J. DIFFERENTIAL GEOMETRY
6 (1971) 129-134

A REMARK ON HOLMGREN'S UNIQUENESS THEOREM

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In a recent note Bony [1] has given a remarkable improvement of Holmgren's uniqueness theorem. The result is as follows. Let $P(x, D)$ be a differential operator with analytic coefficients in a neighborhood X of a point $x_0 \in \boldsymbol{R}^n$, and denote the principal symbol by $P_m(x, \xi)$ where $x \in X$ and $\xi \in \boldsymbol{R}^n$. Let $u \in \mathscr{D}'(X)$ be a solution of the equation $P(x, D)u = 0$ vanishing when $\varphi(x) > \varphi(x_0)$, $x \in X$, where $\varphi \in C^1(X)$ and $N_0 = \operatorname{grad} \varphi(x_0) \neq 0$. Holmgren's uniqueness theorem then states that u must vanish in a neighborhood of x_0 if $P_m(x_0, N_0) \neq 0$. (Schapira [4] has proved that this remains true for hyperfunction solutions.) Bony [1] introduced the smallest ideal $I(P)$ in $C^\infty(X \times (\boldsymbol{R}^n \backslash 0), \boldsymbol{R})$ such that

(i) $Q \in I(P)$ if $Q(x, \xi)$ is positively homogeneous with respect to ξ and vanishes for all $(x, \xi) \in X \times (\boldsymbol{R}^n \backslash 0)$ with $P_m(x, \xi) = 0$,

(ii) $Q_1, Q_2 \in I(P)$ implies $\{Q_1, Q_2\} \in I(P)$ if

$$\{Q_1, Q_2\} = \sum (\partial Q_1/\partial \xi_j \, \partial Q_2/\partial x_j - \partial Q_1/\partial x_j \, \partial Q_2/\partial \xi_j)$$

is the Poisson bracket of Q_1 and Q_2.

Bony's result is that $u = 0$ in a neighborhood of x_0 unless all $Q \in I(P)$ vanish at (x_0, N_0). The idea of the proof is that if the boundary of supp u is smooth, then the functions satisfying condition (i) must vanish on the normal bundle by Holmgren's uniqueness theorem, so the classical integration theory for first order equations shows that the repeated Poisson brackets of such functions must vanish too. The surprising point in the argument of Bony is that although the support of u may be an arbitrary closed set a priori, one can find sufficiently good parametrizations of large parts of a generalized normal bundle in order to carry through this argument. Although very ingenious the proof seems slightly artificial in that it forces one to introduce highly irregular objects into consideration. We shall here give an alternative more elementary proof based on deformations of smooth surfaces passing through the point x_0 such that repeated use of Holmgren's uniqueness theorem gives the desired result. This is analogous to the proof of Theorem 5.3.2 in Hörmander [2], which gives another variant of Holmgren's uniqueness theorem for the case where P_m is real, $\varphi \in C^2$, $P_m(x_0, N_0) = 0$ but the second order derivative of φ along the bicharacteristic curve with initial data (x_0, N_0) is positive. Indeed our proof here will also give

Received April 5, 1971. Supported in part by National Science Foundation grant GP-7952X2.

uniqueness when $Q(x_0, N_0) = 0$ for all $Q \in I(P)$ provided that the second order derivative of φ along the bicharacteristic corresponding to Q is positive for some $Q \in I(P)$.

Following Trèves [5] and Zachmanoglou [6] we could also give a stronger result for the case where the bicharacteristics are tangents of higher order, but it is not clear how to combine the idea of Bony with the more far reaching refinement of [2, Theorem 5.3.2], which was recently given by Hörmander [3]. (For the first order case see Zachmanoglou [7].) However, as in [3] we shall state our uniqueness theorems not for solutions of a differential equation but for distributions with non-analytic wave front set $WF_A(u)$ contained in a given set. When $Pu = 0$ this set can be taken as the set of zeros of P_m [3, Theorem 5.1]. Thus let F be a closed conic set in $T^*(X)\backslash 0$ where X is an open set in $\boldsymbol{R}^n$ (or a manifold of dimension n). We assume that F is symmetric so that $(x, \xi) \in F$ implies $(x, -\xi) \in F$. The uniqueness theorems will concern distributions in

$$\mathscr{D}'_F(X) = \{u \in \mathscr{D}'(X);\ WF_A(u) \subset F\} .$$

Definition 1. By $U(F)$ we shall denote the set of all real valued continuous positively homogeneous functions p on $T^*(X)\backslash 0$ such that if $u \in \mathscr{D}'_F(X)$ and $u = 0$ in an open set $X_0 \subset X$ with $\partial X_0 \in C^2$ at $x_0 \in X \cap \partial X_0$, then $u = 0$ in a neighborhood of x_0 in X provided that $p(x_0, N_0) \neq 0$ for a normal N_0 of ∂X_0 at x_0.

With this terminology, [3, Theorem 4.3] means that $p \in U(F)$ if $p = 0$ on F. Also note that a well known argument (see the proof of [2, Theorem 5.3.1]) shows that when $p \in U(F)$ the conclusion required by the definition when $\varphi \in C^2$ remains valid if φ is just differentiable at x_0. Moreover, $u = 0$ in a neighborhood of x_0 which does not depend on u.

The following two lemmas replace respectively the arguments of Bony [1] and the proof of [2, Theorem 5.3.2].

Lemma 2. *If $p, q \in U(F) \cap C^1$, then $\{p, q\} \in U(F)$.*

Lemma 3. *Let $p \in U(F) \cap C^1$ and $u \in \mathscr{D}'_F(X)$. Assume that there is a function $\varphi \in C^2(X)$ with $N_0 = \operatorname{grad} \varphi(x_0) \neq 0$ such that $u = 0$ in a neighborhood of x_0 when $\varphi(x) > \varphi(x_0)$. Then $u = 0$ in a full neighborhood of x_0 (which is independent of u) if ($p(x_0, N_0) = 0$ and)*

$$\begin{aligned}(1)\qquad &\sum (\partial^2\varphi(x_0)/\partial x_j \partial x_k) p^{(j)}(x_0, N_0) p^{(k)}(x_0, N_0) \\ &\qquad + \sum p^{(j)}(x_0, N_0) p_{(j)}(x_0, N_0) > 0 ,\end{aligned}$$

where $p^{(j)}(x, \xi) = \partial p(x, \xi)/\partial \xi_j$ and $p_{(j)}(x, \xi) = \partial p(x, \xi)/\partial x_j$.

As explained in [2, p. 127] the condition (1) means that the bicharacteristic curve defined by the Hamiltonian p with initial data (x_0, N_0) is neither a tangent of higher order nor contained in the set where $\varphi(x) \leq \varphi(x_0)$ in a neighborhood of x_0. Both lemmas will be proved at the same time below, but first we discuss their consequences.

Modifying a definition used above we now denote by $I(F)$ the ideal in $C^\infty(T^*(X)\backslash 0, \boldsymbol{R})$ generated by the smallest subset which contains all C^∞ homogeneous functions vanishing on F and is closed under Poisson brackets. In view of the identity

$$\{ap, q\} = a\{p, q\} + p\{a, q\} ,$$

this set is itself closed under formation of Poisson brackets. Hence $I(F)$ is the smallest ideal in $C^\infty(T^*(X)\backslash 0)$ which is closed under Poisson brackets and contains the homogeneous functions vanishing on F. By Lemma 2 and [3, Theorem 4.3] we have $I(F) \subset U(F)$, so we obtain the following theorem by using Lemma 3 for the second part.

Theorem 4. *Let $u \in \mathscr{D}'_F(x)$ and assume that u vanishes in an open set $X_0 \subset X$ with a boundary point $x_0 \in X$ where ∂X_0 is differentiable with normal N_0. Then $u = 0$ in a neighborhood of x_0 (independent of u) if either*

a) *$p(x_0, N_0) \neq 0$ for some $p \in I(F)$*

or

b) a) *is not fulfilled, $\partial X_0 \in C^2$ at x_0, and* (1) *is valid for some homogeneous $p \in I(F)$; here $X_0 = \{x \in X, \varphi(x) > \varphi(x_0)\}$, $N_0 = \operatorname{grad} \varphi(x_0) \neq 0$ and $\varphi \in C^2$.*

Again we remark that (1) is the second order derivative at (x_0, N_0) of the lifting φ_1 of φ to $T^*(X)$ along the Hamilton field $H_p = \partial p/\partial\xi\, \partial/\partial x - \partial p/\partial x\, \partial/\partial\xi$ of p. If a) does not hold, then the first order derivative of φ_1 along H_p vanishes at (x_0, N_0) for all $p \in I(F)$. Since the Hamilton field of $\{p_1, p_2\}$ is the commutator of H_{p_1} and H_{p_2}, it follows that $H_{p_1}H_{p_2}\varphi_1 = H_{p_2}H_{p_1}\varphi_1$ at (x_0, N_0) if $p_1, p_2 \in I(F)$. Hence

$$H_{p_1}H_{p_2}\varphi_1(x_0, N_0) = B(H_{p_1}(x_0, N_0), H_{p_2}(x_0, N_0)) ,$$

where B is a symmetric bilinear form on the vector space $V \subset T_{(x_0, N_0)}(T^*(X))$ of Hamilton vectors H_p at (x_0, N_0) when $p \in I(F)$. These occur already for homogeneous p since $H_{ap} = aH_p$ when $p = 0$. Thus condition b) means that the quadratic form $t \to B(t, t)$ is *not negative semi-definite on V*. Note that if p_j, $j \in J$, are homogeneous functions in $C^\infty(T^*(X)\backslash 0)$ vanishing on F which generate an ideal containing all such functions, then V is spanned by all commutators of the vector fields H_{p_j} at (x_0, N_0).

Proof of Lemmas 2 *and* 3. We may assume that $x_0 = 0$ and that $N_0 = (0, \cdots, 0, -1)$, so that the function φ in Lemma 3 is of the form

$$\varphi(x) = \varphi(0) - x_n + A(x') + o(x_n) + o(|x'|^2) ,$$

where $x' = (x_1, \cdots, x_{n-1})$, and A is a quadratic form. In the proof of Lemma 2 we may also assume that the set X_0 occurring in Definition 1 is defined by

$\varphi(x) > \varphi(0)$ when $|x| < 2\delta$ say. Our purpose is to deform the surface $\varphi(x) = \varphi(0)$ slightly so that a surface is obtained for which uniqueness is known by hypothesis. Before doing so we make a change of scale where we consider x_n/ε^2 and $x'/\varepsilon = y$ instead of x near 0, ε being a small positive number.

More precisely, let ψ_t, $0 \leq t \leq 1$, be a continuous function from [0, 1] to $C^1(\boldsymbol{R}^{n-1})$, and assume that for some open bounded neighborhood V of 0 in $\boldsymbol{R}^{n-1}$

$$(2) \qquad \psi_0(y) < A(y),\ y \in V;\ \psi_t(y) < A(y),\ y \in \partial V,\ 0 \leq t \leq 1\ .$$

Assuming as we may that $\partial\varphi/\partial x_n < 0$ when $|x| < \delta$, we set

$$X_t^\varepsilon = \{(x', x_n);\ x_n < \varepsilon^2\, \psi_t(x'/\varepsilon),\ x'/\varepsilon \in V,\ |x| < \delta\}\ .$$

If $x \in X_t^\varepsilon$ and $\psi_t(x'/\varepsilon) \leq A(x'/\varepsilon) - c$ for some $c > 0$, we obtain

$$\varphi(x) - \varphi(x_0) \geq c\varepsilon^2 - o(\varepsilon^2) > c\varepsilon^2/2$$

for sufficiently small ε. In view of (2) it follows for small ε that $X_0^\varepsilon \subset X_0$ and that for $0 \leq t \leq 1$ we have

$$x_n = \varepsilon^2\psi_t(x'/\varepsilon),\ x'/\varepsilon \in V,\ \psi_t(x'/\varepsilon) > A(x'/\varepsilon) - c \quad \text{on} \quad \mathscr{C}X_0 \cap \partial X_t^\varepsilon\ .$$

Thus the normal is $(\varepsilon\psi_t'(y), -1)$ where $y = x'/\varepsilon$. If

$$(3) \quad p(\varepsilon y, \varepsilon^2\psi_t(y), \varepsilon\psi_t'(y), -1) \neq 0 \ \text{for}\ 0 \leq t \leq 1, y \in \overline{V}, \psi_t(y) \geq A(y) - c\ ,$$

when ε is small while c remains a fixed positive number, and if $p \in U(F)$, we may conclude that any $u \in \mathscr{D}_F'(X)$ vanishing in X_0 and in X_t^ε vanishes in a neighborhood of the closure of X_t^ε. Since [0, 1] is connected and $X_0^\varepsilon \subset X_0$, it follows that $u = 0$ in a neighborhood of $\overline{X}_1^\varepsilon$. If $\psi_1(0) \geq 0$, this contains the desired conclusion that $u = 0$ in a neighborhood of 0.

We can now prove Lemma 3. Taking the Taylor expansion in (3) we find that the inequality is valid for small ε provided that

$$(4) \quad a(y) + b(\partial/\partial y)\psi_t \neq 0 \quad \text{if} \quad 0 \leq t \leq 1\ , \quad y \in \overline{V}\ , \quad \psi_t(y) \geq A(y) - c\ .$$

Here we have introduced the notations

$$(5) \qquad a(y) = \sum_1^{n-1} y_j p_{(j)}(0, N_0)\ , \qquad b(\eta) = \sum_1^{n-1} \eta_j p^{(j)}(0, N_0)\ ,$$

and used the assumption that $p(0, N_0) = 0$ which implies $p^{(n)}(0, N_0) = 0$ because of the homogeneity. With these notations the condition (1) is therefore

$$(1)' \qquad b(\partial/\partial y)^2 A + b(\partial/\partial y)a(y) > 0\ .$$

Since $a(y) + b(\partial/\partial y)\psi_t = a(y) + b(\partial/\partial y)A + b(\partial/\partial y)(\psi_t - A)$ and $b(\partial/\partial y)^2 A$

$+ b(\partial/\partial y)a = b(\partial/\partial y)(a + b(\partial/\partial y)A)$, we can reduce ourselves to the case $A = 0$ by considering $\psi_t - A$ instead of ψ_t. Assuming therefore that $A = 0$ we now set, with $\gamma = b(\partial/\partial y)a(y) > 0$ and a constant $c_1 > 0$,

$$\psi_t(y) = c_1(t - 1) + a(y) - a(y)^2/2\gamma + \chi(y) ,$$

where $\chi(0) = 0$, $\chi \leq 0$ and $\chi \to -\infty$ at infinity so slowly that $b(\partial/\partial y)\chi + \gamma > 0$ everywhere. Since $a(y) + b(\partial/\partial y)\psi_t = b(\partial/\partial y)\chi + \gamma$ and ψ_t tends uniformly to $-\infty$ when $y \to \infty$ if $0 \leq t \leq 1$, the conditions (2) and (4) are fulfilled for some V and c_1. Since $\psi_1(0) = 0$, we have proved Lemma 3.

So far the argument is essentially a repetition of the proof of [2, Theorem 5.3.2]. However, we shall now also consider the case where (1)′ does not hold. Again we may assume that $A = 0$. We have

$$b(\partial/\partial y)(\gamma\psi_1(y) + a(y)^2/2) = \gamma(b(\partial/\partial y)\psi_1(y) + a(y)) ,$$

and if $\gamma < 0$ it follows from (2) and (4) that $\gamma\psi_1(y) + a(y)^2/2 > 0$ at ∂V and points in V where the right hand side vanishes. It is therefore clear that $\gamma\psi_1 + a^2 > 0$ in V if (2) and (4) are valid. When $\gamma = 0$ we obtain similarly that $\psi_1 < 0$ when $a = 0$. Summing up the conclusions and stating them for a general A, we have found that if $\gamma = b(\partial/\partial y)^2 A + b(\partial/\partial y)a \leq 0$, it follows from (2) and (4) for $y \in V$ that for some α

$$(6) \qquad \psi_1(y) < A + \alpha(a(y) + b(\partial/\partial y)A)^2/2 , \qquad 1 + \gamma\alpha \geq 0 .$$

On the other hand, we claim that (6) is the only restriction on ψ_1 implied by (2) and (4). To prove this we assume again that $A = 0$ and set

$$\psi_t(y) = -c + f(a(y)^2/2) + \chi(y) ,$$

where $c > 0$ is a constant. Then we have

$$a(y) + b(\partial/\partial y)\psi_t(y) = a(y)(1 + \gamma f'(a(y)^2/2)) + b(\partial/\partial y)\chi(y) .$$

We take for f any function on $\boldsymbol{R}_+$, which is 0 near 0, increases so that $1 + \gamma f'(t) > 0$ up to some point T after which f decreases to $-\infty$. Then the modulus of the first term is at least a positive multiple of $|a(y)|$, so we can choose $\chi \leq 0$ equal to 0 on a large compact set and converging to $-\infty$ at ∞ so slowly that $a(y) + b(\partial/\partial y)\psi_t(y) \neq 0$ where $f(a(y)^2/2) > 0$. This proves that, by means of a family satisfying (2) and (4), we can reach with $\psi_1(y)$ any value satisfying (6).

Assume now that as in Lemma 2 we have two or any finite number of functions $p_j \in U(F) \cap C^1$, $j = 1, \cdots, J$, all vanishing at (x_0, N_0). To each of them we define linear forms $a_j(y)$ and $b_j(\eta)$ as in (5). Since $\sum s_j p_j \in U(F)$ for arbitrary real s_j, we have already proved uniqueness if for some s

$$\gamma(s) = (\sum s_j b_j(\partial/\partial y))^2 A(y) + \sum s_j b_j(\partial/\partial y) \sum s_k a_k(y)$$

is positive. If on the other hand $\gamma(s) \leq 0$ for all s, we can still use deformations satisfying (2) and (4) with respect to $\sum s_j p_j$ to reach any value of $\psi(y)$ with

$$\psi(y) < A + \alpha(\sum s_j a_j(y) + \sum s_j b_j(\partial/\partial y)A)^2/2 , \qquad \alpha\gamma(s) + 1 \geq 0 .$$

We can now repeat the first argument with A replaced by the right hand side and s replaced by some other value σ. Writing

$$B(\sigma, s) = \sum \sigma_j s_k (b_j(\partial/\partial y) b_k(\partial/\partial y) A + b_j(\partial/\partial y) a_k(y)) ,$$

which implies $\gamma(s) = B(s, s)$, we obtain uniqueness if for some σ

$$B(\sigma, \sigma) + \alpha B(\sigma, s)^2 > 0 , \qquad \alpha\gamma(s) + 1 \geq 0 .$$

Thus the only case where uniqueness does not follow now is when for all σ, s and α

$$(7) \quad B(s, s) \leq 0 \quad \text{and} \quad B(\sigma, \sigma) + \alpha B(\sigma, s)^2 \leq 0 \quad \text{if} \quad \alpha B(s, s) + 1 \geq 0 .$$

If $B(s, s)$ vanishes identically, it follows that $B(\sigma, s) = 0$ identically. Otherwise we obtain when $B(s, s) \neq 0$ that $B(\sigma, \sigma)B(s, s) - B(\sigma, s)^2 \geq 0$. Replacing σ by $s + \delta\sigma$ and dividing by δ we obtain when $\delta \to \pm 0$ that $B(\sigma, s) + B(s, \sigma) - 2B(\sigma, s) = 0$, so B is symmetric. But the symmetry of B means that

$$0 = b_j(\partial/\partial y) a_k(y) - b_k(\partial/\partial y) a_j(y) = \{p_j, p_k\}(0, N_0) .$$

If some Poisson bracket does not vanish, we have therefore proved that every $u \in \mathscr{D}'_F(X)$ vanishing when $\varphi(x) > \varphi(0)$ vanishes near 0. This completes the proof of Lemma 2 and therefore of Theorem 4.

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