

NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

A REMARK ON CHARACTERISTIC FUNCTIONS

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1. Let $F(x)$, $-\infty < x < +\infty$, be a distribution function, and

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$$

its characteristic function. It is well known that the existence of $\varphi'(0)$ does not imply the existence of the absolute moment

$$(1) \quad \int_{-\infty}^{+\infty} |x| dF(x).$$

A simple example is provided by the function

$$\varphi(t) = C \sum_{n=2}^{\infty} \frac{\cos nt}{n^2 \log n},$$

where C is a positive constant. Since the series on the right differentiated term by term converges uniformly (see [1]), $\varphi'(t)$ exists (and is continuous) for all values of t , and in particular at the point $t = 0$. Obviously $\varphi(t)$ is the characteristic function of the masses $C/2n^2 \log n$ concentrated at the points $\pm n$ for $n = 2, 3, \dots$. The constant C is such that the sum of all the masses is 1. The divergence of the series $\sum 1/n \log n$ implies that in this particular case the moment (1) is infinite.

In a recent paper (see [2], esp. p. 120, footnote), Fortet raises the problem of whether the existence of $\varphi'(0)$ implies the existence of the first algebraic moment

$$(2) \quad \int_{-\infty}^{+\infty} x dF(x) = \lim_{x \rightarrow +\infty} \int_{-x}^x x dF(x).$$

The main purpose of this note is to show that this is so. We shall even prove a slightly more general result.

A function $\psi(t)$ defined in the neighborhood of a point t_0 is said to be *smooth* at this point if

$$\lim_{h \rightarrow +0} \frac{\psi(t_0 + h) + \psi(t_0 - h) - 2\psi(t_0)}{h} = 0.$$

Clearly, if ψ has a one-sided derivative at the point t_0 , the derivative on the other side also exists and has the same value. Thus the graph of $\psi(t)$ has no angular point for $t = t_0$, and this explains the terminology. If $\psi'(t_0)$ exists and is finite, $\psi(t)$ is smooth for $t = t_0$. The converse is obviously false, since any

function whose graph is symmetric with respect to $t = t_0$ is smooth at that point.

THEOREM 1. *If the characteristic function $\varphi(t)$ is smooth at the point 0, then a necessary and sufficient condition for the existence of $\varphi'(0)$ is the existence of the moment (2). The value of (2) is $-i\varphi'(0)$.*

In particular, the existence and finiteness of $\varphi'(0)$ implies the existence of (2). That the converse is false, is obvious. For if a_0, a_1, a_2, \dots are positive numbers and $a_0 + 2a_1 + 2a_2 + \dots = 1$, then $\psi(t) = a_0 + 2\sum_1^\infty a_n \cos nt$ is the characteristic function of the distribution function $F(x)$ corresponding to masses concentrated at the integer points $\pm n$ and having the values a_n there. Owing to the symmetry of the masses, the number (2) exists, and is zero even if $\varphi(t)$ is non-differentiable for $t = 0$ (we may e.g. take for $\varphi(t)$ the Weierstrass non-differentiable function $C \sum_1^\infty a^n \cos b^n t$, where C is a suitable constant).

PROOF. We may write

$$\varphi(t) = \int_0^\infty \cos xt \, dG(x) + i \int_0^\infty \sin xt \, dG(x) = \psi_1(t) + i\psi_2(t)$$

where

$$G(x) = F(x) - F(-x), \quad H(x) = F(x) + F(-x).$$

Thus

$$(3) \quad 0 \leq |\Delta H| \leq \Delta G.$$

Since $\varphi(t)$ is smooth at the point 0, and since $\psi_1(t)$ is even, $\psi_2(t)$ odd,

$$\begin{aligned} 0 &= \lim_{h \rightarrow +0} \frac{\varphi(h) + \varphi(-h) - 2\varphi(0)}{h} = 2 \lim_{h \rightarrow +0} \frac{\psi_1(h) - \psi_1(0)}{h} \\ &= -2 \lim_{h \rightarrow +0} \int_0^\infty \frac{1 - \cos hx}{h} \, dG(x) \end{aligned}$$

so that, replacing h by $2h$,

$$\int_0^\infty \frac{\sin^2 hx}{h} \, dG(x) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Since the integrand is positive we obtain successively

$$\begin{aligned} &\int_0^{1/h} \frac{\sin^2 hx}{h} \, dG(x) = o(1), \\ &\int_0^{1/h} \frac{\left(\frac{2}{\pi} hx\right)^2}{h} \, dG(x) = o(1), \\ (4) \quad &\int_0^{1/h} x^2 \, dG(x) = o(h^{-1}), \\ &\int_{1/2h}^{1/h} x^2 \, dG(x) = o(h^{-1}), \\ (5) \quad &\int_{1/2h}^{1/h} dG(x) = o(h). \end{aligned}$$

Since $\psi_1(t)$ is even, the smoothness of $\varphi(t)$, and so also of $\psi_1(t)$, at the point $t = 0$ implies that $\psi_1'(0)$ exists and is zero. If $h \rightarrow +0$,

$$\begin{aligned} \frac{\psi_2(h) - \psi_2(0)}{h} &= \int_0^\infty \frac{\sin xh}{h} dH(x) = \int_0^{1/h} + \int_{1/h}^\infty = A_h + B_h, \\ |B_h| &\leq h^{-1} \int_{1/h}^\infty |dH| \leq h^{-1} \left(\int_{1/h}^{2/h} dG + \int_{2/h}^{4/h} dG + \int_{4/h}^{8/h} dG + \dots \right) \\ &= h^{-1} o(h + h/2 + h/4 + \dots) = o(1), \end{aligned}$$

by (3) and (5). Also

$$\begin{aligned} A_h - \int_0^{1/h} x dH &= \int_0^{1/h} \left(\frac{\sin hx}{hx} - 1 \right) x dH = \int_0^{1/h} O(x^2 h^2) x dG \\ &= \int_0^{1/h} O(x^2 h) dG = o(1), \end{aligned}$$

by (3) and (4). Thus

$$\frac{\psi_2(h) - \psi_2(0)}{h} = o(1) + \int_0^{1/h} x dH = o(1) + \int_{-1/h}^{1/h} x dF,$$

and so

$$\frac{\varphi(h) - \varphi(0)}{h} = o(1) + i \int_{-1/h}^{1/h} x dF.$$

It follows that the existence of (2) is equivalent to the existence of the right-hand side derivative of $\varphi(t)$ at the point $t = 0$, or, on account of smoothness, to the existence of $\varphi'(0)$. Moreover, the value of (2) is $-i\varphi'(0)$. This completes the proof of Theorem 1.

2. Suppose that a function $\psi(t)$ defined near the point t_0 satisfies for $h \rightarrow 0$ a relation

$$\psi(t_0 + h) = \alpha_0 + \alpha_1 h/1! + \dots + \alpha_{k-1} h^{k-1}/(k-1)! + [\alpha_k + \sigma(1)] h^k/k!,$$

where $\alpha_0, \alpha_1, \dots, \alpha_k$ are constants. Then α_k is called the k th generalized derivative of ψ at the point t_0 . It will be denoted by $\psi_{(k)}(t_0)$. The existence and finiteness of $\psi^{(k)}(t_0)$ implies the existence of $\psi_{(k)}(t_0)$ and both numbers are equal.

Another generalization of higher derivatives is based on the consideration of the symmetric differences

$$\begin{aligned} \Delta_h \psi(t_0) &= \psi(t_0 + h) - \psi(t_0 - h), \\ \Delta_h^2 \psi(t_0) &= \psi(t_0 + 2h) - 2\psi(t_0) + \psi(t_0 - 2h), \\ \Delta_h^3 \psi(t_0) &= \psi(t_0 + 3h) - 3\psi(t_0 + h) + 3\psi(t_0 - h) - \psi(t_0 - 3h). \\ &\dots \end{aligned}$$

If $\Delta_h^k \psi(t_0)/(2h)^k$ tends to a limit as $h \rightarrow +0$, this limit is called the k th symmetric derivative of ψ at the point t_0 . We shall denote it by $D_k \psi(t_0)$. Clearly, $D_k \psi(t_0)$ exists and equals $\psi^{(k)}(t_0)$, if the latter number exists.

It is a simple matter to prove (see [3]) that if k is a positive even integer, and if the characteristic function $\varphi(t)$ has at $t = 0$ a finite symmetric derivative $D_k \varphi(0)$, then the k th moment $\int_{-\infty}^{+\infty} x^k dF(x)$ exists, and its value is $(-1)^{k/2} D_k \varphi(0)$.

Conversely, the existence of $\int_{-\infty}^{+\infty} x^k dF(x)$ obviously implies (for k even) the existence and continuity of $\varphi^{(k)}(t)$ for all t , and in particular at the point $t = 0$.

In order to obtain an extension of Theorem 1 to the case of derivatives of odd order, we have to generalize the notion of smoothness. We shall say that a function $\psi(t)$ satisfies for $t = t_0$ condition S_k , ($k = 1, 2, \dots$), if

$$\Delta_h^{k+1} \psi(t_0) = o(h^k) \quad \text{as } h \rightarrow +0.$$

For $k = 1$, condition S_k is identical with smoothness at t_0 . Clearly, if $\psi^{(k)}(t_0)$ exists, ψ satisfies condition S_k at t_0 .

THEOREM 2. *Suppose that k is a positive odd integer, and let $\varphi(t)$ be the characteristic function of a distribution function $F(x)$. If φ satisfies condition S_k at the point 0, a necessary and sufficient condition for the existence of $D_k \varphi(0)$ is the existence of the symmetric moment*

$$(6) \quad \int_{-\infty}^{+\infty} x^k dF(x) = \lim_{x \rightarrow +\infty} \int_{-x}^x x^k dF(x)$$

whose value is then equal to $i^{-k} D_k \varphi(0)$. In particular, the existence of $\varphi^{(k)}(0)$ implies that of (6).

The proof of Theorem 2 is analogous to that of Theorem 1. Let $G(x)$ and $H(x)$ have the same meaning as before. Since $k + 1$ is even, condition S_k at the point $t = 0$ gives

$$\begin{aligned} \Delta_h^{k+1} \varphi(0) &= \int_{-\infty}^{+\infty} (e^{ixh} - e^{-ixh})^{k+1} dF(x) = 2^{k+1} (-1)^{(k+1)/2} \int_{-\infty}^{+\infty} (\sin xh)^{k+1} dF(x) \\ &= 2^{k+1} (-1)^{(k+1)/2} \int_0^{\infty} (\sin xh)^{k+1} dG(x) = o(h^k), \end{aligned}$$

so that

$$\int_0^{1/h} (\sin xh)^{k+1} dG(x) = o(h^k)$$

$$(7) \quad \int_0^{1/h} x^{k+1} dG(x) = o(h^{-1})$$

$$(8) \quad \int_{1/2h}^{1/h} dG(x) = o(h^k).$$

On the other hand,

$$\begin{aligned} i^{-k} \frac{\Delta_h^k \varphi(0)}{(2h)^k} &= \int_{-\infty}^{+\infty} \left(\frac{\sin xh}{xh} \right)^k x^k dF(x) = \int_0^{\infty} \left(\frac{\sin xh}{xh} \right)^k x^k dH(x) \\ &= \int_0^{1/h} + \int_{1/h}^{\infty} = A_h + B_h, \end{aligned}$$

say. Here

$$\begin{aligned} |B_h| &\leq h^{-k} \int_{1/h}^{\infty} dG(x) = h^{-k} \left[\int_{1/h}^{2/h} + \int_{2/h}^{4/h} + \dots \right] \\ &= h^{-k} \left[o(h^k) + o\left(\frac{h}{2}\right)^k + \dots \right] = o(1), \end{aligned}$$

by (8). Since

$$\left(\frac{\sin u}{u} \right)^k = \{1 + O(u^2)\}^k = \{1 + O(u)\}^k = 1 + O(u)$$

for small u , we immediately obtain

$$A_h - \int_0^{1/h} x^k dH(x) = \int_0^{1/h} O(hx^{k+1}) dG(x) = o(1),$$

by (7). Collecting the results, we see that

$$i^{-k} \frac{\Delta_h^k \varphi(0)}{(2h)^k} - \int_0^{1/h} x^k dH(x) = i^{-k} \frac{\Delta_h^k \varphi(0)}{(2h)^k} - \int_{-1/h}^{1/h} x^k dF(x) = o(1),$$

which completes the proof of Theorem 2.

One more remark. By Theorem 2, the existence of the first moment is equivalent to the existence of the first symmetric derivative

$$D_{(1)}\varphi(0) = \lim_{h \rightarrow 0} [\varphi(h) - \varphi(-h)]/2h.$$

In Theorem 1 we have a corresponding result for ordinary first derivative

$$\varphi'(0) = \lim_{h \rightarrow 0} [\varphi(h) - \varphi(0)]/h.$$

There is no discrepancy here since at every point where φ is smooth the two notions of derivative are equivalent.

REFERENCES

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- [3] HARALD CRAMÉR, *Mathematical Methods of Statistics*, Princeton Univ. Press, 1946, p. 90.