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A remark on pseudoconvex domains with analytic complements in compact Kähler manifolds

By

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Abstract

For an effective divisor A with support B in a compact Kähler manifold M of dimension ≥ 3, the following are antinomic.

a) $M\backslash B$ has a C^∞ plurisubharmonic exhaustion function whose Levi form has pointwise at least 3 positive eigenvalues outside a compact subset of $M\backslash B$.

b) $[A]|B$, the normal bundle of A, is topologically trivial.

Introduction

The purpose of this note is to ensure the following nonexistence result.

Theorem. *Let M be a compact Kähler manifold and let D be a domain in M. Suppose that $B := M\backslash D$ is a complex analytic subset of pure codimension one such that there exists an effective divisor A with support B for which the line bundle $[A]|B$ is topologically tirivial. Then D admits no C^∞ plurisubharmonic exhaustion function whose Levi form has at least* 3 *positive eigenvalues everywhere outside a compact subset of D. In particular D is not Stein.*

A similar result was obtained in [O-2], where $M\backslash D$ is assumed to be a real hypersurface of class C^ω.

As a crucial step for the proof of Theorem, we shall show : If we suppose the existence of an exhaustion function on D as above, then the sheaf of germs of holomorphic 1-forms on M would admit a subsheaf $\mathcal{L}$ such that the analytic restriction of $\mathcal{L}$ to B is invertible and canonically isomorphic to $[-A]\,|\,B$ as a line bundle. Such a subsheaf induces a foliation on M admitting analytic singularities. Based on this, the rest of the argument towards a contradiction proceeds similarly as in [O-2].

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1. Construction of $\mathcal{L}$

Let (M, D, A, B) be as in the introduction. Suppose that there exists a C^∞ plurisubharmonic exhaustion function φ on D whose Levi form has at least 3 positive eigenvalues outside a compact subset of D.

Let $\mathcal{O}$ $(= \mathcal{O}_M)$ be the structure sheaf of M, let Ω^p $(= \Omega^p_M)$ be the sheaf of holomorphic p-forms, and let $\Omega^p(\log A)$ be the sheaf generated over $\mathcal{O}$ by Ω^p and df/f for the local defining functions of f of A.

We shall identify the natural homomorphism

$$\begin{array}{rcc} \delta : \mathcal{O}(-A)/\mathcal{O}(-A-B) & \longrightarrow & \Omega^1/\mathscr{I}_B\Omega^1 \\ \Psi & & \Psi \\ [f] & \longmapsto & [df] \end{array}$$

with an element of $H^0(B, \mathcal{O}(A) \bigotimes (\Omega^1/\mathscr{I}_B\Omega^1))$, where we denote by $\mathscr{I}_B$ the ideal sheaf of B and put $\mathcal{O}(\pm A) = \mathcal{O}([\pm A])$.

Then δ is contained in the subspace

$$H^0(B, \Omega^1(\log A)/\mathscr{I}_B\Omega^1(\log A)) = H^0(B, \Omega^1(\log A)/\Omega^1)$$

because δ induces the correspondence $[1] \mapsto [df/f]$. Moreover, if we denote by Ω^1_c and $\Omega^1_c(\log A)$ respectively the subsheaves of Ω^1 and $\Omega^1(\log A)$ consisting of d-closed germs, δ is clearly contained in the subspace $H^0(B, \Omega^1_c(\log A)/\Omega^1_c) \subset H^0(B, \mathbb{C})$.

We are going to show the surjectivity of the restriction map

$$H^0(B, \Omega^1_c(\log A)) \to H^0(B, \Omega^1_c(\log A)/\Omega^1_c)$$

by exploiting the existence of φ and the topological triviality of $[A]|B$.

First we note that there exist a neighbourhood $U \supset B$ and a C^∞ map F from U onto the unit disc $\mathbb{D}$ in $\mathbb{C}$ such that $F^{-1}(0) = B$ and dF is nowhere zero on $U \backslash B$, for $[A]|B$ is topologically trivial.

Since $F|(U\backslash B)$ is surjective, it induces an injective homomorphism $F^* : H^1(\mathbb{D}\backslash\{0\}, \mathbb{C}) \to H^1(U\backslash B, \mathbb{C})$. Therefore, since $H^1(U\backslash B, \mathbb{C}) \simeq \mathbb{C}$, the residue homomorphism (or the Gysin map) $H^1(U\backslash B, \mathbb{C}) \to H^0(B, \mathbb{C})$ is surjective.

On the other hand, since D has φ and a Kähler metric, B is connected and the restriction homomorphism

$$H^1(D, \mathbb{C}) \to H^1(U\backslash B, \mathbb{C})$$

is surjective (cf. [O-1], [D], [O-T]).

Hence the residue homomorphism

$$\rho_0 : H^1(M, \iota_*\mathbb{C}) \to H^0(B, \mathbb{C}) \simeq \mathbb{C}$$

is surjective. Here ι denotes the inclusion map $M\backslash B \hookrightarrow M$ and $\iota_*\mathbb{C}$ the direct image of the constant sheaf $\mathbb{C}$.

By the standard exact sequence

$$0 \to \iota_*\mathbb{C} \overset{j}{\hookrightarrow} \widetilde{\mathcal{O}} \overset{d}{\to} \Omega_c^1(\log A) \to 0$$

where $\widetilde{\mathcal{O}}$ deotes the sheaf locally generated by $\log f$ and $\mathcal{O}$ over $\mathbb{C}$, we have an exact sequence

$$H^0(M, \Omega_c^1(\log A)) \to H^1(M, \iota_*\mathbb{C}) \to H^1(M, \widetilde{\mathcal{O}}).$$

It is easy to see that $H^1(M, \widetilde{\mathcal{O}}) \simeq H^1(M, \mathcal{O})$. Here the isomorphism is induced from the inclusion $\mathcal{O} \hookrightarrow \widetilde{\mathcal{O}}$.

Hence by the Hodge theory the image of $H^1(M, \mathbb{C})$ in $H^1(M, \iota_*\mathbb{C})$ is mapped onto $H^1(M, \widetilde{\mathcal{O}})$ by j_*. This means, since $c_1([A]|B) = 0$ by assumption, that the residue map $\rho : H^0(M, \Omega_c^1(\log A)) \to H^0(B, \mathbb{C})$ is also surjective.

Therefore, the injective homomorphism

$$\delta : \mathcal{O}(-A)/\mathcal{O}(-A-B) \to \Omega^1/\mathscr{I}_B\Omega^1$$

can be lifted to a homomorphism say $\tilde{\delta}$ from $\mathcal{O}(-A)$ to Ω^1 of the form $f \to f(df/f + \omega)$ for some $df/f + \omega \in H^0(M, \Omega_c^1(\log A))$.

Thus, by letting $\mathcal{L} = \delta(\mathcal{O}(-A))$, we obtain a desired subsheaf of Ω^1 with $\mathcal{L}/\mathscr{I}_B\mathcal{L} \simeq \mathcal{O}(-A)/\mathcal{O}(-A-B)$ which defines a foliation of codimension one on M, possibly with singularities, which contains B as a leaf.

2. End of the proof

Since $\mathcal{L}/\mathscr{I}_B\mathcal{L} \simeq \mathcal{O}(-A)/\mathscr{I}_B\mathcal{O}(-A)$, $\mathcal{L}$ is invertible on a neighbourhood say V of B, so that one may canonically identify $1 \in H^0(B, \mathbb{C})$ with a section of $\Omega^1(\mathcal{L}^*)$ on V, say s.

By shrinking V if necessary, we may assme that $\mathcal{L}^*$ is topologically trivial on V.

Then, by a vanishing theorem of Grauert and Riemenschneider [G-R], there exists a topologically trivial holomorphic line bundle $\tilde{\mathcal{L}^*}$ over M which extends $\mathcal{L}^*$. (For a more detailed arugument, see [O-2]).

Since M is a compact Kähler manifold, $\tilde{\mathcal{L}^*}$ is unitarily flat. Hence, by the L^2 Hodge theory s is extendable to a holomorphic section $\tilde{s}$ of $\Omega^1(\widetilde{\mathcal{L}^*})$ over M (cf. [O-1], [D], [O-T]). By the Kähler condition again, we have $d\tilde{s} = 0$.

Let $\{U_\alpha\}_{\alpha=1}^m$ be a set of finitely many coordinate neighbourhoods of M such that $\bigcup_{\alpha=1}^m U_\alpha \supset B$ and that s is identified with a system of holomorphic 1-forms $\{s_\alpha\}_{\alpha=1}^m$, s_α being defined on U_α, such that $s_\alpha = e^{i\theta_{\alpha\beta}}s_\beta$ hold on $U_\alpha \cap U_\beta (\neq \emptyset)$ for some $\theta_{\alpha\beta} \in \mathbb{R}$. Here U_α are chosen in such a way that they are biholomorphically equivalent to $\mathbb{D}^n$ and $U_\alpha \cap B$ and $U_\alpha \cap U_\beta$ are connected and contractible.

Let $f_\alpha (1 \leq \alpha \leq m)$ be holomorphic functions on U_α such that $df_\alpha = s_\alpha$ and $f_\alpha \mid U_\alpha \cap B = 0$. Then we have adjacent relations

$$(\#) \qquad f_\alpha = e^{i\theta_{\alpha\beta}} f_\beta$$

on $U_\alpha \cap U_\beta$.

Then we put $T_\epsilon = \bigcup_{\alpha=1}^{m} \{z \in U_\alpha \mid |f_\alpha(z)| = \epsilon\}$ for $\epsilon > 0$. By (#) T_ϵ is a compact set for sufficiently small ϵ. Fix such ϵ and take a point $z_0 \in T_\epsilon$ where $\varphi \mid T_\epsilon$ takes its maximum. Then, since $f_\alpha^{-1}(f_\alpha(z_0)) \subset T_\epsilon$ holds if $U_\alpha \ni z_0$, we have

$$i\partial\overline{\partial}(\varphi \mid f_\alpha^{-1}(f_\alpha(z_0))) \mid_{z=z_0} \leq 0,$$

but this contradicts with the assumption that the Levi form of φ has at least 3 positive eigenvalues near B.

Remark 1. Some non- Kähler manifolds contain D as in the theorem. For instance, let $M = (\mathbb{C}^n\backslash\{0\})/\mathbb{Z}$ $(n \geq 2)$, where two points z, $w \in \mathbb{C}^n\backslash\{0\}$ are identified if and only if

$$\begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} = \begin{pmatrix} e & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} e & e \\ o & e \end{matrix} \end{pmatrix}^m \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{pmatrix}$$

for some $m \in \mathbb{Z}$, let $H = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_n = 0\}$, and let $D = \{[z] \in M \mid z \notin H\}$. Then the boundary B of D obviously satisfies $[B] | B = \mathbb{C} \times B$, but D is Stein because it is biholomorphic to $\mathbb{C}^{n-1} \times \mathbb{C}^*$ by the map

$$(z_1, \ldots, z_n) \mapsto (z_1/z_n, \ldots, z_{n-2}/z_n, e^{2\pi i z_{n-1}/z_n}, z_n e^{-z_{n-1}/z_n})$$

so that D admits an exhaustion function as in the theorem if $n \geq 3$.

Remark 2. There exist Kähler surfaces which contain complex curves of self-intersection zero whose complements are Stein. For instance, the total space X of a holomorphic affine line bundle over a compact Riemann surface C is Stein if and only if it contains no analytic sections, and there exists such an affine line bundle which is at the same time topologially equivalent to $C \times \mathbb{C}$ if the genus C is not zero. By adding to such X the section at infinity, we obtain a Kähler surface containing a Stein domain $D = X$ whose complement is a complex curve of self-intersection zero. See [U] for an analytic theory related to this phenomenon.

Question. Under the assumption of Theorem, is it true that there exist neither 3-dimensional closed Stein subvarieties in $M\backslash B$ nor proper holomorphic maps from $M\backslash B$ onto Stein spaces of dimension ≥ 3 ?

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