Omega_s(x,u)=\frac{\sum_{k_1,k_2,\ldots,k_r}V(k_1,k_2,\ldots,k_r)\prod_{j=1}^r e^{2k_j\pi i\beta_j}/(u+2k_j\pi)^{n+1}}{\sum_{k_1,k_2,\ldots,k_r}V(k_1,k_2,\ldots,k_r)\prod_{j=1}^r e^{2k_j\pi i\alpha_j}/(u+2k_j\pi)^{n+1}}$$

$$=\frac{\sum_{k_1<\cdots<k_r}V(k_1,k_2,\ldots,k_r)\det(e^{2k_m\pi i\beta_l})_{l,m=1}^r\prod_{j=1}^r(u+2k_j\pi)^{-n-1}}{\sum_{k_1<\cdots<k_r}V(k_1,k_2,\ldots,k_r)\det(e^{2k_m\pi i\alpha_l})_{l,m=1}^r\prod_{j=1}^r(u+2k_j\pi)^{-n-1}}$$

$$=\frac{\det(e^{-2(m-1)\pi i\beta_l})_{l,m=1}^r+\left(\frac{u}{u-2r\pi}\right)^{n+1}\det(e^{-2m\pi i\beta_l})_{l,m=1}^r+O\left(\left|\frac{u-2\pi}{u-2r\pi}\right|^{n+1}\right)}{\det(e^{-2(m-1)\pi i\alpha_l})_{l,m=1}^r+\left(\frac{u}{u-2r\pi}\right)^{n+1}\det(e^{-2m\pi i\alpha_l})_{l,m=1}^r+O\left(\left|\frac{u-2\pi}{u-2r\pi}\right|^{n+1}\right)}.$$

Taking the limit as $u\to r\pi$, after some simplification, we obtain

$$\Omega_s(x,u)=\frac{V(e^{-2\pi i\beta_1},\ldots,e^{-2\pi i\beta_r})\left(1+(-1)^{n+1}e^{-2\pi i\sum_{l=1}^r\beta_l}\right)+O((r-2/r)^{n+1})}{V(e^{-2\pi i\alpha_1},\ldots,e^{-2\pi i\alpha_r})\left(1+(-1)^{n+1}e^{-2\pi i\sum_{l=1}^r\alpha_l}\right)+O((r-2/r)^{n+1})}.\tag{4.8}$$

Now suppose n even. If r is even, $\alpha_1=0$ and

$$\sum_{l=1}^r\alpha_l=\frac{r-2}{2}+\frac{1}{2}.\tag{4.9}$$

If r is odd, $\alpha_1>0$ and

$$\sum_{l=1}^r\alpha_l=\frac{r-1}{2}+\frac{1}{2}.\tag{4.10}$$

The result (4.7) then follows from (4.8), (4.9) and (4.10). For odd n,

$$\sum_{l=1}^r\alpha_l=\begin{cases}(r-1)/2 & \text{if } r \text{ is odd,}\\ r/2 & \text{if } r \text{ is even,}\end{cases}$$

and (4.7) follows similarly. □

LEMMA 4.4. *If* $\alpha_1 = 0$, *the following limits hold uniformly*:

$$\lim_{n\to\infty} S_{n,r}(\cos r\pi x) = \cos r\pi x, \tag{4.11}$$

$$\lim_{n\to\infty} S_{n,r}(\sin r\pi x) = \sin r\pi x + (-1)^r 2^{r-1} \prod_{i=1}^{r} \sin \pi(x - \alpha_i). \tag{4.12}$$

If $\alpha_1 > 0$, *the following limits hold uniformly*:

$$\lim_{n\to\infty} S_{n,r}(\cos r\pi x) = \cos r\pi x + (-1)^{r-1} 2^{r-1} \prod_{i=1}^{r} \sin \pi(x - \alpha_i), \tag{4.13}$$

$$\lim_{n\to\infty} S_{n,r}(\sin r\pi x) = \sin r\pi x. \tag{4.14}$$

PROOF. First we write

$$\begin{aligned}\frac{V\left(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2}, \ldots, e^{-2\pi i\beta_r}\right)}{V\left(e^{-2\pi i\alpha_1}, e^{-2\pi i\alpha_2}, \ldots, e^{-2\pi i\alpha_r}\right)} &= \frac{\prod_{1\leqslant l<k\leqslant r} \left(e^{-2\pi i\beta_k} - e^{-2\pi i\beta_l}\right)}{\prod_{1\leqslant l<k\leqslant r} \left(e^{-2\pi i\alpha_k} - e^{-2\pi i\alpha_l}\right)} \\ &= e^{(r-1)\pi i(\alpha_s - x)} \frac{\prod_{1\leqslant l<k\leqslant r} \sin \pi(\beta_k - \beta_l)}{\prod_{1\leqslant l<k\leqslant r} \sin \pi(\alpha_k - \alpha_l)} \\ &= e^{(r-1)\pi i(\alpha_s - x)} \frac{\prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r}{}' \sin \pi(\alpha_s - \alpha_k)},\end{aligned}$$

where $\prod_{k=1}^{\prime r}$ indicates that the factor involving $k = s$ is omitted. Hence it follows from (4.6) and (4.7) that

$$\begin{aligned}\lim_{n\to\infty} S_{n,r}(x, r\pi) &= \sum_{s=1}^{r} e^{r\pi i\alpha_s} \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r}{}' \sin \pi(\alpha_s - \alpha_k)} \\ &\equiv \phi(x) + i\psi(x),\end{aligned}$$

where

$$\phi(x) = \sum_{s=1}^{r} \cos r\pi\alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r}{}' \sin \pi(\alpha_s - \alpha_k)},$$

and

$$\psi(x) = \sum_{s=1}^{r} \sin r\pi\alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r}{}' \sin \pi(\alpha_s - \alpha_k)}.$$

Clearly $\lim_{n\to\infty} S_{n,r}(\cos r\pi x) = \phi(x)$, $\lim_{n\to\infty} S_{n,r}(\sin r\pi x) = \psi(x)$.

Next, we want to simplify $\phi(x)$ and $\psi(x)$. First we consider r even. The Gauss trigonometric interpolation formula gives

(4.15) $$\phi(x) = \cos r\pi x + \lambda \prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k).$$

To determine λ, we write

(4.16)

$$\cos \pi(\alpha_s - x) \prod_{k=1}^{r}{}' \sin \pi(x - \alpha_k) = \frac{1}{2(2i)^{r-1}} (e^{\pi i(x-\alpha_s)} + e^{-\pi i(x-\alpha_s)})$$
$$\cdot \left(\exp\left[\pi i\left((r-1)x - \sum_{k=1}^{r}{}' \alpha_k\right)\right] + \cdots + (-1)^{r-1} \exp\left[-\pi i\left((r-1)x - \sum_{k=1}^{r}{}' \alpha_k\right)\right]\right)$$

and

(4.17) $$\prod_{k=1}^{r} \sin \pi(x - \alpha_k) = \frac{1}{(2i)^r} \left(\exp\left[\pi i\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right] + \cdots + (-1)^r \exp\left[-\pi i\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right]\right).$$

Equating the highest order terms in (4.15), it follows from (4.15)–(4.17) that

$$\frac{1}{2(2i)^{r-1}} \sum_{s=1}^{r} A_s \left\{\cos\left(\pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right) + i \sin\left(\pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right)\right.$$
$$\left. + (-1)^{r-1}\left[\cos\left(\pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right) - i \sin\left(\pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right)\right]\right\}$$
$$= \cos r\pi x + \lambda\left\{\cos \pi\left(rx - \sum_{k=1}^{r} \alpha_k\right) + i \sin \pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right.$$
$$\left. + (-1)^r\left[\cos \pi\left(rx - \sum_{k=1}^{r} \alpha_k\right) - i \sin \pi\left(rx - \sum_{k=1}^{r} \alpha_k\right)\right]\right\},$$

where $A_s = \cos \pi r\alpha_s / \prod_{k=1}^{\prime r} \sin \pi(\alpha_s - \alpha_k)$.

Since r is even it follows that

$$(4.18)\quad \frac{-2}{(2i)^r}\sum_{s=1}^{r} A_s \sin\pi\left(rx-\sum_{k=1}^{r}\alpha_k\right)=\cos r\pi x+\frac{2\lambda\cos\pi\left(rx-\sum_{k=1}^{r}\alpha_k\right)}{(2i)^r}.$$

Now if $\alpha_1 = 0$, then $\Sigma_{k=1}^{r}\alpha_k = r/2-\frac{1}{2}$, so that (4.18) becomes

$$\frac{(-1)^{(r+2)/2}}{(2i)^r}\cos r\pi x\left(\sum_{s=1}^{r} A_s\right)=\cos\pi rx+\frac{(-1)^{(r+2)/2}2\lambda\sin r\pi x}{(2i)^r}.$$

Hence $\lambda = 0$. This proves (4.11) for r even.

If $\alpha_1 \neq 0$, then $\Sigma_{k=1}^{r}\alpha_k = r/2$, so that (4.18) becomes

$$\frac{-2}{2^r}\sin r\pi x\left(\sum_{s=1}^{r} A_s\right)=\cos\pi rx+\frac{2\lambda\cos\pi rx}{2^r}.$$

Hence $\lambda = -2^{r-1}$. This proves (4.13) for r even. The proof of (4.12) and (4.14) for r even are the same.

Next we consider r odd. If $\alpha_1 \neq 0$, we let $\alpha_0 = 0$ and write

$$(4.19)\qquad \phi(x)=\frac{1}{\sin\pi x}\sum_{s=0}^{r}\cos r\pi\alpha_s\cos\pi(x-\alpha_s)\frac{\prod_{k=0}^{r}{}'\ \sin\pi(x-\alpha_k)}{\prod_{k=0}^{r}{}'\ \sin\pi(\alpha_s-\alpha_k)}.$$

The Gauss interpolation formula again gives

$$\sum_{s=0}^{r}\cos r\pi\alpha_s\sin\pi\alpha_s\cos\pi(x-\alpha_s)\frac{\prod_{k=0}^{r}{}'\ \sin\pi(x-\alpha_k)}{\prod_{k=0}^{r}{}'\ \sin(\alpha_s-\alpha_k)}$$
$$=\cos r\pi x\sin\pi x+\lambda\prod_{k=0}^{r}\sin\pi(x-\alpha_k).$$

A similar calculation gives $\lambda = 2^{r-1}$, so that (4.19) gives $\phi(x)=\cos r\pi x + 2^{r-1}\Pi_{k=1}^{r}\sin\pi(x-\alpha_k)$. This proves (4.13) for r odd. The proof of (4.14) is similar.

If $\alpha_1 = 0$, we let $\alpha_{k+1} = \frac{1}{2}$ and write

$$(4.20)\quad \phi(x)=\frac{1}{\cos\pi x}\sum_{s=1}^{r+1}\cos r\pi\alpha_s\cos\pi\alpha_s\cos\pi(x-\alpha_s)\frac{\prod_{k=1}^{r+1}{}'\ \sin\pi(x-\alpha_k)}{\prod_{k=1}^{r+1}{}'\ \sin\pi(\alpha_s-\alpha_k)},$$

and (4.11) and (4.12) for odd r are proved similarly. □

Proof of Theorem 3. Let

$$\alpha_0(u)=\begin{cases}\alpha(-r\pi+0) & \text{if } u=-r\pi,\\ \alpha(u) & \text{if } -r\pi<u<r\pi,\\ \alpha(r\pi-0) & \text{if } u=r\pi.\end{cases}$$

Then $\alpha_0(u)$ has no jumps at $\pm r\pi$. Define

$$f_0(x) = \int_{-r\pi}^{r\pi} e^{iux}\, d\alpha_0(u) \qquad \forall x \in \mathbf{R}.$$

Setting $A_1 = \alpha(-r\pi + 0) - \alpha(-r\pi)$, $A_2 = \alpha(r\pi) - \alpha(r\pi - 0)$, we can write $f(x) = f_0(x) + A_1 e^{-r\pi i x} + A_2 e^{r\pi i x}$, and setting $A = A_1 + A_2$, $B = i(A_2 - A_1)$ we obtain

(4.21) $$f(x) = f_0(x) + A\cos r\pi x + B\sin r\pi x \qquad \forall x \in \mathbf{R}.$$

Now $f_0 \in B^*_{r\pi}$ since f_0 is the uniform limit of the sequence $\{f_j\}$, $f_j \in B_{p_j}$, defined by $f_j(x) = \int_{-p_j}^{p_j} e^{iux}\, d\alpha_0(u)$, with $0 < p_j < r\pi$, $p_j \to r\pi$ as $j \to \infty$. By Lemma 4.2 we conclude that $\lim_{n\to\infty} S_{n,r}(f_0; x) = f_0(x)$ uniformly on $\mathbf{R}$. The theorem now follows from (4.21) and Lemma 4.4. □

Finally we consider the class $\mathcal{AP}$ of almost periodic functions in the sense of Bohr. To every $f \in \mathcal{AP}$ corresponds a Fourier series

$$f(x) \sim \sum_{\nu=1}^{\infty} A_\nu e^{i\lambda_\nu x},$$

where λ_ν are real numbers, called the Fourier exponents of f. Also for $\sigma \geqslant 0$,

$$\mathcal{AP} \cap B_\sigma = \{f : f \in \mathcal{AP},\ -\sigma \leqslant \lambda_\nu \leqslant \sigma\}.$$

PROOF OF THEOREM 4. Suppose $f \in \mathcal{AP} \cap B_{r\pi}$. Then its Fourier exponents λ_ν, $\nu = 1, 2, 3, \ldots$, satisfy $-r\pi \leqslant \lambda_\nu \leqslant r\pi$.

Without loss of generality we may assume that $\lambda_1 = -r\pi$, $\lambda_2 = r\pi$ with the understanding that $A_1 = 0$ if the exponent $-r\pi$ is absent, and similarly that $A_2 = 0$ if exponent $r\pi$ is absent.

Let

$$A_1 e^{-r\pi i x} + A_2 e^{r\pi i x} = A\cos r\pi x + B\sin r\pi x,$$

where $A = A_2 + A_1$, $B = i(A_2 - A_1)$. It follows that the function

(4.22) $$g(x) = f(x) - A\cos r\pi x - B\sin r\pi x$$

has Fourier series $g(x) \sim \sum_{\nu=3}^{\infty} A_\nu e^{i\lambda_\nu x}$ where $-r\pi < \lambda_\nu < r\pi$ $\forall \lambda = 3, 4, 5, \ldots$. A similar argument as in **[10]** shows that $g \in B^*_{r\pi}$. It follows from Lemma 4.2 that

(4.23) $$S_{n,r}(g; x) \to g(x) \quad \text{uniformly on } \mathbf{R}.$$

The theorem then follows from (4.22), (4.23) and Lemma 4.4. □

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