PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 1, 1978

# A REMARK ON FUNCTIONS OF BOUNDED MEAN OSCILLATION AND BOUNDED HARMONIC FUNCTIONS

N. TH. VAROPOULOS

This note is an Addendum to [7]; it is written, however, in such a way that it can be read independently.

We recall that $\mu$ a positive measure on the upper half space $\boldsymbol{R}_+^{n+1} = \{(x, y); x \in \boldsymbol{R}^n, y > 0\}$ (the notations we use are those of [6]) is called a Carleson measure if

$$\mu\{x \in I, 0 < y < h\} \leqq Ch^n$$

for all $I$ hypercube in $\boldsymbol{R}^n$ of side length equal to $h$ (cf. [6] VII 44, [1] [7]). In this note we shall prove the following

THEOREM 1. *Let $u$ be a real bounded harmonic function in $\boldsymbol{R}_+^{n+1}$ such that $\|u\|_\infty \leqq 1$ then there exists $v \in C^\infty(\boldsymbol{R}_+^{n+1})$ a real bounded smooth function such that $|\nabla v| d$ Vol is a Carleson measure in $\boldsymbol{R}_+^{n+1}$ and such that $\|u - v\|_\infty \leqq 1 - \varepsilon_0$ for some numerical constant $\varepsilon_0$* $(0 < \varepsilon_0 < 1)$.

$\nabla v$ denotes of course the gradient of $v$ in $\boldsymbol{R}_+^{n+1}$.

REMARK. Note that $|\nabla u| d$ Vol is not in general a Carleson measure (cf. [5]).

THEOREM 2. *Let $f \in$ BMO $(\boldsymbol{R}^n)$ then there exists $F \in C^\infty(\boldsymbol{R}_+^{n+1})$ such that*

( i ) $F(x, y) \xrightarrow[y \to 0]{} f(x)$ $x \in \boldsymbol{R}^n$ *p.p.*

( ii ) $\sup_{y>0} |F(x, y)| \in L^1_{\text{loc}}(\boldsymbol{R}^n; dx)$

(iii) $|\nabla F| d$ Vol *is a Carleson measure in* $\boldsymbol{R}_+^{n+1}$.

*Conversely if $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$ is such that there exists some $F \in C^1(\boldsymbol{R}_+^{n+1})$ that satisfies* (i), (ii), *and* (iii) *then $f$ is a* BMO *function.*

(For the definition of BMO and background cf. [2].)

In functional terms the above theorem means that the space BMO is the restriction space (i.e., quotient space) of an appropriate space of $C^\infty$ functions in the upper half space (cf. [3] for analogous results).

Both the above theorems can be generalized in other contexts, e.g., the complex ball or more generally strictly pseudoconvex domains in $C^n$. Indeed it was in these contexts that theorems of the above type turned out to be useful for the first time cf. [7].

We shall give the proof of both theorems when $n = 1$, the general case is identical. The proof of theorem is based on a construction that was introduced in [1] by L. Carleson and which has proved to be, a very powerfull tool in the study of bounded harmonic functions. A number of modifications of the above construction due to D. Sarason and D. Marshall can be used to simplify matters.

Let $u$ be a real bounded harmonic function in $\boldsymbol{R}^2_+$ and let us suppose that $\|u\|_\infty \leqq 1$. Let us denote by $S = \{0 < x < 1, 0 < y < 1\}$ the unit square in $\boldsymbol{R}^2_+$.

$$A = \{s \in S; |u(s)| < \alpha\}$$
$$B_1 = \{s \in S; u(s) > \beta\}$$
$$B_2 = \{s \in S; u(s) < -\beta\}$$

and $B = B_1 \cup B_2$, where $0 < \alpha < \beta < 1$ are two numbers that will be chosen appropriately. We have then

LEMMA (L. Carleson and D. Marshall). *For an appropriate choice of $\alpha$ and $\beta$ there exists $\Gamma \subset S$ a closed subset which is the union of countably many horizontal and vertical linear segments in $S$ that satisfies the following two conditions*:

(i) *$\Gamma$ separates $A$ from $B$ in $S$* i.e., *if $\sigma: [0, 1] \to S$ is a continuous curve s.t. $\sigma(0) \in A$ and $\sigma(1) \in B$ then $\exists\ 0 \leqq t \leqq 1$ s.t. $\sigma(t) \in \Gamma$.*

(ii) *The measure $\mu_\Gamma$ induced by the arc length of $\Gamma$ is a Carleson measure. ($\mu_\Gamma$ is defined by $\mu_\Gamma(\Omega)$ = total length of the intersection of the linear segments of $\Gamma$ with $\Omega$, where $\Omega$ is an arbitrary open set of $S$.)*

The above lemma is explicitely proved in [4], therefore no proof will be given here. (The proof is far from easy however, it essentially contains the hard part of the Corona theorem!). The best reference (i.e., the most elementary approach) for the above lemma is to be found in D. Marshall's thesis UCLA (1976). In that reference he avoids the use of Hall's lemma.

*Proof of Theorem* 1. Let $S \backslash \Gamma = \bigcup_{n=1}^\infty S_n$ be the decomposition of $S \backslash \Gamma$ into its connected components, and let us define $F$ a function on $S \backslash \Gamma$ as following

$$F(x) = 1 \quad \forall x \in S_n \quad n \geqq 1 \quad \text{if} \quad S_n \cap B_1 \neq \varnothing$$
$$F(x) = -1 \quad \forall x \in S_n \quad n \geqq 1 \quad \text{if} \quad S_n \cap B_2 \neq \varnothing$$
$$F(x) = 0 \quad \forall x \in S_n \quad n \geqq 1 \quad \text{if} \quad S_n \cap B = \varnothing\ .$$

It is then clear from the lemma that $F$ is well defined and that $|\nabla F|$ taken in the sense of distribution theory is a Carleson measure. It is also evident from the definition that $\|u - F\|_\infty \leqq \max [1 - \beta, \beta]$ in $S$. To obtain the required function in $S$ it suffices to smooth out appropriately the above function. The passage to the whole of $\boldsymbol{R}_+^2$ is obvious. Observe also that in Theorem 1 the function $v$ can be chosen such that $\lim v(x, y)$ as $y \to 0$ exists for almost all $x$.

*Proof of Theorem* 2. Using a very simple argument that involves Banach spaces and a geometric series (which anyone who has seen a proof of the closed graph theorem can supply) we see that Theorem 1 implies the first part of Theorem 2 if $f \in L^\infty(\boldsymbol{R}^n)$. The general case follows then from Theorem 1.1.1 of [7]. The inverse is contained in [7] pt. 1.

One question (of mild interest) that remains open is whether $\varepsilon_0$ in Theorem 1 can be chosen to be an arbitrary positive number such that $0 < \varepsilon_0 < 1$. *Added in proof*: J. Garnett has found a way to do that.

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Received July 15, 1977.

MATHEMATIQUE
UNIVERSITE DE PARIS-SUD
CENTRE D'ORSAY, FRANCE