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A REMARK ON ELLIPTIC UNITS

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§ 0. Introduction

Let p be a prime number such that $p\equiv 3 \bmod 4$ and $p>3$. Put $K=\boldsymbol{Q}(\sqrt{-p})$ and let H be the absolute class field of K. In [5], Gross defined units u_σ ($\sigma\in\mathrm{Gal}(H/K)$) in a class field of HT of a CM-field T containing K. He gave a question about a property of these units. In this paper, following Robert [8], we give the explicit method to calculate u_σ. In particular when $p=23$ we calculate them concretely to show that Gross' question is correct.

I would like to thank Prof. T. Kanno for encouragement and suggestion.

§ 1.

First we define the notations and recall the problem of Gross [5]. Let p be a prime number such that $p\equiv 3 \bmod 4$ and $p>3$. Let $K=\boldsymbol{Q}(\sqrt{-p})$ with the integer ring $O=O(K)$. Let H be the absolute class field of K with the integer ring $O(H)$. Let I_K (resp. I_H) be the idele group of K (resp. H). Let E be an elliptic curve defined over H with complex multiplication by O. We fix a Weierstrass model for E, $y^2=4x^3-g_2x=g_3$ where $g_2, g_3\in O$. Let j_E be the absolute invariant of E

i.e.
$$j_E=\frac{1728g_2^3}{g_2^3-27g_3^2}.$$

Let v be a finite place of H where E has good reduction. Let H_v be the completion at v, and let k_v be the residue field of H_v. Let $\tilde{E}_v$ be the reduction of E at v. The reduction of endomorphisms gives an injection:

$$\theta_v: K \xrightarrow{\sim} \mathrm{End}_H(E)\otimes\boldsymbol{Q} \longrightarrow \mathrm{End}_{k_v}(E_v)\otimes\boldsymbol{Q}$$

whose image contains the Frobenius endomorphism π_v. Let α_v be the unique element of K with $\theta_v(\alpha_v)=\pi_v$.

Let χ_E be the Grössen character of E. This is a continuous homomorphism of I_H to the multiplicative group $K^\times$, which is the uniquely characterized by the following conditions:

1) If $a=(\alpha)$ is a principal idele, $\chi_E(a)=N_{H/K}(\alpha)$.

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2) If $a=(a_v)$ is an idele with $a_v=1$ at all infinite places of H and at those places where E has bad reduction,

$$\chi_E(a)=\prod \alpha_v^{v(a_v)}$$

where the product is taken over the places of H at which E has good reduction.

Let h be the class number of K. It is known that the absolute invariant is H-isomorphism invariant and there are just h absolute invariants of elliptic curves whose endomorphism rings are isomorphic to O. We denote this set of absolute invariants by J. The character χ_E is H-isogeny invariant.

We say a curve E over H with complex multiplication by O is a $\boldsymbol{Q}$-curve if it is isogenous over H to all of its Galois conjugates E^τ ($\tau\in\mathrm{Aut}(H)$).

Recall the $\boldsymbol{Q}$-curve $A=A(p)$ which was studied in [2][4][5].

Let χ_p be the unique continuous homomorphism of I_H to K which satisfies

1) If $a=(\alpha)$ is a principal idele, $\chi_p(a)=N_{H/K}(\alpha)$.

2) If $a=(a_v)$ is an idele with $a_v=1$ for all $v\mid\infty, p$ and $\mathfrak{p}_v$ is prime at v, then

$$\chi_p(a)=\prod_{v\nmid\infty, p}\alpha_v^{v(a_v)}$$

where ε is the composition of the natural isomorphism from $(O/\sqrt{-p}O)^\times$ to $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ and quadratic residue homomorphism from $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ to $\{\pm 1\}$, and α_v is the element of O such that $N_{H/K}\mathfrak{p}_v=(\alpha_v)$ and $\varepsilon(\alpha_v)=1$. (In this case this determines α_v uniquely.)

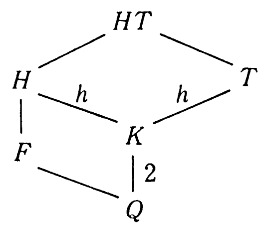
There exists an elliptic curve with complex multiplication by O defined over $F=\boldsymbol{Q}(j)$ ($j\in J$) with the absolute value j, the Grössen character χ_p and the minimal discriminant $(-p^3)$ over F. It is determined uniquely up to F-isomorphism and we denote this curve by $A=A(p)$. (In fact $A(p)$ is F-isomorphic to the following elliptic curve.

$$y^2=x^3+\frac{mp}{2^4\cdot 3}x-\frac{np^2}{2^5\cdot 3^3}$$

where $m^3=j_{A(p)}$

$$n^2=\frac{j-1728}{-p} \qquad \text{sign } n=\left(\frac{2}{p}\right) \quad \text{(c. f. Gross [5])}$$

Let $B=B(p)=\mathrm{Res}_{H/K}A(p)=\prod_{\sigma\in\mathrm{Gal}(H/K)}A(p)^\sigma$ be the Weil restriction of $A(p)$ which is an abelian variety of dimension h. Then $T=\mathrm{End}_K(B)\otimes\boldsymbol{Q}$ is CM-field of degree $2h$ and



Here we can define the Grössen character of B, χ_B. $\chi_B : I_K \to T^\times$: continuous homomorphism

s.t. 1) If $a=(\alpha)$ is a principal idele, $\chi_B(a)=\alpha$

2) If $a=(a_v)$ is an idele with $a_v=1$ when $v|\infty$ or B is bad reduction at v, then

$$\chi_B(a)=\prod \alpha_v^{v(a_v)}$$

where the product is taken over the places of K at which B has good reduction and α_v is the inverse image of the Frobenius endomorphism as in the elliptic case.

From now on in this section, we write $\mathfrak{a}$, $\mathfrak{b}$ for integral ideals of K which are prime to p and write α for an integer of K which is prime to p.

By the definition of χ_B, we get an integer $\chi_B(\mathfrak{a})$ of T. If we write $O(T)$ for the integer ring of T, a principal ideal $\chi_B(\mathfrak{a})O(T)$ is $\mathfrak{a}O(T)$ and the following identities hold:

(1) $\chi_B(\alpha)=\alpha$

(2) $\chi_B(\mathfrak{a}\mathfrak{b})=\chi_B(\mathfrak{a})\chi_B(\mathfrak{b})$.

The restriction $f=\chi_B(\mathfrak{a})|_A$ is an isogeny from A to $A^{\sigma_\mathfrak{a}}$, where $\sigma_\mathfrak{a}$ is $(\mathfrak{a}, H/K)$. Let $f_\mathfrak{a}$ be an element of H s.t. $f^*(\omega^{\sigma_\mathfrak{a}})=f_\mathfrak{a}\omega$, where f^* is the pull back of f. Then the principal idele $f_\mathfrak{a}O(H)$ is $\mathfrak{a}O(H)$ and the following identities hold:

(1) $f_{(\alpha)}=\alpha$

(2) $f_{\mathfrak{a}\mathfrak{b}}=f_\mathfrak{a}f_\mathfrak{b}$.

By the above we get units $u_\mathfrak{a} \overset{\text{def}}{=} \chi_B(\mathfrak{a})/f_\mathfrak{a}$ of HT and $u_{\mathfrak{a}\mathfrak{b}}=u_\mathfrak{a}u_\mathfrak{b}^{\sigma_\mathfrak{a}}$.

Since $u_\mathfrak{a}$ depends only on the ideal class of $\mathfrak{a}$, we denote $u_{\sigma_\mathfrak{a}}=u_\mathfrak{a}$. Let U_{HT} be the unit group of HT. By the above

$$\begin{array}{cccc} u : \mathrm{Gal}(HT/T) \cong & \mathrm{Gal}(H/K) & \longrightarrow & U_{HT} \\ \Downarrow & & & \Downarrow \\ \sigma & & \longrightarrow & u_\sigma|_H \end{array}$$

is 1-cocycle. Gross gave the following two questions.

Q 1 Is the cocycle u a coboundary? i.e. $u \in B^1(\mathrm{Gal}(HT/T), U_{HT})$?

Q 2 Does the summation of $u(\sigma)$ over $\mathrm{Gal}(HT/T)$ belong to U_{HT}?

i. e. $$\sum_{\sigma\in \mathrm{Gal}(HT/T)} u(\sigma)\in U_{HT}?$$

§ 2. The explicit algorithm for u

For a prime p we have h different $A(p)$ (so do $\{u_\sigma\}$) followed by the choice of j, where h is the class number of $K=\boldsymbol{Q}(\sqrt{-p})$. But from the definition they are conjugate and we may only examine the case when $j\in\boldsymbol{R}$, and we may suppose the coefficients of the defining equation of $A(p)$ are integers.

From now on $j\in\boldsymbol{R}$

$$A(p): y^2=4x^3-g_2x-g_3 \qquad g_2,\ g_3\in O \qquad \omega=dx/y$$

It is easy to calculate $\chi_B(\mathfrak{a})$ from the definition of χ_B and χ_p. We give the algorithm for $f_{\mathfrak{a}}$, followed by Robert [8].

First we give a few notations.

$$L=\left\{\int_\gamma \omega \mid \gamma\in H_1(A(\boldsymbol{C}),\ \boldsymbol{Z})\right\} \quad L_\sigma=\left\{\int_\gamma \omega^\sigma \mid \gamma\in H_1(A^\sigma(\boldsymbol{C}),\ \boldsymbol{Z})\right\} \quad (\sigma\in \mathrm{Gal}(H/K))$$

$$G_2(\mathcal{L})=\lim_{\substack{s\to 0\\ s>0}} \sum_{\lambda\in\mathcal{L}-\{0\}} \frac{1}{\lambda^2\cdot|\lambda|^{2s}} \quad G_k(\mathcal{L})=\sum_{\lambda\in\mathcal{L}-\{0\}}\frac{1}{\lambda^k} \quad (k>2)$$

($\mathcal{L}$: a lattice)

Then $G_k(L)\in H$ $(k\geqq 2)$

$$\wp(z,\ \mathcal{L})=\frac{1}{z^2}+\sum_{\lambda\in\mathcal{L}-\{0\}}\left\{\frac{1}{(z-\lambda)^2}-\frac{1}{\lambda^2}\right\} : \text{the Weierstrass } \wp\text{-function}$$

Then

$$\wp(z,\ \mathcal{L})=\frac{1}{z^2}+\sum_{k\geqq 1}(2k+1)G_{2k+2}(\mathcal{L})z^{2k} \quad (0<|z|<\min_{\omega\in\mathcal{L}-\{0\}}|\omega|)$$

($\mathcal{L}$: a lattice)

$$\wp_{\mathfrak{a},L}=\sum_{0\neq\lambda\in\mathfrak{a}^{-1}L/L}\wp(\lambda,\ L)\in H.$$

We use the q-expansions and the integral conditions to calculate u explicitely as follows.

1. the determination of $G_2(L)$

1. approximate value of $G_2(L)$

(1) $$\left(\frac{w_1}{2\pi}\right)^2 G_2(\mathfrak{a}^{-1})=\frac{1}{12}\left(1-24\sum_{n\geqq 1}\frac{nq^n}{1-q^n}\frac{3}{\pi\,\mathrm{Im}(w_2/w_1)}\right),$$

(2) $$\left(\frac{w_1}{2\pi}\right)^2 G_4(\mathfrak{a}^{-1})=\frac{1}{720}\left(1+240\sum_{n\geqq 1}\frac{n^3q^n}{1-q^n}\right),$$

(3) $\left(\frac{w_1}{2\pi}\right)^2 G_6(\mathfrak{a}^{-1})=\frac{1}{30240}\left(1-504\sum_{n\geq 1}\frac{n^5q^n}{1-q^n}\right),$

where $\mathfrak{a}$ is an integral ideal of K s. t. $\mathfrak{a}^{-1}=(w_1, w_2)\,\mathrm{Im}\,(w_1/w_2)>0$
$q=\exp(2\pi i(w_2/w_1))$.

From the complex multiplication theory, their exists

$$\rho(\mathfrak{a})=\rho(\mathfrak{a}, L)\in \boldsymbol{C} \qquad \text{s. t. } L_{\sigma_\mathfrak{a}}=\rho(\mathfrak{a})\mathfrak{a}^{-1}$$

(4) $\rho(\mathfrak{a})^2=\frac{140}{60}\cdot\frac{g_2}{g_3}\cdot\frac{G_6(\mathfrak{a}^{-1})}{G_4(\mathfrak{a}^{-1})},$

(5) $G_2(L)^{\sigma_\mathfrak{a}}=G_2(L_{\sigma\mathfrak{a}})=\rho(\mathfrak{a})^{-2}G_2(\mathfrak{a}^{-1})$.

2. the integral condition of $G_2(L)$

(6) $2\sqrt{-p}G_2(L)\in O(H)$.

In general it is difficult to determine the integer ring when the degree is high, but in this case when j is real we can do it slightly more easily.

(7) $2pG_2(L)\in O(F)$: the integer ring of $F=\boldsymbol{Q}(j)$.

2. the determination of $\wp_{\mathfrak{a}, L}$
1. approximate value of $(\wp_{\mathfrak{a}, L})^{\sigma_\mathfrak{b}}$

(8) $\left(\frac{w_3}{2\pi i}\right)\wp(z, \mathfrak{b})=\frac{1}{12}\sum_{m\in Z}\frac{q^m q_z}{(1-q^m q_z)^2}-2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$

where $\mathfrak{b}$ is an integral ideal s. t. $\mathfrak{b}=(w_3, w_4)$ $\mathrm{Im}\,(w_3/w_4)>0$
$q=\exp(2\pi i(w_4/w_3))$ $q_z=\exp(2\pi i(z/w_3))$

(9) $(\wp_{\mathfrak{a}, L})^{\sigma_\mathfrak{b}}=\wp_{\mathfrak{a}, L\sigma_\mathfrak{b}}=\rho(\mathfrak{b})^{-2},$

2. the integral condition of $\wp_{\mathfrak{a}, L}$

(10) $2\wp_{\mathfrak{a}, L}\in O(H)$

Especially when $N\mathfrak{a}=2$

(11) $4\wp_{\mathfrak{a}, L}^3-g_2\wp_{\mathfrak{a}, L}-g_3=0$

3. 1. the determination of $G_2(\mathfrak{a}^{-1}L)$

(12) $G_2(\mathfrak{a}^{-1}L)-N\mathfrak{a}G_2(L)=\wp_{\mathfrak{a}, L}$

From **1**, **2** and (12) we can determine $G_2(\mathfrak{a}^{-1}L)$
2. the determination of $f_\mathfrak{a}$

(13) $G_2(\mathfrak{a}^{-1}L)=f_\mathfrak{a}^2 G_2(L)$

From **1** and (13) we can determine $f_{\mathfrak{a}}$.
Proof of (1)~(13)
(1) (3), (8) See Lang [7] Chap. 4 and Kubert and Lang [6] Chap. 10
(12) Define

$$\sigma(z, L)=z\prod_{\lambda\in L-\{0\}}\left(1-\frac{z}{\lambda}\right)\exp\left(\frac{z}{\lambda}+\frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right): \text{ the Weierstrass } \sigma\text{-function.}$$

Then

$$\mathscr{P}(z, L)=-\frac{\partial^2}{\partial z^2}\log\sigma(z, L)$$

Define

$$\theta(z, L)=\Delta(L)\sigma^{12}(z, L)\exp(-6G_2(L)z^2)$$

where $\Delta(L)=(2\pi)^{12}((60G_4(L))^3-27(140G_6(L))^2)$.
Then

$$z\frac{\partial}{\partial z}\log\theta(z, L)=-12G_2(L)z^2+12\frac{\sigma'(z, L)}{\sigma(z, L)}z$$
$$=12(1-\sum_{\substack{k>0\\2\mid k}}G_k(L)z^k)$$

Let $\mathfrak{a}$ be an integral ideal of K.
Define

$$\theta(z, L;\mathfrak{a})=\theta(z, L)^{N\mathfrak{a}}/\theta(z, \mathfrak{a}^{-1}L).$$

Then

$$z\frac{\partial}{\partial z}\log(z, L;\mathfrak{a})=12(N\mathfrak{a}-1+\sum_{\substack{k>0\\2\mid k}}(G_k(\mathfrak{a}^{-1}L)-N\mathfrak{a}G_k(L))z^k)$$

On the other hand, $\theta(z, L;\mathfrak{a})$ is an elliptic function $w.r.$ to L and an even function. Comparing zeros, poles and the first coefficient of power series expansion at $z=0$, we get the next equation:

$$\theta(z, L;\mathfrak{a})=\frac{\Delta(L)}{\Delta(\mathfrak{a}^{-1}L)}\prod_{\lambda\in\mathfrak{a}^{-1}L/L-\{0\}}\frac{\Delta(L)}{(\mathscr{P}(z, L)-\mathscr{P}(\lambda, L))^6}$$

We compare two expression of z^2-coefficient of $z(\partial/\partial z)\log\theta(z, L;\mathfrak{a})$ and we get the result.
(5), (9) From the definition and (12).
(4) From the homogeneity of G_4 and G_6,

$$\rho(\mathfrak{a})^2=\left(\frac{G_6(\mathfrak{a}^{-1})}{G_6(L)}\right)\left(\frac{G_4(\mathfrak{a}^{-1})}{G_4(L)}\right)^{-1}=\frac{140}{60}\cdot\frac{g_2}{g_3}\cdot\frac{G_6(\mathfrak{a}^{-1})}{G_4(\mathfrak{a}^{-1})}$$

(6) In (12) we take $\mathfrak{a}=(\alpha)$. $\alpha\in O$

$$O\ni 2\mathscr{P}_{\mathfrak{a},L}=2(G_2(\mathfrak{a}^{-1}L)-N\mathfrak{a}G_2(L))=\alpha(\alpha-\bar{\alpha})G_2(L).$$

Since the greatest common ideal of $\alpha(\alpha-\bar{\alpha})$ is $(\sqrt{-p})$, $2\sqrt{-p}G_2(L)\in O$

(7) Since $H=\boldsymbol{Q}(j, \sqrt{-p})$,

$$2\sqrt{-p}G_2(L)=x_0+x_1\sqrt{-p}+x_2j+x_3j\sqrt{-p}+x_4j^2+x_5j^2\sqrt{-p} \quad x_i\in\boldsymbol{Q}$$

Since j is real, $G_2(L)$ is also real and

$$2\sqrt{-p}G_2(L)=x_1\sqrt{-p}+x_3j\sqrt{-p}+x_5j^2\sqrt{-p}$$

From (6)

$$O(F)\ni N_{H/K}(2\sqrt{-p}G_2(L))=p(x_1+x_3j+x_5j^2)^2$$

$$O(F)\ni p(x_1+x_3j+x_5j^2)=2pG_2(L)$$

(10) See Cassels [3]

(11) From

$$A(p): y^2=4x^3-g_2x-g_3$$

(13) From $f_{\mathfrak{a}}L=\mathfrak{a}L$ and (5).

§ 3. Calculation example when $p=23$.

If the class number of $K=\boldsymbol{Q}(\sqrt{-p})$ is 1, then $u_\sigma=1$ and Q 1 and Q 2 are trivially correct. There doesn't exist K of the class number 2 under the assumption of p. Under the condition that the class number of K is 3, $p=23$ is minimal. In this case we calculate u_σ concretely by the method of § 2, and show that Q 1 and Q 2 are correct.

Let $p=23$ and $K=\boldsymbol{Q}(\sqrt{-23})$, then we have the absolute class field $H=K(\alpha)$ for $\alpha\in\boldsymbol{R}$ such that $\alpha^3-\alpha-1=0$.

Set

$$O=\boldsymbol{Z}+((1+\sqrt{-23})/2)\boldsymbol{Z},$$

$$\mathfrak{a}=2\boldsymbol{Z}+((1+\sqrt{-23})/2)\boldsymbol{Z}$$

and

$$\tilde{\mathfrak{a}}=2\boldsymbol{Z}+((1-\sqrt{-23})/2)\boldsymbol{Z}.$$

Then $\mathrm{Gal}(H/K)=\{\sigma_O, \sigma_{\mathfrak{a}}, \sigma_{\tilde{\mathfrak{a}}}\}$ and since $N\mathfrak{a}=N\tilde{\mathfrak{a}}=2$, in this case we can use (11). And

$$j=-\alpha^{12}5^3(2\alpha-1)^3(3\alpha+2)^3$$

$$A(23): y^2=4x^3-2^23^3c_4x-2^33^3c_6$$

$$\text{where } c_4=5\cdot 23^2\alpha^4(2\alpha-1)(3\alpha+2)$$

$$c_6=\frac{7\cdot 23^2\alpha^8(4\alpha^2+2\alpha-3)(3\alpha+1)}{2\alpha+3}.$$

As for the numerical value above, see Berwick [1] or Gross [4]. From the algorithm of §2

$$G_2(L)=33\alpha^2+30\alpha+9$$

$$\mathcal{P}_{\mathfrak{a},L}=-\frac{3}{2}(13\alpha^2+38\alpha+45-3(\alpha^2-2\alpha+1)\sqrt{-23})$$

$$f_{\mathfrak{a}}=-\frac{1}{2}\alpha^2+\frac{1}{2}\alpha-\frac{1}{2}-\frac{7}{46}\alpha^2\sqrt{-23}-\frac{1}{46}\alpha\sqrt{-23}-\frac{3}{46}\sqrt{-23}$$

And from the definition of χ_B

$$\chi_B(\mathfrak{a})=\left(\frac{3-\sqrt{-23}}{2}\right)^{1/3} \quad \text{(Fix a cubic root of unity.)}$$

Therefore

$$u_{\sigma_O}=1$$

$$u_{\sigma_{\mathfrak{a}}}=\left(\frac{1}{2}\alpha^2-\frac{1}{4}\alpha-\frac{3}{4}-\frac{1}{46}\alpha^2\sqrt{-23}+\frac{3}{92}\alpha\sqrt{-23}+\frac{9}{92}\sqrt{-23}\right)\left(\frac{3-\sqrt{-23}}{2}\right)^{1/3}$$

$$u_{\sigma_{\bar{\mathfrak{a}}}}=\bar{u}_{\sigma_{\mathfrak{a}}}.$$

To see that $u_{\sigma_O}+u_{\sigma_{\mathfrak{a}}}+u_{\sigma_{\bar{\mathfrak{a}}}}$ is unit, we examine

$$\frac{1}{u_{\sigma_O}+u_{\sigma_{\mathfrak{a}}}+u_{\sigma_{\bar{\mathfrak{a}}}}}$$

$$=(-\alpha^2+1)+\frac{1}{92}(23\alpha+12\alpha^2\sqrt{-23}+5\alpha\sqrt{-23}-8\sqrt{-23})\left(\frac{3-\sqrt{-23}}{2}\right)^{1/3}$$

$$+\frac{1}{92}(23\alpha-12\alpha^2\sqrt{-23}-5\alpha\sqrt{-23}+8\sqrt{-23})\left(\frac{3-\sqrt{-23}}{2}\right)^{1/3}$$

is an integer.

To do so, we may examine that the elementary symmetric polynomials of the conjugates of $1/\sum u_\sigma$ over T. They all are in T^+ (the maximal real subfield of T). Since $[T^+:\boldsymbol{Q}]=3$, the integer ring of T^+ is determined by Tornheim [9], for example. In this case the integer ring of T^+ is $\boldsymbol{Z}[\chi_B(\mathfrak{a})+\chi_B(\mathfrak{a})^{-1}]$ and they all are integers and Q 1, 2 are correct.

Thus we have

PROPOSITION. *Let $K=\boldsymbol{Q}(\sqrt{-23})$ and let H be the absolute class field of K. Let $A(23)$ be the $\boldsymbol{Q}$-curve as in* §1. *Let $B=\prod_{\sigma\in \mathrm{Gal}(H/K)} A(23)^\sigma$ be the Weil restriction of $A(23)$. Let $T=\mathrm{End}_K(B)\otimes\boldsymbol{Q}$. Let U_{HT} be the unit group of HT. Let u be the* 1*-cocycle of* $\mathrm{Gal}(HT/T)$ *to U_{HT} as in* §1.

Then u is contained in $B^1(\mathrm{Gal}(HT/T), U_{HT})$ and $\sum_{\sigma\in\mathrm{Gal}(HT/T)} u(\sigma)$ is contained in U_{HT}.

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