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A REMARK ON MINIMAL IMBEDDING OF SURFACES IN E^4

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1[1]. In [1] Prof. T. Ōtsuki introduced some kinds of curvature and torsion form for surfaces in a higher dimensional Euclidean space and proved some interesting formulas and theorems, one of them is given as follows:

THEOREM. *Let x: $M^2 \to E^4$ be an immersion of an oriented closed surface M^2 in 4-dimensional Euclidean space E^4, then we have*

$$\int_{S_0^3} m_1(e) d\Sigma_3 = -\pi \int_{M_2} G(p) dV + 2\int_{M-} \left\{ -\left(\frac{\pi}{2} - \alpha\right) G + \sqrt{-\lambda\mu} \right\} dV, \tag{1}$$

and

$$\int_{S_0^3} m_1(e) d\Sigma_3 = \pi \int_{M_1} G(p) dV + 2\int_{M_-} \left\{ \alpha G(p) + \sqrt{-\lambda\mu} \right\} dV - 4(1-g)\pi^2, \tag{2}$$

where g denotes the genus of M^2, $M_1 = \{p \in M^2, \mu(p) \geqq 0\}$ and $M_2 = \{p \in M^2, \lambda(p) \leqq 0\}$.

The aim of the present paper is to use the above results to prove the followings:

PROPOSITION. *If x: $M^2 \to E^4$ is an immersion of an oriented closed surface of genus g in E^4 with $\lambda\mu \geqq 0$, and $G(p)$ denotes the Gaussian curvature of M^2 at p, then the following inequalities hold*:

$$\int_U G(p) dV \geqq 4\pi, \tag{3}$$

and

$$\int_V G(p) dV \leqq -4g\pi, \tag{4}$$

where $U = \{p \in M^2, G(p) \geqq 0\}$, and $V = \{p \in M^2, G(p) \leqq 0\}$.

THEOREM 1. *Let x: $M^2 \to E^4$ be an immersion of an oriented closed surface of genus g in E^4 with $\lambda\mu \geqq 0$, then x: $M^2 \to E^4$ is a minimal imbedding if and only if the equalities in* (3) *and* (4) *hold.*

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1) We follow the notations in [1].

2. Proof of Proposition. Since by the assumption, $\lambda\mu\geqq 0$, we have

$$M_{-}=\{p\in M^2,\ \lambda(p)\mu(p)<0\}=\phi,\ M_1=U \text{ and } M_2=V. \tag{5}$$

Therefore formulas (1) and (2) reduce to the following forms:

$$\int_{S_0^3} m_1(e)d\Sigma_3=-\pi\int_V G(p)dV \tag{6}$$

and

$$\int_{S_0^3} m_1(e)d\Sigma_3=\pi\int_U G(p)dV-4(1-g)\pi^2. \tag{7}$$

Now, by virtue of the Morse's inequalities we have

$$\begin{gathered} m_0(e)\geqq 1, \qquad m_1(e)-m_0(e)\geqq 2g-1, \\ m_2(e)-m_1(e)+m_0(e)=2(1-g)=\chi(M^2) \end{gathered} \tag{8}$$

for any $e\in S_0^3$, except a set of measure zero, so that we get

$$m_0(e)\geqq 1, \qquad m_1(e)\geqq 2g \quad\text{and}\quad m_2(e)\geqq 1. \tag{9}$$

This gives us

$$\int_{S_0^3} m_1(e)d\Sigma_3\geqq 2gc_3=4g\pi^2. \tag{10}$$

Substitute (10) into (6) and (7), we get

$$4g\pi^2\leqq\pi\int_U G(p)dV-4(1-g)\pi^2,$$

and

$$4g\pi^2\leqq-\pi\int_V G(p)dV,$$

these imply the inequalities (3) and (4).

3. Proof of Theorem 1. Let x: $M^2\to E^4$ be an immersion of an oriented closed surface of genus g in E^4 with $\lambda\mu\geqq 0$, If x: $M^2\to E^4$ is a minimal imbedding, then by the definition of minimal imbedding, we have

$$m_0(e)=m_2(e)=1 \qquad\text{and}\qquad m_1(e)=2g \tag{11}$$

for any $e\in S_0^3$, except a set of measure zero. Now, substitute these equalities into (6) and (7), we can easily get

$$\int_U G(p)dV=4\pi, \tag{12}$$

and

$$\int_V G(p)dV = -4g\pi. \tag{13}$$

Conversely, if the equalities (12) and (13) hold, then let us substitute (12) and (13) into (1) and (2), we get

$$\int_S m_1(e) = 4g\pi^2 = 2gc_3,$$

therefore

$$m_1(e) = 2g \quad \text{almost everywhere on} \quad S_0^3, \tag{14}$$

Substitute (14) into (8), we have

$$m_0(e) + m_2(e) = 2 \tag{15}$$

for any $e \in S_0^3$, except a set of measure zero. Therefore by the fundamental formula (16) in [1], we know x: $M^2 \to E^4$ is a minimal imbedding. This completes the proof of Theorem 1.

With use of the result in §1 due to Ōtsuki, we can easily prove that if M^2 is an oriented closed surface immersed in E^4, then the set $\{p \in M^2;\ \lambda(p) > 0\}$ is a positive measure set. The further results of $G(p)$ see [4].

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