A REMARK ON RATIONAL CHEREDNIK ALGEBRAS AND DIFFERENTIAL OPERATORS ON THE CYCLIC QUIVER

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Abstract. We show that the spherical subalgebra $U_{k,c}$ of the rational Cherednik algebra associated to $S_n \wr C_\ell$, the wreath product of the symmetric group and the cyclic group of order ℓ, is isomorphic to a quotient of the ring of invariant differential operators on a space of representations of the cyclic quiver of size ℓ. This confirms a version of [EG, Conjecture 11.22] in the case of cyclic groups. The proof is a straightforward application of work of Oblomkov, [O], on the deformed Harish–Chandra homomorphism, and of Crawley–Boevey, [CB1] and [CB2], and Gan and Ginzburg, [GG], on preprojective algebras.

1. Introduction

1.1. The representation theory of symplectic reflection algebras has links with a number of subjects including algebraic combinatorics, resolutions of singularities, Lie theory and integrable systems. There is a family of symplectic reflection algebras associated to any symplectic vector space V and finite subgroup $\Gamma \leq Sp(V)$, but a simple reduction allows one to study those subgroups Γ which are generated by symplectic reflections (i.e. by elements whose set of fixed points is of codimension two in V). This essentially focuses attention on two cases:

(1) $\Gamma = W$, a finite complex reflection group, acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is a reflection representation of W;
(2) $\Gamma = S_n \wr K$, where K is a finite subgroup of $SL_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$.

The representation theory in the first case is mysterious at the moment: several important results are known but there is no general theory yet. On the other hand a geometric point of view on the representation theory in the second case is beginning to emerge. A key fact is that in this case the singular space V/Γ admits a crepant resolution of singularities: the representation theory of the symplectic reflection algebra is then expected to be closely related to the resolution. In the case $\Gamma = S_n$ (i.e. K is trivial) there are two approaches to this: the first is via noncommutative algebraic geometry, [GS], the second via sheaves of differential operators, [GG]. In this paper we extend the second approach to the groups $\Gamma = \Gamma_n = S_n \wr C_\ell$.

1.2. To state the result here we need to introduce a little notation. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow.

We will consider representation spaces of these quivers. Let $\delta = (1, 1, \dots, 1)$ be the affine dimension vector of Q, and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞. Let $\mathrm{Rep}(Q, n\delta)$ and $\mathrm{Rep}(Q_\infty, \epsilon)$ be the representation spaces of these quivers with the given dimension vectors. There is an action of

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$G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ on both these spaces. In fact, the action of the scalar matrices in G is trivial on $\mathrm{Rep}(Q, n\delta)$ (but not on $\mathrm{Rep}(Q_\infty, \epsilon)$) so in this case the action descends to an action of $PG = G/\mathbb{C}^*$.

Let $\mathfrak{X} = \mathrm{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$. There is an action of PG on $\mathfrak{X}$.

1.3. Let $D(\mathrm{Rep}(Q_\infty, \epsilon))$ denote the ring of differential operators on the affine space $\mathrm{Rep}(Q_\infty, \epsilon)$, $D_{\mathfrak{X}}(nk)$ the sheaf of twisted differential operators on $\mathfrak{X}$ and $D(\mathfrak{X}, nk)$ its algebra of global sections. The group action of G (respectively PG) on $\mathrm{Rep}(Q_\infty, \epsilon)$ (respectively $\mathfrak{X}$) differentiates to an action of $\mathfrak{g} = \mathrm{Lie}(G)$ (respectively $\mathfrak{pg} = \mathrm{Lie}(PG)$) by differential operators. This gives mappings

$$\hat{\tau} : \mathfrak{g} \longrightarrow D(\mathrm{Rep}(Q_\infty, \epsilon)), \qquad \tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk).$$

1.4. Let $U_{k,c}$ be the spherical subalgebra of type $S_n \wr C_\ell$ (this is defined in Section 3.4).

Theorem. *For all (k, c) there are isomorphisms of algebras*

$$\left(\frac{D(\mathrm{Rep}(Q_\infty, \epsilon))}{I_{k,c}}\right)^G \cong \left(\frac{D(\mathfrak{X}, nk)}{I_c}\right)^{PG} \cong U_{k,c},$$

where $I_{k,c}$ is the left ideal of $D(\mathrm{Rep}(Q_\infty, \epsilon)$ generated by $(\hat{\tau} - \chi_{k,c})(\mathfrak{g})$ and I_c is the left ideal of $D(\mathfrak{X}, nk)$ generated by $(\tau - \chi_c)(\mathfrak{pg})$ for suitable characters $\chi_{k,c} \in \mathfrak{g}^*$ and $\chi_c \in \mathfrak{pg}^*$ (which are defined in Section 4).

Note that it is a standard fact that the left hand side is an algebra. The proof of the theorem has two parts. One part constructs a filtered homomorphism from the left hand side to the right hand side using as its main input the work of Oblomkov, [O]. The other part proves that the associated graded homomorphism is an isomorphism and is a simple application of results of Crawley–Boevey, [CB1] and [CB2], and of Gan–Ginzburg, [GG].

1.5. We give an application of this result in Section 4. For related pairs (k, c) and (k', c') we construct "shift functors"

$$U_{k,c}\text{-mod} \longrightarrow U_{k',c'}\text{-mod}$$

using differential operators. We expect these to be a useful tool in the representation theory of Cherednik algebras, deserving of careful study.

1.6. While writing this down, we were informed that the general version of [EG, Conjecture 11.22] has been proved in [EGGO]. That result is more general than the work presented here and requires a new approach and ideas to overcome problems that simply do not arise for the case $\Gamma = S_n \wr C_\ell$.

2. Quivers

2.1. Once and for all fix integers ℓ and n. We assume that both are greater than 1. Set $\eta = \exp(2\pi i/\ell)$.

2.2. Let Q be the cyclic quiver with ℓ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) 0. Then let Q_∞ be the quiver obtained by adding one vertex named ∞ to Q that is joined to 0 by a single arrow. Let $\overline{Q}$ and $\overline{Q}_\infty$ denote the double quivers of Q and Q_∞ respectively, obtained by inserting an arrow a^* in the opposite direction to every arrow a in the quiver.

We will consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of Q, and set $\epsilon = e_\infty + n\delta$, a dimension vector for Q_∞. Recall that

$$\operatorname{Rep}(Q, n\delta) = \bigoplus_{r=0}^{\ell-1} \operatorname{Mat}_n(\mathbb{C}) = \{(X_0, X_1, \ldots, X_{\ell-1}) : X_r \in \operatorname{Mat}_n(\mathbb{C})\} = \{(X)\}$$

and

$$\operatorname{Rep}(Q_\infty, \epsilon) = \bigoplus_{r=0}^{\ell-1} \operatorname{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n = \{(X_0, X_1, \ldots, X_{\ell-1}, i) : X_r \in \operatorname{Mat}_n(\mathbb{C}), i \in \mathbb{C}\} = \{(X, i)\}.$$

Let $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ be the base change group. If $g = (g_0, \ldots, g_{\ell-1})$ then g acts on $\operatorname{Rep}(Q, n\delta)$ by

$$g \cdot (X_0, X_1, \ldots, X_{\ell-1}) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \ldots, g_{\ell-1} X_{\ell-1} g_0^{-1})$$

and on $\operatorname{Rep}(Q_\infty, \epsilon)$ by

$$g \cdot (X_0, X_1, \ldots, X_{\ell-1}, i) = (g_0 X_0 g_1^{-1}, g_1 X_1 g_2^{-1}, \ldots, g_{\ell-1} X_{\ell-1} g_0^{-1}, g_0 i).$$

The action of the scalar subgroup $\mathbb{C}^*$ is trivial in the first action (but not the second), so we can consider the first action as a PG–action where $PG = G/\mathbb{C}^*$. Let $\mathfrak{g}$ and $\mathfrak{pg}$ be the Lie algebras of G and PG respectively.

2.3. Let $\mathfrak{h}^{\mathrm{reg}} \subset \mathbb{C}^n$ be the affine open subvariety consisting of points $x = (x_1, \ldots, x_n)$ such that

(i) if $i \neq j$ then $x_i \neq \eta^m x_j$ for all $m \in \mathbb{Z}$,

(ii) for each $1 \leq i \leq n$ $x_i \neq 0$.

This is the subset of $\mathbb{C}^n$ on which $\Gamma_n = S_n \wr C_\ell$ acts freely.

2.4. We can embed $\mathfrak{h}^{\mathrm{reg}}$ into $\operatorname{Rep}(Q, n\delta)$ by first considering a point $x = (x_1, \ldots, x_n) \in \mathfrak{h}^{\mathrm{reg}}$ as a diagonal matrix $X = \operatorname{diag}(x_1, \ldots, x_n)$ and then sending this to $\underline{X} = (X, X, \ldots, X)$. We denote the image of $\mathfrak{h}^{\mathrm{reg}}$ in $\operatorname{Rep}(Q, n\delta)$ by $\mathcal{S}$.

Let T_Δ be the subgroup of G with elements $(T, T, \ldots, T)$ where T is a diagonal matrix in $GL_n(\mathbb{C})$. Then T_Δ is the stabiliser of $\mathcal{S}$. So consider the mapping

$$\pi : G/T_\Delta \times \mathfrak{h}^{\mathrm{reg}} \longrightarrow \operatorname{Rep}(Q, n\delta)$$

given by $\pi(gT_\Delta, x) = g \cdot \underline{X}$. If we let G act on $G/T_\Delta \times \mathfrak{h}^{\mathrm{reg}}$ by left multiplication then π is a G–equivariant mapping.

Lemma. *π is an étale mapping with covering group Γ_n. In fact its image $\operatorname{Rep}(Q, n\delta)^{reg}$ is open in $\operatorname{Rep}(Q, n\delta)$ and we have an isomorphism*

$$\omega : G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{reg} \longrightarrow \operatorname{Rep}(Q, n\delta)^{reg}.$$

Proof. Let $\mathcal{S} = \{\underline{X} : x \in \mathfrak{h}^{\mathrm{reg}}\}$ and set $N_G(\mathcal{S}) = \{g \in G : g \cdot \mathcal{S} = \mathcal{S}\}$ and $Z_G(\mathcal{S}) = \{g \in G : g \cdot \underline{X} = \underline{X}$ for all $\underline{X} \in \mathcal{S}\}$.

Suppose $g \cdot \underline{X} = \underline{Y}$ for some $\underline{X}, \underline{Y} \in \mathcal{S}$. This implies that for each $0 \leq i \leq \ell - 1$

$$g_i \operatorname{diag}(x)^\ell g_i^{-1} = \operatorname{diag}(y)^\ell.$$

The hypotheses on $\mathfrak{h}^{\mathrm{reg}}$ imply that both $\mathrm{diag}(x)^\ell$ and $\mathrm{diag}(y)^\ell$ are regular semisimple in $\mathbb{C}^n$. Two such elements are conjugate if and only if $g_i \in N_{GL_n(\mathbb{C})}(T) = T \cdot S_n$ where T is the diagonal subgroup of $GL_n(\mathbb{C})$. So there exists $\sigma \in S_n$ such that for all i we have $g_i = t_i\sigma$ for some $t_i \in T$, and for all $1 \leq r \leq n$ we have that $x^\ell_{\sigma(r)} = y^\ell_r$. Hence $x_{\sigma(r)} = \eta^{m_r} y_r$ for some $m_r \in \mathbb{Z}$. Now we find that $\underline{Y} = g \cdot \underline{X}$ implies that $\mathrm{diag}(y_r) = t_i t_{i+1}^{-1} \mathrm{diag}(\eta^{m_r} y_r)$. Since $y_r \neq 0$ this shows that $t_{i+1} = \mathrm{diag}(\eta^{m_r})t_i$ for each i. Hence we find that $gT_\Delta = (\sigma, \mathrm{diag}(\eta^{m_r})\sigma, \ldots, \mathrm{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

In particular, if $\underline{X} = \underline{Y}$ we see from above that each $m_r = 0$, so that $Z_G(\mathcal{S}) = T_\Delta$. Thus the group Γ_n is isomorphic to $N_G(\mathcal{S})/Z_G(\mathcal{S})$ via the homomorphism that sends $(\eta^{m_1}, \ldots, \eta^{m_r})\sigma$ to $(\sigma, \mathrm{diag}(\eta^{m_r})\sigma, \ldots, \mathrm{diag}(\eta^{m_r})^{\ell-1}\sigma)T_\Delta$.

Now suppose that $\pi(gT_\Delta, x) = \pi(hT_\Delta, y)$. Then $(h^{-1}g) \cdot \underline{X} = \underline{Y}$ and so we see that $h^{-1}g \in N_G(\mathcal{S})$. This shows that π is the composition

$$G/T_\Delta \times \mathfrak{h}^{\mathrm{reg}} \longrightarrow G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\mathrm{reg}} \xrightarrow{\sim} \mathrm{Rep}(Q, n\delta)^{reg}.$$

The first mapping factors out the action of Γ_n, and since Γ_n acts freely on $\mathfrak{h}^{\mathrm{reg}}$ this is an étale mapping. Hence, to finish the lemma, it suffices to show that $\mathrm{Rep}(Q, n\delta)^{reg}$ is open in $\mathrm{Rep}(Q, n\delta)$.

We claim first that $\mathrm{Rep}(Q, n\delta)^{reg}$ is the set O of representations of Q which decompose into n simple modules of dimension δ and whose endomorphism ring is n–dimensional. To prove this observe that any element of $\mathrm{Rep}(Q, n\delta)^{reg}$ is isomorphic to a representation of the form $\underline{X}$ and so it decomposes into the n indecomposable modules $\underline{X}_1, \ldots \underline{X}_n$ of dimension δ where $\underline{X}_i = (x_i, x_i, \ldots, x_i)$ (the condition $x_i \neq 0$ implies simplicity). Now the representation $\underline{X}_i$ is isomorphic to the representation $(1, 1, \ldots, 1, x_i^\ell)$. By hypothesis $x_i^\ell \neq x_j^\ell$ so we deduce that the representations $\underline{x}_i$ are pairwise non–isomorphic which ensures that the endomorphism ring of $\underline{X}$ is n–dimensional. This proves the inclusion $\mathrm{Rep}(Q, n\delta)^{reg} \subseteq O$. On the other hand, if V belongs to O then $V = V_1 \oplus \ldots \oplus V_n$ where each V_i is isomorphic to a representation $(1, 1, \ldots, 1, \nu_i)$ for some non–zero scalars ν_i. Moreover, since $\dim \mathrm{End}(V) = n$ the ν_i must be pairwise distinct. Now, let η_i be an ℓ–th root of ν_i. Then V_i is isomorphic to $(\eta_i, \ldots, \eta_i)$. Therefore V is isomorphic to the representation $\underline{X}$ where $x = (\eta_1, \ldots, \eta_n)$.

Now we must show that O is open in $\mathrm{Rep}(Q, n\delta)$. We use first the fact that the canonical decomposition of the vector $n\delta$ is $\delta + \delta + \cdots + \delta$, [Scho, Theorem 3.6]. This means that the representations of $\mathrm{Rep}(Q, n\delta)$ whose indecomposable components all have dimension δ form an open set. Now, consider the morphism f from $\mathrm{Rep}(Q, \delta)$ to $\mathbb{C}$ which sends the representation $(\lambda_1, \ldots, \lambda_\ell)$ to the product $\lambda_1 \ldots \lambda_\ell$. The open set $f^{-1}(\mathbb{C}^*)$ consists of the simple representations of dimension vector δ. Therefore the subset of $\mathrm{Rep}(Q, n\delta)$ consisting of representations which decompose as the sum of n simple representations of dimension vector δ is open. On the other hand, the function from $\mathrm{Rep}(Q, n\delta)$ to $\mathbb{N}$ which sends a representation V to $\dim \mathrm{End}(V)$ is upper semi–continuous. Thus $\{V : \dim \mathrm{End}(v) \leq n\}$ is an open set in $\mathrm{Rep}(Q, n\delta)$. Intersecting these two sets shows that O is open, as required. $\square$

2.5. Now we're going to move from Q to Q_∞. So let's start with the following

$$\{([gT_\Delta, x], i) : g_0^{-1} i \text{ is a cyclic vector for } \mathrm{diag}(x)\} \subset (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\mathrm{reg}}) \times \mathbb{C}^n.$$

By applying $\omega^{-1} \times \mathrm{id}_{\mathbb{C}^n}$ this corresponds to an open subset of $\mathrm{Rep}(Q, n\delta) \times \mathbb{C}^n = \mathrm{Rep}(Q_\infty, \epsilon)$. Call that set U_∞. This is a G–invariant open set since the G–action on triples is given by

$$h \cdot ([gT_\Delta, x], i) = ([hgT_\Delta, x], h_0 i)$$

so $g_0^{-1} i$ is cyclic for $\mathrm{diag}(x)$ if and only if $(h_0 g_0)^{-1} h_0 i$ is cyclic for $\mathrm{diag}(x)$. Observe too that U_∞ is an affine variety. Indeed it is defined by the non–vanishing of the morphism

$$s : (G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\mathrm{reg}}) \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

which sends $([gT_\Delta, x], i)$ to $(g_0^{-1} i) \wedge \mathrm{diag}(x) \cdot (g_0^{-1} i) \wedge \cdots \wedge \mathrm{diag}(x)^{n-1} \cdot (g_0^{-1} i)$.

Lemma. *The G–action on U_∞ is free and projection onto the second component*

$$\pi_2 : U_\infty \longrightarrow \mathfrak{h}^{reg}/\Gamma_n$$

is a principal G–bundle.

Proof. Suppose that $h \cdot ([gT_\Delta, x], i) = ([gT_\Delta, x], i)$.Then $[g^{-1}hgT_\Delta, x] = [T_\Delta, x]$, so by Lemma 2.4 $g^{-1}hg \in T_\Delta$.

We have that $h_0 i = i$. Setting $i' = g_0^{-1} i$ implies that $g_0^{-1} h_0 g_0 i' = i'$. By hypothesis i' is a cyclic vector for $\mathrm{diag}(x)$. So in the standard basis i' decomposes as $\sum \lambda_j e_j$ where each λ_j is non–zero. Therefore the only diagonal matrix that fixes i' is the identity element. In other words $g_0^{-1} h_0 g_0 = I_n$. Since $g^{-1}hg \in T_\Delta$ this implies that $g^{-1}hg = \mathrm{id}$. Thus $h = \mathrm{id}$ and this proves that the action is free.

It remains to prove that each fibre of π_2 is a G–orbit. So take $([gT_\Delta, x], i) \in \pi_2^{-1}([x])$. This equals $g \cdot ([T_\Delta, x], g_0^{-1} i)$. Now $g_0^{-1} i$ is a cyclic vector for $\mathrm{diag}(x)$ so it has the form $\sum \lambda_j e_j$ with each λ_j non–zero. Let $t = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and consider $\underline{t} = (t, \ldots, t) \in T_\Delta$. We have

$$([gT_\Delta, x], i) = g\underline{t}\underline{t}^{-1}([T_\Delta, x], g_0^{-1} i) = g\underline{t}([T_\Delta, x], \sum_{j=1}^{n} e_j).$$

This proves that each fibre of π_2 is indeed a G–orbit. □

2.6. Consider the representation space for the doubled quiver $\overline{Q}_\infty$:

$$\mathrm{Rep}(\overline{Q}_\infty, \epsilon) = \{(X_0, \ldots, X_{\ell-1}, Y_0, \ldots, Y_{\ell-1}, i, j) : X_r, Y_r \in \mathrm{Mat}_n(\mathbb{C}), i \in \mathbb{C}, j \in \mathbb{C}^*\} = \{(X, Y, i, j)\}.$$

We can naturally identify it with $T^* \mathrm{Rep}(Q_\infty, \epsilon)$. The group G acts on the base and hence on the total space of the cotangent bundle. The resulting moment map

$$\mu : \mathrm{Rep}(\overline{Q}_\infty, \epsilon) \longrightarrow \mathfrak{g}^* \cong \mathfrak{g}$$

is given by

$$\mu(X, Y, i, j) = [X, Y] + ij.$$

Theorem (Gan–Ginzburg, Crawley–Boevey)**.** *Let $\mu^{-1}(0)$ denote the scheme–theoretic fibre of μ.*

(1) *$\mu^{-1}(0)$ is reduced, equidimensional and a complete intersection.*
(2) *The moment map μ is flat.*
(3) $\mathbb{C}[\mu^{-1}(0)]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.

Proof. (i) This is [GG, Theorem 3.2.3].

(ii) This follows from [CB1, Theorem 1.1] and the dimension formula in [GG, Theorem 3.2.3(iii)].

(iii) This is [CB2, Theorem 1.1] □

2.7. Let $\mathfrak{X} = \{(X,i) \in \mathrm{Rep}(Q,n\delta) \times \mathbb{P}^{n-1}\}$. This space is the quotient of the (quasi–affine) open subvariety

$$U = \{(X,i) : i \neq 0\} \subset \mathrm{Rep}(Q_\infty, \epsilon)$$

by the scalar group $\mathbb{C}^*$. Thus there is an action of PG on $\mathfrak{X}$.

Since

$$T^*\,\mathbb{P}^{n-1} = \{(i,j) : i \neq 0,\, ji = 0\}/\mathbb{C}^*$$

we have

$$T^*\mathfrak{X} = \{(X,Y,i,j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0,\, ji = 0\}/\mathbb{C}^*.$$

The PG action on $\mathfrak{X}$ gives rise to a moment map

$$\mu_{\mathfrak{X}} : T^*\mathfrak{X} \longrightarrow \mathfrak{pg}^* \cong \mathfrak{pg}\,.$$

Let

$$\mu_{\mathfrak{X}}^{-1}(0) = \{(X,Y,i,j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0,\, ji = 0, \quad [X,Y] + ij = 0\}/\mathbb{C}^*$$

denote the scheme theoretic fibre of 0.

Proposition. *There is an isomorphism* $\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}$.

Proof. Consider the G–equivariant open subvariety of $\mu^{-1}(0)$ given by the non–vanishing of i. The variety $\mu^{-1}(0)$ is determined by the conditions $[X,Y] + ij = 0$, so if we take the trace of this equation then we see that $0 = Tr(ij) = Tr(ji) = ji$. Thus we see that $\{(X,Y,i,j) \in \mathrm{Rep}(\overline{Q}_\infty, \epsilon) : i \neq 0,\, ji = 0\} \cap \mu^{-1}(0)$ is an open subvariety of $\mu^{-1}(0)$ so in particular reduced by Theorem 2.6(1). Hence factoring out by the action of $\mathbb{C}^* \leq G$ shows that $\mu_{\mathfrak{X}}^{-1}(0)$ is reduced and that there is a PG–equivariant morphism

$$\mu_{\mathfrak{X}}^{-1}(0) \longrightarrow \mu^{-1}(0)//\mathbb{C}^*.$$

This induces an algebra map

$$\alpha : \mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG}.$$

We now follow some of the proof of [GG, Lemma 6.3.2]. Write O_1 for the conjugacy class of rank one nilpotent matrices in $\mathfrak{gl}(n)$, and let $\overline{\mathsf{O}}_1$ denote the closure of O_1 in $\mathfrak{gl}(n)$. The moment map $\upsilon : T^*\,\mathbb{P}^{n-1} \longrightarrow \mathfrak{gl}(n)^* \cong \mathfrak{gl}(n)$ that sends (i,j) to ij gives a birational isomorphism $T^*\,\mathbb{P}^{n-1} \longrightarrow \overline{\mathsf{O}}_1$. Let $J \subset \mathbb{C}[\mathfrak{gl}(n)] = C[Z]$ be the ideal generated by all 2×2 minors of the matrix Z and also by the trace function. Then J is a prime ideal whose zero scheme is $\overline{\mathsf{O}}_1$ and the pullback morphism $\upsilon^* : \mathbb{C}[\mathfrak{gl}(n)]/J \longrightarrow \mathbb{C}[T^*\,\mathbb{P}^{n-1}]$ is a graded isomorphism.

Now the moment map $\mu_{\mathfrak{X}} : T^*\mathfrak{X} \longrightarrow \mathfrak{g}^*$ factors as the composite

$$T^*\mathfrak{X} = T^*\,\mathrm{Rep}(Q,n\delta) \times T^*\,\mathbb{P}^{n-1} \longrightarrow T^*\,\mathrm{Rep}(Q,n\delta) \times \overline{\mathsf{O}}_1 \xrightarrow{\theta} \mathfrak{pg}^*$$

where the first mapping is $\mathrm{id}\times\upsilon$ and the second mapping θ sends (X,Y,Z) to $[X,Y]+Z_0$ where Z_0 indicates that we place the matrix Z on the copy of $\mathfrak{gl}(n)$ associated to vertex 0. We have a graded algebra isomorphism

$$\mathbb{C}[T^*\operatorname{Rep}(Q,n\delta)]\otimes\mathbb{C}[\mathfrak{gl}(n)]/J\longrightarrow\mathbb{C}[T^*\mathcal{X}].$$

Now write $\mathbb{C}[X,Y,Z]=\mathbb{C}[T^*\operatorname{Rep}(Q,n\delta)\ \times\mathfrak{gl}(n)]$, and let $\mathbb{C}[X,Y,Z]([X,Y]+Z_0)$ denote the ideal in $\mathbb{C}[X,Y,Z]$ generated by all matrix entries of the ℓ matrices $[X,Y]+Z_0$. Let $\mathbf{I}$ denote the ideal $\mathbb{C}[X,Y,Z]([X,Y]+Z_0)+\mathbb{C}[X,Y]\otimes J\subset\mathbb{C}[X,Y,Z]$. From the above we have

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]\cong\mathbb{C}[T^*\operatorname{Rep}(Q,n\delta)\times\overline{\mathsf{O}}_1]/\mathbb{C}[T^*\operatorname{Rep}(Q,n\delta)\times\overline{\mathsf{O}}_1]\theta^*(\mathfrak{gl}(n))=\mathbb{C}[X,Y,Z]/\mathbf{I}.$$

Define an algebra homomorphism $r:\mathbb{C}[X,Y,Z]\longrightarrow\mathbb{C}[X,Y]$ by sending $P\in\mathbb{C}[X,Y,Z]$ to the function $(X,Y)\mapsto P(X,Y,-[X,Y]_0)$. Obviously r induces an isomorphism $\mathbb{C}[X,Y,Z]/\mathbb{C}[X,Y,Z]([X,Y]+Z_0)\cong\mathbb{C}[X,Y]/I_1$ where I_1 is the ideal of $\mathbb{C}[\operatorname{Rep}(\overline{Q},n\delta)]=\mathbb{C}[X,Y]$ generated by the elements

$$\sum_{h(a)=i}X_aX_{a^*}-\sum_{t(a)=i}X_{a^*}X_a$$

for all i not equal to zero. Observe that the linear function $P:(X,Y,Z)\mapsto TrZ=Tr([X,Y]+Z_0)$ belongs to the ideal $\mathbb{C}[X,Y,Z]([X,Y]+Z_0)$. We deduce that the mapping r sends $\mathbb{C}[X,Y]\otimes J$ to the ideal generated by

$$\operatorname{rank}(\sum_{h(a)=0}X_aX_{a^*}-\sum_{t(a)=0}X_{a^*}X_a)\le 1.$$

Thus we obtain algebra isomorphisms

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]\cong\mathbb{C}[X,Y,Z]/\mathbf{I}\cong\mathbb{C}[T^*\operatorname{Rep}(Q,n\delta)]/I_2$$

where I_2 is ideal generated by the elements

$$\sum_{h(a)=i}X_aX_{a^*}-\sum_{t(a)=i}X_{a^*}X_a$$

for all $1\le i\le\ell-1$, and

$$\operatorname{rank}(\sum_{h(a)=0}X_aX_{a^*}-\sum_{t(a)=0}X_{a^*}X_a)\le 1.$$

By [LP, Theorem 1] the G–invariant (respectively PG–invariant) elements of $\mathbb{C}[\operatorname{Rep}(\overline{Q}_\infty,\epsilon)]$ (respectively $\mathbb{C}[\operatorname{Rep}(\overline{Q},n\delta)]$) are generated by traces along oriented cycles. Since all oriented cycles in $\overline{Q}$ are oriented cycles in $\overline{Q}_\infty$ we have a surjective composition of algebra homomorphisms

$$\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\Gamma_n}\cong\mathbb{C}[\mu^{-1}(0)]^G\longrightarrow\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG}\longrightarrow\left(\frac{\mathbb{C}[\operatorname{Rep}(\overline{Q},n\delta)]}{I_2}\right)^{PG},\tag{2.7.1}$$

where the first isomorphism is Theorem 2.6(3). The left hand side is a domain of dimension $2\dim\mathfrak{h}$, so to see that the mapping is an isomorphism it suffices to prove that the right hand side also has dimension $2\dim\mathfrak{h}$.

Let I_3 be the ideal of $\mathbb{C}[\operatorname{Rep}(\overline{Q},n\delta)]$ generated by the elements

$$\sum_{h(a)=i}X_aX_{a^*}-\sum_{t(a)=i}X_{a^*}X_a$$

for all i. This is the ideal of the zero fibre of the moment map for the PG–action on $\mathrm{Rep}(\overline{Q}, n\delta)$. This ideal contains I_2 since the rank condition on the matrices is implied by the commutator condition. So there is a surjective mapping

$$\frac{\mathbb{C}[\mathrm{Rep}(\overline{Q}, n\delta)]^{PG}}{I_2^{PG}} \longrightarrow \frac{\mathbb{C}[\mathrm{Rep}(\overline{Q}, n\delta)]^{PG}}{I_3^{PG}}.$$

We do not know yet whether the right hand side is reduced or not, but by [CB2, Theorem 1.1] the reduced quotient of the right hand side is the ring of functions of the variety $(\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. As this variety has dimension $2 \dim \mathfrak{h}$ we deduce that the composition in (2.7.1) is an isomorphism, and hence that

$$\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.$$

$\square$

Remark. *In passing let us note that the commutativity of the following diagram*

$$\begin{tikzcd}
\mathbb{C}[T^* \mathrm{Rep}(Q, n\delta)] \arrow[r, "\iota"] \arrow[drr, bend right, "pr"'] & \mathbb{C}[T^* \mathrm{Rep}(Q, n\delta)] \otimes \mathbb{C}[T^* \mathbb{P}^{n-1}] \arrow[r] \arrow[d, "\upsilon^*"] & \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)] \arrow[d, "\wr"] \\
& \mathbb{C}[T^* \mathrm{Rep}(Q, n\delta)] \otimes \mathbb{C}[\overline{\mathcal{O}}_1] \arrow[r, "r"] & \mathbb{C}[T^* \mathrm{Rep}(Q, n\delta)]/I_2
\end{tikzcd}$$

where $\iota(f) = f \otimes 1$, *shows that* $\mathrm{im}\, \iota$ *maps surjectively onto* $\mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]$.

3. Differential operators

3.1. **Symplectic reflection algebras.** Let C_ℓ be the cyclic subgroup of $SL_2(\mathbb{C})$ generated by $\sigma = \mathrm{diag}(\eta, \eta^{-1})$. The vector space $V = (\mathbb{C}^2)^n$ admits an action of $S_n \wr C_\ell = S_n \ltimes (C_\ell)^n$: $(C_\ell)^n$ acts by extending the natural action of C_ℓ on $\mathbb{C}^2$, whilst S_n acts by permuting the n copies of $\mathbb{C}^2$. For an element $\gamma \in C_\ell$ and an integer $1 \leq i \leq n$ we write γ_i to indicate the element $(1, \ldots, \gamma, \ldots, 1) \in C_\ell^n$ which is non–trivial in the i–th factor.

3.2. The elements $S_n \wr C_\ell$ whose fixed points are a subspace of codimension two in V are called symplectic reflections. In this case their conjugacy classes are of two types:

- (S) the elements $s_{ij}\gamma_i\gamma_j^{-1}$ where $1 \leq i, j \leq n$, $s_{ij} \in S_n$ is the transposition that swaps i and j, and $\gamma \in C_\ell$.
- (C_ℓ) the elements γ_i for $1 \leq i \leq n$ and $\gamma \in C_\ell \setminus \{1\}$.

There is a unique conjugacy class of type (S) and $\ell - 1$ of type (C_ℓ) (depending on the non–trivial element we choose from C_ℓ). We will consider a conjugation invariant function from the set of symplectic reflections to $\mathbb{C}$. We can identify it with a pair (k, c) where $k \in \mathbb{C}$ and c is an $\ell - 1$–tuple of complex numbers: the function sends elements from (S) to k and the elements $(\sigma^m)_i$ to c_m.

3.3. There is a symplectic form on V which is induced from n copies of the standard symplectic form ω on $\mathbb{C}^2$. If we pick a basis $\{x, y\}$ for $\mathbb{C}^2$ such that $\omega(x, y) = 1$ then we can extend this naturally to a basis $\{x_i, y_i : 1 \leq i \leq n\}$ of V such that the x's and the y's form Lagrangian subspaces and $\omega(x_i, y_j) = \delta_{ij}$. We let TV denote the tensor algebra on V: with our choice of basis this is just the free algebra on generators x_i, y_i for $1 \leq i \leq n$. The symplectic reflection algebra $H_{k,c}$ associated to $S_n \wr C_\ell$ is the quotient of $TV * (S_n \wr C_\ell)$ by the following relations:

$$\begin{array}{rcll} x_i x_j = x_j x_i, & & y_i y_j = y_j y_i & \text{for all } 1 \leq i, j \leq n \\ y_i x_i - x_i y_i = 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in C_\ell} s_{ij} \gamma_i \gamma_j^{-1} + \sum_{\gamma \in C_\ell \setminus \{1\}} c_\gamma \gamma_i & & & \text{for } 1 \leq i \leq n \\ y_i x_j - x_j y_i = -\frac{k}{2} \sum_{m=0}^{\ell-1} \eta^m s_{ij} (\sigma^m)_i (\sigma^m)_j^{-1} & & & \text{for } i \neq j. \end{array}$$

(NB: my k is $-k$ for Oblomkov.)

3.4. **The spherical algebra.** The symmetrising idempotent of the group algebra $C(S_n \wr C_\ell)$ is

$$e = \frac{1}{|S_n \wr C_\ell|} \sum_{w \in S_n \wr C_\ell} w.$$

The subalgebra $eH_{k,c}e$ is denoted by $U_{k,c}$ and called the *spherical algebra*. It will be our main object of study.

3.5. **Rings of differential operators.** Recall the definition of $\mathfrak{X}$ from 2.7. Let $D_{\mathfrak{X}}(nk)$ denote the sheaf of twisted differential operators on $\mathfrak{X}$ and let $D(\mathfrak{X}, nk)$ be its algebra of global sections. This is simply the tensor product $D(\mathrm{Rep}(Q, n\delta)) \otimes D_{\mathbb{P}^{n-1}}(nk)$. (The twisted differential operators on $\mathbb{P}^{n-1}$ can be defined as follows. Let $A_n = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ be the n–th Weyl algebra. This is a graded algebra with $\deg(x_i) = 1$ and $\deg(\partial_i) = -1$. The degree zero component is the subring generated by the operators $x_i \partial_j$ which, under the commutator, generate the Lie algebra $\mathfrak{gl}(n)$. Call this subring R. Let $\mathbf{E} = \sum_{i=1}^n x_i \partial_i \in R$ be the Euler operator. Then $D(\mathbb{P}^{n-1}, nk)$ is the quotient of R by the two–sided ideal generated by $\mathbf{E} - nk$.)

The group action of PG on $\mathfrak{X}$ differentiates to an action of $\mathfrak{pg}$ on $\mathfrak{X}$ by differential operators. This gives a mapping

$$\tau : \mathfrak{pg} \longrightarrow D_{\mathfrak{X}}(nk). \tag{3.5.1}$$

(One way to understand this is to start back with $U \subset \mathrm{Rep}(Q_\infty, \epsilon)$ and look at the G action on U. Differentiating the G–action gives an action of $\mathfrak{g}$ by differential operators on U, $\hat{\tau} : \mathfrak{g} \longrightarrow D_U$. Since $\mathbb{C}^*$ acts trivially on $\mathrm{Rep}(Q, n\delta)$ and by scaling on $i \in \mathrm{Rep}(Q_\infty, \epsilon)$ we find that $\hat{\tau}(\mathrm{id}) = 1 \otimes \mathbf{E}$ where $\mathrm{id} = (I_n, I_n, \ldots, I_n) \in \mathbb{C} \subset \mathfrak{g}$. Thus we get an action of $\mathfrak{pg}$ on $(D_U / D_U(1 \otimes \mathbf{E} - nk))^{\mathbb{C}^*} = D_{\mathfrak{X}}(nk)$.)

3.6. Recall the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and its quotient $\mathfrak{pg} = \mathrm{Lie}(PG)$ which is simply $\mathfrak{g}/\mathbb{C} \cdot \mathrm{id}$ where $\mathrm{id} = (I_n, \ldots, I_n) \in \mathfrak{g}$. Let $\chi_c : \mathfrak{g} \longrightarrow \mathbb{C}$ send an element $(X) = (X_0, \ldots, X_{\ell-1}) \in \mathfrak{g}$ to

$$\chi_c(X) = \sum_{r=0}^{\ell-1} C_r \, \mathrm{Tr}(X_r)$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. Observe that

$$\chi_c(\mathrm{id}) = \mathrm{Tr}(I_n) \sum_{r=0}^{\ell-1} C_i = n \sum_{r=0}^{\ell-1} \sum_{m=0}^{\ell-1} -\eta^{rm} c_m = 0.$$

In particular χ_c is actually a character of $\mathfrak{pg}$.

Let $\chi_k : \mathfrak{g} \longrightarrow \mathbb{C}$ send an element $(X) = (X_0, \ldots, X_{\ell-1})$ to $\chi_k(c) = k \,\mathrm{Tr}(X_0)$.

We will be regularly using the character $\chi_{k,c} \in \mathfrak{g}^*$ defined by $\chi_{k,c} = \chi_c + \chi_k$.

3.7. Let us recall Oblomkov's deformed Harish–Chandra homomorphism, [O]. By Lemma 2.4 $\mathcal{S} = \omega(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n)$ is a subset of $\mathrm{Rep}(Q, n\delta)^{reg}$ which is a slice for the PG–action on $\mathrm{Rep}(Q, n\delta)$. Let

$$W_k' = (y_1 \ldots y_n)^{-k} \mathbb{C}_{(0)}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}],$$

a space of multivalued functions on $(\mathbb{C}^*)^n$. The Lie algebra $\mathfrak{g}$ acts on W_k' by projection onto its 0–th summand $\mathfrak{gl}(n)$, and then by the natural action of $\mathfrak{gl}(n)$ on polynomials (so E_{ij} acts as $y_i \partial/\partial y_j$). With this action the identity matrix in $\mathfrak{gl}(n)$ becomes the Euler operator $\mathbf{E}$ which acts by multiplication by $-nk$. Thus we can make W_k' a $\mathfrak{pg}$-module by twisting W_k' by the character χ_k since then id acts trivially. If we call this module W_k then $W_k = W_k' \otimes \chi_k$. Now define Fun' to be the space of functions on $\mathrm{Rep}(Q, n\delta)$ of the form

$$f = \tilde{f} \prod_{i=0}^{\ell-1} \det(X_i)^{r_i}$$

where $\tilde{f}$ is a rational function on $\mathrm{Rep}(Q, n\delta)^{reg}$ regular on $\mathcal{S}$, $r_i = \sum_{j=0}^{i} C_j + \sigma$ and $\sigma = \ell^{-1} \sum_{s=0}^{\ell-1} sC_s$. Then $(Fun' \otimes W_k)^{\mathfrak{pg}}$ is a space of $(\mathfrak{pg}, \chi_c)$–semiinvariant functions defined on a neighbourhood of $\mathcal{S}$ which take values in W_k. This space is a free $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^{\Gamma_n}$–module of rank 1, the isomorphism being given by restriction to $\mathcal{S}$. (Note that the determinant of an element of the form $(X, \ldots, X)$ is $\det(X)^{\sum r_i} = 1$ as $\sum r_i = 0$.) Any $\mathfrak{pg}$–invariant differential operator, D, acts on such a function, f. Oblomkov defines his homomorphism to be the restriction of $D(f)$ to $\mathcal{S}$.

3.8. We can view the above procedure in terms of $\mathrm{Rep}(Q_\infty, \epsilon)$. Thanks to Lemma 2.5 we use $\mathcal{S}_\infty = \mathcal{S} \times (1, \ldots, 1) \in U_\infty$ as a slice for the G–action. The space $\mathcal{S} \times (\mathbb{C}^*)^n$ is a closed subset of U_∞ since the condition that i be cyclic for $\mathrm{diag}(x_1, \ldots, x_n)$ is equivalent to $i \in (\mathbb{C}^*)^n$. Thus functions on a neighbourhood of $\mathcal{S}_\infty$ in U_∞ can be identified with functions from a neighbourhood of $\mathcal{S}$ taking values in functions on $(\mathbb{C}^*)^n$. In particular, we can consider elements on $(Fun' \otimes W_k)^{\mathfrak{pg}}$ first as $(\mathfrak{g}, \chi_{k,c})$–semiinvariant functions from a neighbourhood of $\mathcal{S}$ taking values in W_k' and hence as $(\mathfrak{g}, \chi_{k,c})$–semiinvariant functions on an open set in a neighbourhood of $\mathcal{S}_\infty$. We can apply any element of $D \in D(U_\infty)^{\mathfrak{g}}$ to these $(\mathfrak{g}, \chi_{k,c})$–semiinvariant functions and then restrict to S_∞ to get a homomorphism

$$\mathfrak{F}_{k,c} : D(U_\infty)^{\mathfrak{g}} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n).$$

3.9. Since $\mathrm{Rep}(Q_\infty, \epsilon) = \mathrm{Rep}(Q, n\delta) \times \mathbb{C}^n$ there is a mapping

$$\mathfrak{G} : D(\mathrm{Rep}(Q, n\delta))^{\mathfrak{pg}} \longrightarrow D(U_\infty)^{\mathfrak{g}}$$

which sends $D \in D(\mathrm{Rep}(Q, n\delta))^{\mathfrak{pg}}$ to $(D \otimes 1)$. Oblomkov's homomorphism is $\mathfrak{F}_{k,c} \circ \mathfrak{G}$.

3.10. Differentiating the G–action on U_∞ gives a Lie algebra homomorphism $\hat{\tau} : \mathfrak{g} \longrightarrow \mathsf{Vect}(U_\infty)$ which we extend to an algebra map

$$\hat{\tau} : U(\mathfrak{g}) \longrightarrow D(U_\infty).$$

By Lemma 2.5 U_∞ is a principal G–bundle over $\mathfrak{h}^{\mathrm{reg}}/\Gamma_n$, so (a generalisation of) [Schw, Corollary 4.5] shows that the kernel of $\mathfrak{F}_{k,c}$ is $(D(U_\infty)(\hat{\tau} - \chi_{k,c})(\mathfrak{g}))^{\mathfrak{g}}$. Moreover, since the finite group Γ_n acts freely on $\mathfrak{h}^{\mathrm{reg}}$ we can identify $D(\mathfrak{h}^{\mathrm{reg}}/\Gamma_n)$ with $D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}$.

3.11. Recall that

$$D_{\mathfrak{X}}(nk) \cong \left(\frac{D_U}{D_U(\hat{\tau} - \chi_k)(\mathbb{C}\cdot \mathrm{id})}\right)^{\mathbb{C}^*}.$$

Hence we have

$$\left(\frac{D_U}{D_U(\hat{\tau} - \chi_{k,c})(\mathfrak{g})}\right)^{G} \cong \left(\frac{D_{\mathfrak{X}}(nk)}{D_{\mathfrak{X}}(nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG}, \tag{3.11.1}$$

where $U = \{(X,i) : i \neq 0\} \subset \mathrm{Rep}(Q_\infty, n\delta)$ as in 2.7. Now we consider the restriction mapping $D_U \longrightarrow D(U_\infty)$. Composing the global sections of the above isomorphism with this restriction and the homomorphism $\mathfrak{F}_{k,c}$ gives

$$\mathfrak{R}'_{k,c} : \left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}.$$

3.12. Let

$$\delta_{k,c}(x) = \delta^{-k-1}\delta_\Gamma^\sigma$$

where $\delta = \prod_{1\le i<j\le n}(x_i^\ell - x_j^\ell)$ and $\delta_\Gamma = \prod_{i=1}^n x_i$. Define a twisted version of $\mathfrak{R}'_{k,c}$ above

$$\mathfrak{R}_{k,c}(D) = \delta_{k,c}^{-1} \circ \mathfrak{R}'_{k,c}(D) \circ \delta_{k,c}$$

for any differential operator D.

3.13. Our main result is the following.

Theorem. *For all values of k and c, the homomorphism $\mathfrak{R}_{k,c}$ has image $\mathrm{im}\,\theta_{k,c}$. In particular we have an isomorphism*

$$\theta_{k,c}^{-1} \circ \mathfrak{R}_{k,c} : \left(\tfrac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{\mathfrak{pg}} \xrightarrow{\ \sim\ } U_{k,c}.$$

Proof. Let us abuse notation by writing $U_{k,c}$ for the image of $U_{k,c}$ in $D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}$ under $\theta_{k,c}$.

Since $\mathfrak{X} = \mathrm{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$ there is a mapping

$$D(\mathrm{Rep}(Q, n\delta))^{PG} \longrightarrow D(\mathfrak{X}, nk)^{PG} \longrightarrow D(\mathfrak{h}^{\mathrm{reg}})^{\Gamma_n}$$

which sends $D \in D(\mathrm{Rep}(Q, n\delta))^{PG}$ to $\mathfrak{R}_{k,c}(D \otimes 1)$. Recall τ from (3.5.1). Since $\mathrm{gr}\,\tau = \mu_{\mathcal{X}}^*$ we have an inclusion $\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg}) \subseteq \mathrm{gr}(D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg}))$. This gives a graded surjection

$$p : \left(\frac{\mathrm{gr}\, D(\mathfrak{X}, nk)}{\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})}\right)^{PG} \longrightarrow \mathrm{gr}\left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG}.$$

By Remark 2.7 the composition

$$\mathrm{gr}\, D(\mathrm{Rep}(Q, n\delta))^{PG} \longrightarrow \mathrm{gr}\, D(\mathfrak{X}, nk)^{PG} \longrightarrow \left(\frac{\mathrm{gr}\, D(\mathfrak{X}, nk)}{\mathrm{gr}(D(\mathfrak{X}, nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})}\right)^{PG} \longrightarrow \mathrm{gr}\left(\frac{D(\mathfrak{X}, nk)}{D(\mathfrak{X}, nk)(\tau - \chi_c)(\mathfrak{pg})}\right)^{PG}$$

is surjective. Thus the homomorphism

$$D(\mathrm{Rep}(Q,n\delta))^{PG} \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG}$$

is also surjective. In particular, by 3.9 this implies that the image of $\mathfrak{R}_{k,c}$ equals the image of Oblomkov's Harish–Chandra homomorphism, which, by [O, Theorem 2.5], is $U_{k,c}$.

Thus we have a filtered surjective homomorphism

$$\mathfrak{R}_{k,c} : \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG} \longrightarrow U_{k,c}.$$

Thus the dimension of the left hand side is at least $2\dim\mathfrak{h} = \dim U_{k,c}$. By Proposition 2.7

$$\left(\frac{\mathrm{gr}\, D(\mathfrak{X},nk)}{\mathrm{gr}(D(\mathfrak{X},nk))\mu_{\mathcal{X}}^*(\mathfrak{pg})}\right)^{PG} \cong \mathbb{C}[\mu_{\mathfrak{X}}^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\Gamma_n}.$$

Hence p is a surjection from a domain of dimension $2\dim\mathfrak{h}$ onto an algebra of dimension at least $2\dim\mathfrak{h}$ and is hence an isomorphism. Thus $(D(\mathfrak{X},nk)/D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg}))^{\mathfrak{pg}}$ is a domain of dimension $2\dim\mathfrak{h}$. This implies that $\mathfrak{R}_{k,c}$ is an isomorphism. $\square$

4. Application: Shift functors

4.1. **The Holland-Schwarz Lemma.** We want to understand the space

$$\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}.$$

As we observed in the proof of Theorem 3.13 there is a natural surjective homomorphism

$$\frac{\mathrm{gr}\, D(\mathrm{Rep}(Q_\infty,\epsilon))}{\mathrm{gr}\, D(\mathrm{Rep}(Q_\infty,\epsilon))\mu^*(\mathfrak{g})} \longrightarrow \mathrm{gr}\left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right). \tag{4.1.1}$$

It turns out that this is an isomorphism.

Lemma (Schwarz, Holland)**.** *The homomorphism* (4.1.1) *is an isomorphism of* $\mathbb{C}[T^*\,\mathrm{Rep}(Q_\infty,\epsilon)]$*–modules.*

Proof. This is [H, Lemma 2.2] since, by Theorem 2.6(2), the moment map μ is flat. $\square$

4.2. This lets us prove the first part of the isomorphism in the statement of Theorem 1.4.

Lemma. *There is an algebra isomorphism*

$$\left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon))}{D(\mathrm{Rep}(Q_\infty,\epsilon))(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^{G} \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG}$$

Proof. We have a natural $\mathfrak{pg}$–equivariant mapping

$$D(\mathrm{Rep}(Q_\infty,\epsilon))^{\mathbb{C}^*} \longrightarrow D_U^{\mathbb{C}^*} \longrightarrow D_{\mathfrak{X}}(nk)$$

which induces a homomorphism

$$D(\mathrm{Rep}(Q_\infty,\epsilon))^{G} \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(PG)}\right)^{\mathfrak{pg}}.$$

This is surjective since, as we observed in the proof of Theorem 3.13, the image of $D(\mathrm{Rep}(Q,n\delta))^{PG} \subset D(\mathrm{Rep}(Q_\infty,\epsilon)^G$ spans the right hand side. By (3.11.1) the kernel of this homomorphism includes the ideal $(D(\mathrm{Rep}(Q,\infty),\epsilon)(\hat{\tau}-\chi_{k,c})(\mathfrak{g}))^G$. Hence we have a surjective homomorphism

$$\left(\frac{D(\mathrm{Rep}(Q_\infty,\epsilon)}{D(\mathrm{Rep}(Q,\infty),\epsilon)(\hat{\tau}-\chi_{k,c})(\mathfrak{g})}\right)^{G} \longrightarrow \left(\frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau-\chi_c)(\mathfrak{pg})}\right)^{PG}. \tag{4.2.1}$$

By Lemma 4.1 and Proposition 2.7 there is an isomorphism

$$\left(\operatorname{gr}\frac{D(\operatorname{Rep}(Q_\infty,\epsilon)}{D(\operatorname{Rep}(Q,\infty),\epsilon)(\hat\tau-\chi_{k,c})(\mathfrak g)}\right)^G \cong \left(\frac{\operatorname{gr}D(\operatorname{Rep}(Q_\infty,\epsilon))}{\operatorname{gr}D(\operatorname{Rep}(Q_\infty,\epsilon))\mu^*(\mathfrak g)}\right)^G = \mathbb C[\mu^{-1}(0)]^G = \mathbb C[\mathfrak h\oplus\mathfrak h^*]^{\Gamma_n}.$$

This shows that the algebra on the left is a domain of dimension of $2\dim\mathfrak h$ and so (4.2.1) is also injective, as required. □

4.3. **Shifting.** The previous two lemmas provide us with an interesting series of bimodules. Given a character Λ of G we define

$$B^\Lambda_{k,c} = \left(\frac{D(\operatorname{Rep}(Q_\infty,\epsilon))}{D(\operatorname{Rep}(Q_\infty,\epsilon))(\hat\tau-\chi_{k,c})(\mathfrak g)}\right)^\Lambda$$

to be the set of (G,Λ)–semiinvariants. Thanks to Lemma 4.2 and Theorem 3.13 this is a right $U_{k,c}$–module. Now observe that if $x\in\mathfrak g$ and $D\in D(\operatorname{Rep}(Q_\infty,\epsilon))^\Lambda$ then

$$[\tau(x),D]=\lambda(x)D$$

where $\lambda = d\Lambda$. It follows that $B^\Lambda_{k,c}$ is also a left $(D(\operatorname{Rep}(Q_\infty,\epsilon))/D(\operatorname{Rep}(Q_\infty,\epsilon))(\hat\tau-\chi_{k,c}-\lambda)(\mathfrak g))^G$-module. So tensoring sets up a *shift functor*

$$S^\Lambda_{k,c} : \left(\frac{D(\operatorname{Rep}(Q_\infty,\epsilon))}{D(\operatorname{Rep}(Q_\infty,\epsilon))(\hat\tau-\chi_{k,c})(\mathfrak g)}\right)^G\text{-mod} \longrightarrow \left(\frac{D(\operatorname{Rep}(Q_\infty,\epsilon))}{D(\operatorname{Rep}(Q_\infty,\epsilon))(\hat\tau-\chi_{k,c}-\lambda)(\mathfrak g)}\right)^G\text{-mod}.$$

4.4. The character group of G is isomorphic to $\mathbb Z^\ell$ via

$$(i_0,\dots,i_{\ell-1})\mapsto((g_0,\dots,g_{\ell-1})\mapsto\prod_{r=0}^{\ell-1}\det(g_r)^{i_r}).$$

Corresponding to the standard basis element ϵ_i is the character χ_i of $\mathfrak g$ which sends $X\in\mathfrak g$ to $\operatorname{Tr}(X_i)$.

Lemma. *The bimodule $B^{\epsilon_i}_{k,c}$ above is a $(U_{k,c},U_{k',c'})$–bimodule where $k'=k+1$ and $c'=c+(1-\eta^{-i},1-\eta^{-2i},\dots,1-\eta^{-(\ell-1)i})$.*

Proof. Recall that (k,c) corresponds to the character of $\mathfrak g$ we called $\chi_{k,c}$ which is defined as

$$\chi_{k,c}(X)=(C_0+k)\operatorname{Tr}(X_0)+\sum_{j=1}^{\ell-1}C_j\operatorname{Tr}(X_j),$$

where $C_r=\ell^{-1}(1-\sum_{m=1}^{\ell-1}\eta^{mr}c_m)$ for $1\le r\le\ell-1$ and $C_0=\ell^{-1}(1-\ell-\sum_{m=1}^{\ell-1}c_m)$. We need to calculate (k',c') so that $\chi_{k,c}+\chi_i=\chi_{k',c'}$. So we have

$$(\chi_{nk,c}+\chi_i)(X)=(C_0+k)\operatorname{Tr}(X_0)+\operatorname{Tr}(X_i)+\sum_{j=1}^{\ell-1}C_j\operatorname{Tr}(X_j)=(C_0'+k')\operatorname{Tr}(X_0)+\sum_{j=1}^{\ell-1}C_j'\operatorname{Tr}(X_j).$$

Calculation shows that $k'=k+1$ and that if $i=0$ then $C_j'=C_j$ and otherwise

$$C_j' = C_j + \begin{cases}-1 & \text{if } j=0\\ 1 & \text{if } j=i\\ 0 & \text{otherwise.}\end{cases}$$

These unpack to give $c_m'=c_m+1-\eta^{-mi}$. □

4.5. **Question.** Thus for each $0 \leq i \leq \ell - 1$ we have a *shift functor*

$$S_i : U_{k,c}\text{-mod} \longrightarrow U_{k+1,c'}\text{-mod}$$

where c' is as above. When is this an equivalence of categories?

Remarks. (1) *We have been able to prove this is an equivalence when* (k, c) *can be reached from* $(0, 0)$ *by shifting.*

(2) *Shift functors are also constructed in* [BC] *and* [V]*. Hopefully they agree with the functors here.*

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