

Vlastimil Pták

A remark on approximation of continuous functions

Czechoslovak Mathematical Journal, Vol. 8 (1958), No. 2, 251–256

Persistent URL: <http://dml.cz/dmlcz/100299>

Terms of use:

© Institute of Mathematics AS CR, 1958

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

A REMARK ON APPROXIMATION OF CONTINUOUS
FUNCTIONS

VLASTIMIL PTÁK, Praha

(Received September 16, 1957)

The author gives a direct geometrical proof of Haar's theorem on approximation of continuous functions and of the Čebyšev characterization of the polynomial of best approximation.

In his beautiful paper [2] on the application of Minkowskian geometry to the theory of approximation, A. HAAR has given a necessary and sufficient condition that the best approximation of any continuous function by means of linear combinations of n given continuous functions be unique. His proof is based on the following idea. Suppose we have n linearly independent continuous functions x_1, \dots, x_n defined on $\langle 0, 1 \rangle$. In E_n we consider the set K of all vectors $[\xi_1, \dots, \xi_n]$ which fulfil the inequality

$$\max_{0 \leq t \leq 1} |\xi_1 x_1(t) + \dots + \xi_n x_n(t)| \leq 1.$$

It is easy to see that K is a convex body in E_n in the sense of MINKOWSKI. The discussion of the best approximation by linear combinations of x_1, \dots, x_n is then reduced to the study of properties of the Minkowskian geometry defined by K . The proof, though simple and clear enough, is by no means a short one. It is especially the sufficiency of Haar's condition which requires more subtle considerations. Even modern proofs (see, e. g. [1]) devote to the sufficiency more than three pages.

It is the purpose of the present remark to give a simple proof of Haar's theorem using only the simplest geometrical notions.

Theorem 1. *Let T be a compact Hausdorff space and let us denote by $C(T)$ the space of all continuous functions on T with the usual norm. Let E be a given n -dimensional subspace of $C(T)$. The best approximation of every $x \in C(T)$ by means of elements of E is unique if and only if there does not exist an $e \in E$, $e \neq 0$, such that the equation $e(t) = 0$ has at least n distinct solutions.*

We begin with a simple

Lemma. Let L be a p -dimensional subspace of $C(T)$. Let f be a linear functional on L of norm one. Then there exist p distinct points $t_i \in T$ and numbers λ_i such that $f = \sum_{i=1}^p \lambda_i t_i$ and $\sum_{i=1}^p |\lambda_i| = 1$. The equation $f = \sum \lambda_i t_i$ is, of course, taken to mean $\langle x, f \rangle = \sum \lambda_i x(t_i)$ for every $x \in L$.

To prove this lemma, we note first that, with every point $t \in T$, we may associate a linear functional $\varphi(t)$ on L defined by the relation $\langle x, \varphi(t) \rangle = x(t)$ for every $x \in L$. Clearly φ is a continuous mapping of T into L' . At the same time, the norm of $\varphi(t)$ is at most one for every $t \in T$ and it is easy to see that the unit sphere of L' coincides with the closed symmetrical convex envelope of the set $\varphi(T)$. Indeed, suppose we have an $x' \in L'$, $|x'| \leq 1$ such that x' does not belong to the closed symmetrical convex envelope of $\varphi(T)$. It follows¹⁾ that there exists a point $x \in L$ such that $\sup |\langle x, \varphi(T) \rangle| < \langle x, x' \rangle$. This is, however, a contradiction, since $|x'| \leq 1$ and $\sup |\langle x, \varphi(T) \rangle| = |x|$. The mapping φ being continuous, $\varphi(T)$ is compact and it follows from a result of CARATHÉODORY²⁾ that every point f of the unit sphere of L' may be expressed in the form $\sum_{i=1}^p \lambda_i \varphi(t_i)$ with $\sum |\lambda_i| \leq 1$. If the norm of f is one, $\sum |\lambda_i| < 1$ is impossible which concludes the proof.

Before going into the proof it is convenient to state Haar's condition in another equivalent form. We have the following equivalence. There exists in

¹⁾ If W is a closed convex subset of a finite-dimensional vector space and w a point outside W , there exists, according to a well-known theorem, a hyperplane separating W and w . Since x' does not belong to the closed symmetrical convex envelope of $\varphi(T)$, it does not belong to the closed convex envelope of the union of $\varphi(T)$ and $-\varphi(T)$. According to the separation theorem there exists a linear functional g on L' such that $\sup |g(\varphi(T))| < g(x')$. The space L' being finite dimensional, linear functionals on L' may be identified with elements of L .

²⁾ The result of Carathéodory referred to is the following: Let E be a p -dimensional vector space and let M be a compact subset of E . The closed convex envelope K of M consists of all vectors of the form $\sum_{i=1}^{p+1} \lambda_i m_i$, where $m_i \in M$, $\lambda_i \geq 0$ and $\sum_{i=1}^{p+1} \lambda_i = 1$.

If k is a point of the boundary of K , it may be expressed in the form $k = \sum_{i=1}^p \lambda_i m_i$

where $m_i \in M$, $\lambda_i \geq 0$ and $\sum_{i=1}^p \lambda_i = 1$. Now suppose we have a compact set B and

a point x of its closed symmetrical convex envelope S . There exists a number $\delta \geq 1$ such that δx belong to the boundary of S . Since S clearly coincides with the closed

convex envelope of the union of B and $-B$, we may express δx in the form $\delta x = \sum_{i=1}^p \omega_i \varepsilon_i b_i$,

where $b_i \in B$, $\varepsilon_i = \pm 1$, $\omega_i \geq 0$ and $\sum_{i=1}^p \omega_i = 1$. If we put $\lambda_i = \frac{\omega_i \varepsilon_i}{\delta}$, we have $x = \sum_{i=1}^p \lambda_i b_i$

and $\sum_{i=1}^p |\lambda_i| \leq 1$. For a simple proof of these results, using the definition of convexity only, see a recent paper of the author's [3].

E a nonzero element e with at least n distinct zero points t_1, \dots, t_n if and only if there exists a nonzero linear combination $f = \sum_{i=1}^n \lambda_i t_i$ which vanishes on E . Indeed, if e_1, \dots, e_n is a basis of E , both conditions express the fact that $\det e_i(t_j) = 0$.

Proof of Haar's theorem. Suppose first that the best approximation is not always unique. Then there exist points $x_0 \in C(T)$, $e_0 \in E$ and a nonzero $e \in E$ such that, for every ε small enough in absolute value, the point $e_0 + \varepsilon e$ is the best approximation of x_0 .

Let us denote by L the linear span of E and x_0 , so that L has $n + 1$ dimensions. If we put $w_0 = x_0 - e_0$, it follows that there exists a linear functional f of norm one on L vanishing on E and assuming the value $|w_0|$ on w_0 . It follows from our lemma that f may be expressed in the form $\sum_{i=1}^{n+1} \lambda_i t_i$ where $t_i \in T$ and $\sum_{i=1}^{n+1} |\lambda_i| = 1$.

We shall distinguish two cases:

1° We have $\lambda_i \neq 0$ for every i . Since

$$|w_0| = \langle w_0, f \rangle = \sum_{i=1}^{n+1} \lambda_i w_0(t_i) \quad \text{and} \quad \sum_{i=1}^{n+1} |\lambda_i| = 1,$$

we have $|w_0(t_i)| = |w_0|$ for every i . Since $|w_0 + \varepsilon e| = |w_0|$ for small ε , it follows that $e(t_i) = 0$ for every i .

2° We have $\lambda_i = 0$ for some i . We may clearly suppose that $\lambda_{n+1} = 0$. In this case, $\sum_{i=1}^n \lambda_i t_i$ is a nonzero linear combination of n points vanishing on E .

The sufficiency of Haar's condition is thus proved. On the other hand, suppose that Haar's condition is not fulfilled. Then there exists a nonzero point $e_0 \in E$, $|e_0| \leq 1$ which vanishes in n distinct points $t_i \in T$ and a nonzero linear combination $f = \sum_{i=1}^n \lambda_i t_i$ which vanishes on E . We may clearly suppose

that $\sum_{i=1}^n |\lambda_i| = 1$. Now choose an arbitrary $a \in C(T)$ such that $|a| \leq 1$ and $a(t_i) = \text{sign } \lambda_i$ whenever $\lambda_i \neq 0$. Let us define the function x_0 by the relation $x_0(t) = a(t)(1 - |e_0(t)|)$. We have clearly $x_0(t_i) = a(t_i) = \text{sign } \lambda_i$ for $\lambda_i \neq 0$, $|x_0| = 1$ and $|x_0(t) + |e_0(t)| \leq 1$ for every $t \in T$. It follows that $|x_0 - e_0| \leq \max_{t \in T} (|x_0(t) + |e_0(t)|) \leq 1$. We have, however, $|x_0 - e_0| \geq 1$ for every $e \in E$.

Indeed, if $|x_0 - e| < 1$ for some e , it would follow

$$1 = \langle x_0, f \rangle = \langle x_0 - e, f \rangle \leq |x_0 - e| |f| < 1,$$

which is a contradiction. It follows that both points 0 and e_0 are best approximations of x_0 .

We conclude this remark with a few words concerning the characterization of the polynomial of best approximation. First of all, let us have a compact Hausdorff space T and an n -dimensional subspace $E \subset C(T)$ fulfilling Haar's condition. Let $x \in C(T)$, $x \notin E$ be given; the best approximation $e \in E$ of x is thus uniquely determined. We assert now that *the equation $|x(t) - e(t)| = |x - e|$ is fulfilled for at least $n + 1$ distinct points t* . In fact, we know that there exist $n + 1$ points t_1, \dots, t_{n+1} and real numbers $\lambda_1, \dots, \lambda_{n+1}$ with $\sum_{i=1}^{n+1} |\lambda_i| = 1$ such that, for $f = \sum_{i=1}^{n+1} \lambda_i t_i$ we have $\langle x - e, f \rangle = |x - e|$ and $\langle E, f \rangle = 0$. From the first equation it follows that $|x(t_i) - e(t_i)| = |x - e|$ whenever $\lambda_i \neq 0$. We have, however, $\lambda_i \neq 0$ for every i since, in the contrary case, E would not fulfil Haar's condition (see the section 2^o of the preceding proof).

A more precise result is hardly to be expected in the general case. It may be shown on examples that there may be exactly $n + 1$ points t_i where $|x(t_i) - e(t_i)| = |x - e|$ and $x(t_i) - e(t_i) = |x - e|$ for every i .

Let us consider now the case where T is a compact interval $\langle a, b \rangle$. In this case the above method yields a particularly simple proof of the classical theorem of ČEBYŠEV. We intend to show that the theorem of Čebyšev as well as the related result of DE LA VALLÉE-POUSSIN are both immediate consequences of the following lemma:

Let $T = \langle a, b \rangle$ and let E an n -dimensional subspace of $C(T)$ fulfilling Haar's condition. If $t_1 < t_2 < \dots < t_{n+1}$ are given points of T , there exists exactly one (apart from a scalar factor) nonzero linear combination $f = \sum_{i=1}^{n+1} \lambda_i t_i$ vanishing on E . All numbers λ_i are different from zero and they alternate in sign.

Proof. The space E being n -dimensional, the $n + 1$ functionals t_i are linearly dependent on E . There exists a nonzero linear combination $f = \sum_{i=1}^{n+1} \lambda_i t_i$ such that $\langle E, f \rangle = 0$. The assumption that $\lambda_i = 0$ for some i would lead to a contradiction with our assumption concerning E . First of all, let $n = 1$ and let e be a nonzero element of E . It follows from Haar's condition that $e(t)$ is different from zero on the whole of T . Suppose now that $\lambda_1 e(t_1) + \lambda_2 e(t_2) = 0$. We have $\text{sign } e(t_1) = \text{sign } e(t_2)$ so that λ_1 and λ_2 cannot be of the same sign. Suppose now that $n > 1$ and that there is an index p such that λ_p and λ_{p+1} are of the same sign. It follows that there exists an i such that at least one of the numbers λ_{i-1} or λ_{i+1} is of the same sign as λ_i . Clearly we may suppose that $\lambda_i > 0$. Choose now two positive numbers α_{i-1} and α_{i+1} such that

$$\lambda_{i-1} \alpha_{i-1} + \lambda_{i+1} \alpha_{i+1} > 0.$$

Now there exists an $e \in E$ such that $e(t_{i-1}) = \alpha_{i-1}$, $e(t_{i+1}) = \alpha_{i+1}$ and $e(t_j) = 0$ for every j different from $i - 1, i, i + 1$. Since $\langle E, f \rangle = 0$, we have, in particular, $\langle e, f \rangle = 0$.

This reduces to

$$\lambda_{i-1}\alpha_{i-1} + \lambda_i e(t_i) + \lambda_{i+1}\alpha_{i+1} = 0.$$

It follows that $e(t_i) < 0$. Since $e(t_{i-1})$ and $e(t_{i+1})$ are positive, there exist points $s_{i-1} \in (t_{i-1}, t_i)$ and $s_{i+1} \in (t_i, t_{i+1})$ with $e(s_{i-1}) = e(s_{i+1}) = 0$. This is a contradiction with Haar's condition.

Theorem 2. *Let $T = \langle a, b \rangle$ and let E be an n -dimensional subspace of $C(T)$ fulfilling Haar's condition. Let $x \in C(T)$ be given. The following condition is sufficient and necessary for a point $e \in E$ to be the best approximation of x :*

There exist $n + 1$ points $t_1 < t_2 < \dots < t_{n+1}$ of T and a number ε with $|\varepsilon| = 1$ such that $x(t_i) - e(t_i) = (-1)^i \varepsilon |x - e|$.

Proof. Let e be the best approximation of x . We have then a functional $f = \sum_{i=1}^{n+1} \lambda_i t_i$ with $\sum_{i=1}^{n+1} |\lambda_i| = 1$ vanishing on E and such that $\langle x - e, f \rangle = |x - e|$. Hence $x(t_i) - e(t_i) = |x - e| \text{sign } \lambda_i$ whenever $\lambda_i \neq 0$. If the t_i are arranged in increasing order, it follows from the preceding lemma that the λ_i are different from zero and alternate in sign. The rest is easy.

The second part of the theorem is a consequence of the following result of de la Vallée-Poussin.

Let $T = \langle a, b \rangle$ and let E be an n -dimensional subspace of $C(T)$ fulfilling Haar's condition. Let $x_0 \in C(T)$ and $e_0 \in E$ be given. Suppose there exist $n + 1$ points $t_1 < t_2 < \dots < t_{n+1}$, positive numbers ε_i and a number ε with $|\varepsilon| = 1$ such that

$$x_0(t_i) - e_0(t_i) = (-1)^i \varepsilon \varepsilon_i.$$

We have then

$$\min_{e \in E} |x_0 - e| \geq \min_{1 \leq i \leq n+1} \varepsilon_i.$$

Proof. According to the preceding lemma there exist positive numbers $\lambda_1, \dots, \lambda_{n+1}$ with $\sum_{i=1}^{n+1} \lambda_i = 1$ such that

$$f = \sum_{i=1}^{n+1} (-1)^i \varepsilon \lambda_i t_i$$

vanishes on E . For every $e \in E$, we have

$$|x_0 - e| \geq \langle x_0 - e, f \rangle = \langle x_0 - e_0, f \rangle = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \geq \min \varepsilon_i$$

which concludes the proof.

BIBLIOGRAPHY

- [1] *H. И. Ахиезер*: Теория аппроксимаций, Moskva-Leningrad 1947.
- [2] *A. Haar*: Minkowskische Geometrie und Annäherung an stetige Functionen, Math. Ann. 78 (1918), 293—311.
- [3] *V. Pták*: O absolutně konvexním obalu množiny v konečně dimensionálním vektorovém prostoru, Čas. pro pěst. mat., 83 (1958), No 3.

Резюме

ЗАМЕТКА ОБ АППРОКСИМАЦИИ НЕПРЕРЫВНЫХ ФУНКЦИЙ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага

(Поступило в редакцию 16/IX 1957 г.)

В статье дается прямое геометрическое доказательство теоремы Хаара о приближении непрерывных функций.

Теорема. Пусть T — компактное хаусдорфово пространство; обозначим через $C(T)$ пространство всех непрерывных функций на T с обычной нормой. Пусть E — данное n -мерное подпространство пространства $C(T)$. Наилучшее приближение каждого $x \in C(T)$ при помощи элементов пространства E будет однозначно определенным тогда и только тогда, если не будет существовать ненулевой элемент $e \in E$, имеющий не менее чем n различных нулей.